Connection to work of Chen-Bo Zhu

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The connection to Chen-Bo is the following article, dealing with an entirely different problem:
Representations with scalar K-types and applications
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Theorem 3.2. *Suppose that*

$$\psi(S^r(\mathfrak{g})^G) + \mathcal{U}(\mathfrak{g})\mathfrak{k} \supseteq \psi(S^r(\mathfrak{p})^K), \qquad r \in \mathbb{Z}_{\geq 0}$$

(true at least for all classical G), and $\rho \in \hat{K}$ is one-dimensional. For an irreducible Harish-Chandra module V with $V_\rho \neq 0$, the infinitesimal character of V determines V up to infinitesimal equivalence.

Chen-Bo uses this to derive results about the dual pairs correspondence.

Spherical Dual of U(p,q) I

This is about research that I did in 1984 while visiting Marseille. It was presented at MSRI in 1989, and was only published later in Noncommutative Harmonic Analysis in honor of Jacques Carmona, Patrick Delorme and Michelle Vergne editors, Progress in Mathematics, volume 220, 2024.

It resolves the question of determining the spherical unitary dual of $U(p,q)$ with integral infinitesimal character. For the quasisplit groups $U(n,n)$ and $U(n+1,n)$ the classification of the spherical unitary dual is complete.

While discussing related matters with Daniel Wong last week, I/we realized that the results extend to the case of representations containing a $1-$dimensional lowest $K-$type.

Spherical Dual of U(p,q) II

Theorem

Let $X(\gamma)$ be a spherical principal series with integral infinitesimal character and Langlands subquotient $\overline{X}(\gamma)$. Then $\overline{X}(\gamma)$ is unitary if and only if there is a $\theta-$stable parabolic $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ and an irreducible 1-dimensional unitary representation W on L such that

$$\mathcal{R}^i_{\mathfrak{q}}(W) = \begin{cases} 0 & i \neq dim(\mathfrak{u} \cap \mathfrak{k}) \\ \text{has } \overline{X}(\gamma) \text{ as a subquotient for} & i = s := dim(\mathfrak{u} \cap \mathfrak{k}) \end{cases}$$

The sharper statement that $\mathcal{R}^s_{\mathfrak{q}}(W) = \overline{X}(\gamma)$ should hold.

Summary I

In the case of the quasisplit groups, $U(n,n)$ and $U(n+1,n)$, we determine the *full* spherical unitary spectrum.
The necessary conditions for unitarity are obtained in sections 4 and 5 of [B1]. The connection is as follows. Let M be the Levi component of the minimal parabolic subgroup of G. To establish whether a spherical module π is unitary, one has to check that for any K type (V,μ) occuring in π, a certain form on $(V^*)^M$ is positive definite. To get necessary conditions for unitarity we compute the signature of this hermitian form on a certain set of K-types which we call *relevant*. The Weyl group W acts on $(V^*)^M$, and for a relevant K-type, the hermitian form is completely determined by the action of W. The W representations that come from relevant K types are called *relevant W-types*. The hermitian form is the same as the one for the affine Hecke algebras of type B and C considered in [B1].
To show that the necessary conditions for unitarity obtained from the relevant K-types are also sufficient, we have to prove certain

Summary II

irreducibility and unitarity results for unipotent representations. For the unitary groups these representations are cohomologically induced, so we use [KV] and the references therein. The conclusion is that the unitary spherical dual for $U(n, n)$, coincides with the unitary spherical dual of the affine Hecke algebra of type C, while the spherical unitary dual of $U(n+1, n)$ coincides with the spherical dual of the afffine Hecke algebra of type B.

A separate section deals with Langlands parameters containing nontrivial fine K-types. These occur in $U(n, n)$ only. The same techniques as for the spherical case imply that the unitary dual for such parameters is contained in the spherical unitary dual of the affine Hecke algebra of type D. There are some irreducibility results that are not addressed but ought to be accessible with the results in [V1] and [KV].

Unipotent Representations for Sp(2n,R) I

These results are from work with Pandžić on $U(pq)$ and $Sp(2n, \mathbb{R})$. The representations are well known to the specialists, particularly Chen-Bo, from his own work and the generalizations in [BMSZ].

I quote from the work B-Pandžić. For $U(p, q)$ there is a series of cohomologically induced modules possibly not in good range. We calculate the Dirac cohomology.

We first study the special case $p + q = 2k \leq n$. Let $\epsilon, \eta \in \{0, 1\}$. Denote by $\mathbb{C}_\epsilon$ the character det^ϵ of $O(p)$, and by $\mathbb{C}_\eta$ the character det^η of $O(q)$. (If $p = 0$, we require $\epsilon = 0$, and if $q = 0$, we require $\eta = 0$.) Let $\mathbb{C}_{\epsilon,\eta}$ be the character of $O(p, q)$ with restriction to $O(p) \times O(q)$ equal to $\mathbb{C}_\epsilon \otimes \mathbb{C}_\eta$. The representation $X = X(p, q; \epsilon, \eta)$ of G is obtained by theta lifting the character $\mathbb{C}_{\epsilon,\eta}$ from $O(p, q)$ to G.
For n odd, there is another series of special unipotent representations $X'(p, q, \epsilon, \eta)$, with $p + q = 2k = n + 1$. Here (ϵ, η)

Unipotent Representations for Sp(2n,R) II

can be $(0,0)$, $(0,1)$ or $(1,0)$ in case p and q are both nonzero, and if p or q is zero, then there is just one case, $(\epsilon, \eta) = (0,0)$.
The infinitesimal character of $X'(p,q,\epsilon,\eta)$ is

$$\Lambda = (k-1, k-2, \ldots, -k+1).$$

These modules are obtained from the dual pair correspondence $Sp(2n,\mathbb{R}) \times O(p,q)$.

Theorem 3.11. *Let $X = X(p, q; \epsilon, \eta)$ where $p + q = 2k$. The general formula for the cohomology is*

$$H_D(X) = \sum_{\tau \ special} [E_\tau : H_D(X(p, q, \epsilon, \eta))]E_\tau + \\ + \sum_{\tau \ not \ special} [E_{-w_0\tau} : H_D(X(q, p, \eta, \epsilon))]E_\tau.$$

For $\tau = x\Lambda - \rho_{\mathfrak{k}}$ special as in (3.4), the multiplicity is as follows. (For τ not special, see Lemma 3.2.)

(1) *p, q even and positive. (For p or q equal to zero, see Proposition 3.8.)*

Ia: *$\epsilon + \eta \equiv n(\mathrm{mod}\ 2)$ and $u_\tau \equiv \epsilon(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is*

$$\binom{k-1}{\frac{p}{2}} = \binom{k-1}{\frac{q-2}{2}}.$$

Ib: *$\epsilon + \eta \equiv n(\mathrm{mod}\ 2)$ and $u_\tau \equiv \epsilon + 1(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is*

$$\binom{k-1}{\frac{p-2}{2}} = \binom{k-1}{\frac{q}{2}}.$$

IIa: *$\epsilon + \eta \equiv n + 1(\mathrm{mod}\ 2)$ and $u_\tau \equiv \epsilon + 1(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is*

$$\binom{k}{\frac{p}{2}}.$$

IIb: *$\epsilon + \eta \equiv n + 1(\mathrm{mod}\ 2)$ and $u_\tau \equiv \epsilon(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is 0.*

(2) *p, q odd.*

I: *$\epsilon + \eta \equiv n(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is*

$$\binom{k-1}{\frac{q-1}{2}} = \binom{k-1}{\frac{p-1}{2}}.$$

II: *$\epsilon + \eta \equiv n + 1(\mathrm{mod}\ 2)$. The multiplicity of τ in $H_D(X)$ is 0. In this case, $H_D(X) = 0$.*

Theorem 3.14. *Let $X' = X'(p,q;\epsilon,\eta)$ where $p+q = 2k = n+1$. Then the Dirac cohomology of X' is*

$$H_D(X') = [E_0 : H_D(X')]E_0,$$

where E_0 is the trivial $\widetilde{K}-$module. The multiplicity is as follows.

(1) *p,q even and positive.*

I: $(\epsilon,\eta) = (1,0)$. *The multiplicity of E_0 in $H_D(X')$ is*

$$\binom{k-1}{\frac{p}{2}} = \binom{k-1}{\frac{q-2}{2}}.$$

II: $(\epsilon,\eta) = (0,0)$. *The multiplicity of E_0 in $H_D(X')$ is*

$$\binom{k}{\frac{p}{2}} = \binom{k}{\frac{q}{2}}.$$

III: $(\epsilon,\eta) = (0,1)$. *The multiplicity of E_0 in $H_D(X')$ is*

$$\binom{k-1}{\frac{q}{2}} = \binom{k-1}{\frac{p-2}{2}}.$$

(2) *p,q odd.*

I: $(\epsilon,\eta) = (1,0)$ *or* $(0,1)$. *The multiplicity of E_0 in $H_D(X')$ is*

$$\binom{k-1}{\frac{p-1}{2}} = \binom{k-1}{\frac{q-1}{2}}.$$

II: $(\epsilon,\eta) = (0,0)$. *The multiplicity of E_0 in $H_D(X')$ is 0, so $H_D(X') = 0$.*

Best Wishes Chen-Bo

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