Crystals of Representations

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- What can we do with symmetries? Informally, we should be able to compose and invert them. For this talk, we'll be concerned with 2×2 matrices with determinant 1:

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - bc = 1 \right\},$$

  where the composition is matrix multiplication. This is almost the set of symmetries of the Riemann sphere from complex analysis.

- Since the determinant is 1, any matrix in $SL(2)$ is invertible, and their inverse also lies in $SL(2)$. These properties make $SL(2)$ into a **group**.

- We can multiply a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by a matrix in $SL(2)$. So an element of our group corresponds to a linear transformation of a vector space and the collection of these linear transformations is what we'll call a **representation.**
- You might think that $SL(2)$ consists of 2×2 matrices, so you can only apply it to vectors in a 2-dimensional vector space, but ...

- We can also act on row vectors

$$\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax + cy & bx + dy \end{pmatrix}$$

- Let $V(k)$ denote the (k-dimensional) vector space of polynomials in x and y of degree $k-1$, that is,

$$V(k) = \mathrm{Span}\left\{x^{k-1}, x^{k-2}y, \ldots, y^{k-1}\right\}.$$

- Let $p(x, y) \in V(k)$. Then we can define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x, y) = p(ax + cy, bx + dy).$$

This makes $V(k)$ into a representation for $SL(2)$.

Our action on $V(k)$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(ax+cy, bx+dy).$$

For example,

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot (x^2+2xy) = (2x+3y)^2 + (2x+3y)(x+2y)$$
$$= 4x^2+12xy+9y^2+2x^2+4xy+3xy+5y^2$$
$$= 6x^2+15xy+14y^2.$$

$SL(2)$ is better than most finite groups, since it is continuous, and even differentiable in a sense. If you know what a manifold is, it is a good exercise to check that $SL(2)$ is one.

- We can notice that the identity matrix fixes every polynomial, so we might hope that matrices that are "near" it only perturb the polynomials "a little bit".
- To make this precise, we bring in some calculus. For example, for t small, matrices of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ are "close" to the identity matrix.
- Let's compute the derivative of this action at $t = 0$ (the identity matrix)

$$\frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot p(x, y) = \frac{d}{dt}\bigg|_{t=0} p(x, tx + y).$$

for $p(x, y) \in V(3)$.

- Since $V(3)$ has basis $\{x^2, xy, y^2\}$, we compute

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot x^2 = x^2 = x^2 \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} 0$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot xy = x(tx+y) = tx^2 + xy \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} x^2$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot y^2 = (tx+y)^2 = t^2x^2 + 2txy + y^2 \xrightarrow{\left.\frac{d}{dt}\right|_{t=0}} 2xy$$

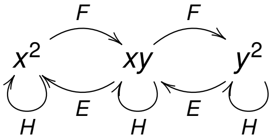
- So we may say that (if we are thinking about the action on $V(3)$)

$$E := \left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = x\frac{\partial}{\partial y}.$$

- Similarly we might notice that for t close to 0, matrices of the form $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ or $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ are also close to the identity matrix.
- Similarly, we compute:

$$\begin{aligned} F &:= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} &= y\frac{\partial}{\partial x}. \\ H &:= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \end{aligned}$$

- Using $E = x\frac{\partial}{\partial y}, F = y\frac{\partial}{\partial x}, H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, we can make a graph:



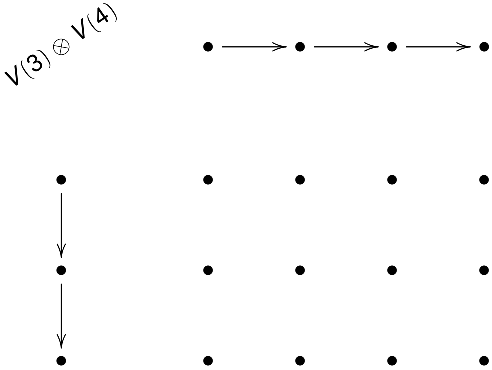
- Notice that our basis consists of eigenvectors for H, and that E and F are almost inverses to each other, so it suffices to record the action of F. Then our picture of $V(3)$ becomes

$$\bullet \rightarrow \bullet \rightarrow \bullet$$

  This graph is the **crystal** of $V(3)$.
- Similarly, the crystal of $V(k)$ consists of k vertices arranged on a line with arrows going to the right.

- Representations are vector spaces, so we can consider the direct sum $V \oplus W$ of two representations V and W.
- These are more interesting than before, for example, we can decompose $\mathbb{R}^3 \cong \mathbb{R}^1 \oplus \mathbb{R}^1 \oplus \mathbb{R}^1$ as vector spaces, but, for example, $V(3) \not\cong V(1) \oplus V(1) \oplus V(1)$ (Hint: what is $V(1)$? How do E, F, H act on it?).
- The above observation means that $\mathbb{R}^1$ is the only **indecomposable** vector space, but there are are lots of indecomposable representations of $SL(2)$ (actually the $V(k)$ for $k \geq 1$ are all of them).
- The crystal of a direct sum of representations is the disjoint union of their crystals.

- Similarly, the tensor product $V \otimes W$ of two representations V and W is also a representation. It is an important question in representation theory to write $V \otimes W$ as a direct sum of indecomposables.
- We can answer this question using crystals! We have to define what we mean by "tensoring" two crystals.



From the picture on the previous slide, we can deduce the Clebsch-Gordan formula (valid for $k \leq l$)

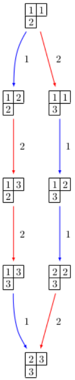
$$V(k) \otimes V(l) \cong V(k-l+1) \oplus V(k-l+3) \oplus \ldots \oplus V(k+l-3) \oplus V(k+l-1)$$

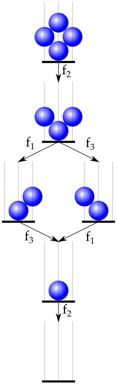
that arises in many contexts, even in angular momentum coupling in quantum mechanics. For example,

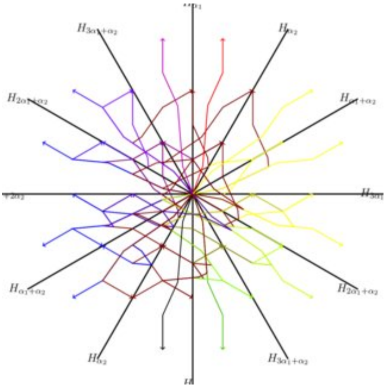
$$V(2) \otimes V(2) = V(3) \oplus V(1)$$

describes the combination of two fermions into a bosonic composite.

Crystals for groups larger than $SL(2)$ will have edges of different colors, and there are many combinatorial models for them (these three are different crystals):







Crystals themselves have interesting symmetries, and these lead down all sorts of rabbit holes to quantum groups, knot theory or geometry.

- In a project work in progress with Anne Dranowski, Joel Kamnitzer, Tanny Libman and Calder Morton-Ferguson, I study the relationship between two models for crystals, one involving dropping beads on runners, and one involving components of Nakajima quiver varieties. It would be interesting to compare the actions of the cactus group (this is the natural symmetry group for crystals) on the two sides of this equivalence.
- In [CGP16], Chmutov, Glick and Pylyavskyy proved that the action of the cactus group and the action of the Berenstein-Kirillov group on $SL(n)$-crystals coincide. Using the model for crystals involving beads on runners, we again have two natural group actions on crystals for other groups. Do these two actions coincide?

- In [AE20] work with Tair Akhmejanov, I proved a “cyclic sieving” result about tensor products $V_1 \otimes \cdots \otimes V_m$ of certain representations of $SL(n)$ where the cyclic action is given by rotation of tensor factors, where the cyclic sieving polynomial is given by a certain generalized Kostka polynomial. We used the fancy technology of the Geometric Satake equivalence to establish this, but there should be a proof using a cyclic action on crystals.
- In [ST18], Schumann and Torres prove a branching rule (an important problem from representation theory) for $\mathfrak{sp}(2n, \mathbb{C}) \subset \mathfrak{sl}(2n, \mathbb{C})$ in terms of Littelmann paths. There are some natural candidates where one could hope to obtain a similar result.

[AE20] Tair Akhmejanov and Balázs Elek.
Promotion and Cyclic Sieving on Rectangular δ-Semistandard Tableaux.
arXiv e-prints, page arXiv:2010.13930, October 2020.

[CGP16] Michael Chmutov, Max Glick, and Pavlo Pylyavskyy.
The Berenstein-Kirillov group and cactus groups.
arXiv e-prints, page arXiv:1609.02046, Sep 2016.

[ST18] Bea Schumann and Jacinta Torres.
A non-Levi branching rule in terms of Littelmann paths.
Proc. Lond. Math. Soc. (3), 117(5):1077–1100, 2018.