Heaps, Crystals and Preprojective algebra modules

Balázs Elek
(joint with Anne Dranowski, Joel Kamnitzer, Tanny Libman and Calder Morton-Ferguson)

Cornell University,
Department of Mathematics

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Let \( g = \mathfrak{sl}_n(\mathbb{C}) \) be the Lie algebra of trace 0 matrices and \( V = \mathbb{C}^n \). The standard basis vectors \( v_1, \ldots, v_n \) form a basis for \( V \) that has several favorable properties:

1. Each basis vector is an eigenvector for the action of the subalgebra \( \mathfrak{h} \) of diagonal matrices, i.e.
   \[
   \text{diag}(t_1, \ldots, t_n) \cdot v_k = t_k v_k
   \]

2. The matrices \( E_{i,j} = (e_{mn}) \) s.t. \( e_{mn} = \begin{cases} 1 & \text{if } (m,n) = (i,j) \\ 0 & \text{else} \end{cases} \) for \( i \neq j \) “almost permute” these vectors, i.e. \( E_{i,j} \cdot v_j = v_i \) and \( E_{i,j} \cdot v_k = 0 \) for \( k \neq j \).

3. We only need to use the matrices \( F_i = E_{i+1,i} \) to reach any basis vector from \( v_1 \).
Thus we can encode the representation as a colored directed graph, for example, \( \mathfrak{sl}_3 \) acting on \( \mathbb{C}^3 \) could be represented like this:

\[
\begin{align*}
\mathbf{v}_1 & \xrightarrow{F_1} \mathbf{v}_2 \xrightarrow{F_2} \mathbf{v}_3
\end{align*}
\]

Our aim is to generalize this idea and we’d hope that the nice basis we found is compatible with things we want to do with \( g \)-representations, like tensor product decompositions and branching.
This works only as long as each weight space is one-dimensional. We already run into trouble with the adjoint representation of $\mathfrak{sl}_3$, as $\ker F_1$, $\ker F_2$, $\mathrm{im} F_1$, $\mathrm{im} F_2$ are all different subspaces of $\mathfrak{h}$. 
Fortunately, thanks to Kashiwara [Kas91], there is a way of fixing this problem: by going first to the quantized universal enveloping algebra $U_q(g)$ and then taking a limit as $q \to 0$ in a suitable sense. It turns out that in this setting, choosing a good basis is always possible. This object, the directed graph with vertices the basis elements and edges labeled by the action of the lowering operators is called a crystal. Since the representation theory of $U_q(g)$ is very similar to that of $U(g)$, we can use this combinatorial gadget to study representations.
Why do we like crystals? Because the rules for tensoring and branching are purely combinatorial. For $\mathfrak{g} = \mathfrak{sl}_2$-crystals, tensor product decompositions are given by:
We know that for an irreducible $\mathfrak{sl}_n$-representation $V_\lambda$ of highest weight $\lambda$, $\dim(V_\lambda) = \#SSYT(\lambda)$ with entries up to $n$. The crystal of the adjoint representation of $\mathfrak{sl}_3$ is

**Figure:** The crystal $B(\omega_1 + \omega_2)$ for $A_2$
Definition 1 (Stembridge [Ste96])

Let $W$ be a Coxeter group. An element $w$ is **fully commutative** if any reduced word for $w$ can be obtained from any other by using only the Coxeter relations that involve commuting generators.

Example 2

If $W = S_n$, then $w$ is fully commutative if and only if it is 321-avoiding.
We will mainly be interested in fully commutative elements associated to minuscule representations. Recall that a fundamental weight $\omega_p$ is minuscule if $W$ acts transitively on the set of weights appearing in the representation $V(\omega_p)$. Let $P_p$ be the maximal parabolic subgroup associated to $\omega_p$, then the (unique) minimal length representative $w_0^P$ for $w_0 W_{P_p}$ in $W/W_{P_p}$ is fully commutative.

**Example 3**

Let $g = sl_4$. All fundamental weights are minuscule, and $V(\omega_2) \cong \wedge^2 \mathbb{C}^4$. Then $w_0^P = s_2 s_1 s_3 s_2$, which is indeed fully commutative.
Following Stembridge [Ste96], given a word \( w = r_1 r_2 \cdots r_k \) in \( W \), we define the **heap** \( H(w) \) of \( w \) to be the pair consisting of:

1. The poset on \( \{1, \ldots, k\} \), where we declare \( i \preceq j \) if \( i > j \) and the corresponding entry of the Cartan matrix \( a_{ij} \neq 0 \) and we take transitive closure of this relation.

2. The labeling function \( \pi \) that sends \( i \) to \( s_i \).

One can visualize a heap as a configuration of beads on runners arranged according to the Dynkin diagram as in Figure 2, where are dropping the beads one by one, and bead \( i \) is dropped on runner \( r_{k-i+1} \).

**Figure:** The heap of the element \( s_2 s_1 s_3 s_2 \) in type \( A_3 \)
If \( w \) is a fully commutative element and \( w \) is a reduced word for \( w \), then the heap \( H(w) \) is independent of \( w \), so we’ll refer to it as the heap \( H(w) \) of \( w \).
Let \( \omega_p \) be a minuscule fundamental weight, then the weights occurring in \( V(\omega_p) \) are in bijection with \( W/W_{P_p} \). In this case, all minimal length representatives for elements of \( W/W_{P_p} \) are fully commutative, moreover, they are all elements \( v \) of \( W \) such that

\[
  v \leq_I w_{0p}^P
\]

where \( \leq_I \) denotes the left weak order, i.e. \( v \leq_I w \) if some terminal substring of a reduced word for \( w \) is a reduced word for \( v \).
As an example, consider $V(\omega_2)$ for $A_3$. Then $w_0^{P_2} = s_2 s_1 s_3 s_2$, and the poset $W/W_{P_2}$ is as follows:

![Diagram showing the poset $W/W_{P_2}$]

Note that the heaps of these elements correspond to **order ideals** in $H(w_0^{P_2})$. 

**Figure:** The heaps corresponding to elements of $W/W_{P_2}$
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Heaps

Crystals from heaps

We can use these observations to describe a model for crystals of minuscule representations $B(\omega_p)$, where the underlying set is the order ideals (which are heaps themselves) $J(H(w_0^P))$ of $H(w_0^P)$ and the lowering operators have an easy description: to apply $f_j$ to a heap $\phi$, try to remove a bead from runner $j$. If this is not possible because another bead on a neighboring runner is blocking it, then $f_j(\phi) = 0$, otherwise, $f_j(\phi)$ is $\phi$ with the highest bead on runner $j$ removed.

Figure: The crystal $B(\omega_2)$ using Young tableaux

Figure: The crystal $B(\omega_2)$ using $J(H(s_2 s_1 s_3 s_2))$
So far we only considered minuscule representations, but we can use the language of heaps to construct models of more general crystals in a type-independent way.

**Theorem 4**

Let \( \omega_p \) be a minuscule fundamental weight. Consider the set of \( k \)-fold tensor products of order ideals of \( H(w_0^{P_p}) \). The subset

\[
H(w_0^{P_p}) \otimes^k \leq \left\{ \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k \mid \phi_j \in J(H(w_0^{P_p})), \phi_i \subseteq \phi_{i+1} \right\},
\]

with lowering operators defined using the tensor product rule for crystals, is a model for the crystal \( B(k\omega_p) \).
Definition 5

The set $RPP(H(w_0^{P_p}), k)$ of order-reversing maps from the poset $H(w_0^{P_p})$ to $\{0, \ldots k\}$ (with the standard ordering) is called a reverse plane partition of shape $H(w_0^{P_p})$.

There is a bijection between elements of $H(w_0^{P_p})^{\leq k}$ and $RPP(H(w_0^{P_p}), k)$, for example

$$
\begin{pmatrix}
1 \\
1 \\
3
\end{pmatrix}
$$

corresponds to the rpp

\[ \begin{array}{ccc}
& & \\
& \times & \\
& & \\
\end{array} \ 
\begin{array}{ccc}
& & \\
& \times & \\
& & \\
\end{array} \ 
\begin{array}{ccc}
& & \\
& \times & \\
& & \\
\end{array} \]
The lowering operator $f_i$ acts on rpps by decreasing an entry on the $i$-th column. For example, for $RPP(H(s_2s_1s_3s_2), k)$ in type $A_3$, we have (as long as the resulting array is an rpp)

\[
\begin{align*}
    f_1 \begin{pmatrix} b & a & c \\ d & \end{pmatrix} &= \begin{pmatrix} b - 1 & a & c \\ d & \end{pmatrix} \\
    f_2 \begin{pmatrix} b & a & c \\ d & \end{pmatrix} &= \begin{cases} \\
        \begin{pmatrix} a - 1 & & \\ b & c & \\ d & a & \end{pmatrix} & \text{if } a + d \leq b + c \\
        \begin{pmatrix} b & d & c \\ b & a & \\ d - 1 & & \end{pmatrix} & \text{if } a + d > b + c \\
    \end{cases} \\
    f_3 \begin{pmatrix} b & a & c \\ d & \end{pmatrix} &= \begin{pmatrix} b & a & c - 1 \\ d & \end{pmatrix}
\end{align*}
\]
To summarize our discussion up to this point, we have constructed a model for crystals of the form $B(k\omega_p)$ where $\omega_p$ is a minuscule fundamental weight using either $k$-fold tensor products of heaps, or as reverse plane partitions of shape $H(w_0^{P\rho})$. We found that (at least in type $A_3$) the lowering operators have a nice description in terms of the rpps.
Let $Q$ be an orientation of $g$’s Dynkin diagram with vertex set $I$, and $Q^*$ be the opposite orientation. Consider the doubled quiver $\overline{Q} = Q \cup Q^*$. Let $\mathbb{C}\overline{Q}$ be the path algebra of $\overline{Q}$. Consider the element

$$\rho = \sum_{e \in E(\overline{Q})} \varepsilon(e) e^* e$$

where $\varepsilon(e) = 1$ if $e \in E(Q)$ and $-1$ if $e \in E(Q^*)$. The algebra $\Lambda(Q) = \mathbb{C}\overline{Q}/(\rho)$ is called the preprojective algebra of $Q$. 
Given a heap of the form $H(w_0^{P_0})$, we make the following construction:

1. Replace each element in the heap by $C$.
2. Replace each covering relation in $H(w_0^{P_0})$ by the identity map $1 : \mathbb{C} \to \mathbb{C}$.
3. Define an $I$-graded vector space $L(\omega_p)$ by letting $L(\omega_p)_j$ be the direct sum of the 1-dimensional spaces which are labeled by $j \in I$.
4. Define a map from $L(\omega_p)_j$ to $L(\omega_p)_i$ by extending the above-defined maps linearly.

The resulting quiver representation $L(\omega_p)$ is a $\Lambda(Q)$-module, and it is the projective cover of the simple quiver representation $S(p)$ supported at vertex $p$. 
As an example, consider the heap $H(s_2 s_1 s_3 s_2)$ for $A_3$. Following the construction we get

and we see that this is indeed a (projective) $\Lambda(Q)$-module.
We can repeat the above steps with Weyl group elements other than $w_0^{P_p}$. We need $w$ to be **dominant minuscule**, meaning that there is a weight $\lambda$ and a reduced word $s_{i_1} \cdots s_{i_k}$ for $w$ such that

$$s_{i_j}s_{i_{j+1}} \cdots s_{i_k}(\lambda) = \lambda - \alpha_{i_k} - \alpha_{i_{k-1}} - \cdots - \alpha_{i_j}$$

(this is stronger than fully commutative). In this case, the set $RPP(H(w), k)$ serves as the underlying set of the Demazure crystal $B_w(\lambda)$. The resulting quiver representation is still a $\Lambda(Q)$-module. For simplicity, we’ll continue to assume that $w = w_0^{P_p}$. 
The construction of the module $L(\omega_p)$ from the heap $H(w_0^P)$ reveals an additional piece of structure. Consider the following map on the elements of $H(w_0^P)$: send each bead to the bead on the same runner just below it, or, if a bead is the lowest bead on the runner, send it to 0. Extend this to a nilpotent endomorphism $T$ of $L(\omega_p)$. 

![Diagram](image)
Consider the space $L(k\omega_p) = \{ M \subseteq L(\omega_p)^{\oplus k} \}$ of $\Lambda(Q)$-submodules. $L(k\omega_p)$ has connected components indexed by possible dimension vectors of $M$. We’ll also denote the nilpotent endomorphism $T^{\oplus k}$ of $L(k\omega_p)$ by $T$. We can obtain a finer decomposition of $L(k\omega_p)$ by considering the Jordan type of $T$ restricted to each $M_i$. This gives us a partition over each vertex, but there are also conditions between the Jordan types over neighboring vertices, and the Jordan type of $T$ restricted to $M_i$ is given by an rpp of shape $H(w_0^{P_p})$ with the sum of the entries in column $i$ adding up to $\dim(M_i)$. 
Conjecture 6

The irreducible components of $L(k\omega_p)$ are indexed by reverse plane partitions.

For example, consider $L(2\omega_2)$ in type $A_3$. Let $M$ be a submodule with dimension vector $(1, 2, 1)$. Write $M_i$ for the subspace corresponding to the $i$-th node of the Dynkin diagram. To choose $M_2$, we have to choose a 2-dimensional subspace of $\mathbb{C}^2 \oplus \mathbb{C}^2$ stable under the linear map

$$T(x, y, z, w) = (0, 0, x, y).$$
If $M_2 = \ker T$, and we can choose $M_1$ and $M_3$ arbitrarily, this corresponds to the rpp

$$
\begin{pmatrix}
0 \\
1 & 1 \\
2
\end{pmatrix}
$$

If $M_2 \neq \ker T$, then $M_1$ and $M_3$ are determined, this corresponds to the rpp

$$
\begin{pmatrix}
1 \\
1 & 1 \\
1
\end{pmatrix}
$$
**Conjecture 7**

The lowering operator \( f_i \) on the rpps corresponds to taking a generic submodule with quotient the simple module \( S(i) \).

Consider the case \( L(k\omega_2) \) in type \( A_3 \). Let \( \phi = \begin{pmatrix} b & a \\ c & d \end{pmatrix} \) be an rpp, and let \( M \) be a module in the component indexed by \( \phi \). Note that in this case, this just means that \( \dim(\ker T \cap M_2) = d \).

For simplicity, we identify the subspaces \( M_1 = B \) and \( M_3 = C \) with their images in \( M_2 = A + D \). Visually \( M \) looks like this:

```
A
 ↙   ↙
B   C
 ↙   ↙
D
```
We are looking for a submodule of $M$ that fits into the SES

$$0 \rightarrow f_2(M) \rightarrow M \rightarrow S(2) \rightarrow 0$$

then we have to choose an $a + d - 1$-dimensional subspace of $M_2 = A + D$. To be a submodule of $M$, this subspace needs to contain $B$ and $C$, and therefore $B + C$. Generically, this subspace will not contain all of $D$, unless $B + C = D$, in which case we are forced to contain all of $D$. 
We claim that

$$\dim(B + C) = \min(b + c - a, d).$$

To get these upper bounds we use the rank-nullity theorem for the operators $T^0$ and $T^1$ restricted to $M_2$.

1. To see that $\dim(B + C) \leq b + c - a$, note that

$$\dim(B + C) = \dim(B) + \dim(C) - \dim(B \cap C) \leq b + c - a$$

2. To see that $\dim(B + C) \leq d$, note that

$$\dim(B + C) = \dim(T(B + C)) + \dim((B + C) \cap (\ker T)) \leq 0 + d$$

(in general we get an upper bound from rank-nullity applied to each operator $T^0, T^1, \ldots T^{m-1}$ where $m$ is the number of beads on the runner in the heap $H(w_0^{P_0})$).
Therefore \( B + C = D \) if and only if \( b + c - a \geq d \), or equivalently, if
\[
a + d \leq b + c.
\]
and we see that this is the same rule as the lowering operator on the rpps (16).
In general, to compute \( f_i(M) \), we need to know which subspaces of the form \( M_i \cap \ker T^j \) must be contained in a generic submodule with quotient \( S(i) \). This coincides with the upper bound coming from the rank-nullity theorem applied to \( T^j \) being attained, and we want the largest \( j \) for which this happens. Then \( f_i(M) \) will contain \( M_i \cap \ker T^j \) but not \( M_i \cap \ker T^{j+1} \), so we know the Jordan type of \( T \) restricted to \( f_i(M) \).
On crystal bases of the $Q$-analogue of universal enveloping algebras.

On the fully commutative elements of Coxeter groups.