2014 Fall Olivetti: Schubert Calculus

Balázs Elek

September 16, 2014

1 Introduction

Recall Ravi's Oliver talk on elliptic curves and $\otimes\mathbb{C}$. Interesting part: the curve $C : y^2 = x^3 + ax + b$ has a group law given by (draw picture). This only works "because of Schubert Calculus", i.e. any line intersects C at 3 points.

A typical question of *enumerative geometry*: How many points are there in an intersection?

Hermann Schubert was interested in questions like: How many lines in 3-space intersect 4 given lines? He would "specialize" the lines so that $L_1 \cap L_2 = P$ and $L_3 \cap L_4 = Q$ for $P, Q \in \mathbb{C}^3$. Then a line intersecting all of $L_1, \ldots, L_4$ must either go through P and Q or else it must be the intersection of the two planes determined by L_1, L_2 and L_3, L_4. Scubert would appeal to the "principle of continuity" to argue that there must be two such lines in the generic case as well ("principle of conservation of number").

Problem: Hilbert's 15th: put this on rigorous foundation. Plan:

1. Make the set of lines in 3-space into a manifold, so "specializing" is just picking a representative of a continuous family.
2. Conveniently express the condition that a line intersects a given subspace in a certain way.
3. Arrive at the precise notion of what exactly is "conserved" (H^* class).

2 $\mathrm{Gr}(k, n)$

Some issues:

1. With $\mathbb{R}$: things which should intersect do not (draw parabola and line in $\mathbb{R}^2$). So do it over $\mathbb{C}$.
2. With tangency (draw circle with secant and tangent lines). So count multiplicities (not as cool as the 27 lines on any smooth cubic surface in $\mathbb{P}^3$).
3. With affine space $\mathbb{C}^n$: parallel lines don't intersect in $\mathbb{C}^2$. So move to $\mathbb{P}^n$.

Speaking of $\mathbb{P}^n$, define

$$\mathbb{P}^n = \text{set of lines in } \mathbb{C}^{n+1} = \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}/\sim,$$

where

$$(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$$

for $\lambda \in \mathbb{C}^\times$.

A line in $\mathbb{P}^n$ is a set of points of the form

$$\{[x_0 + ty_0 : \ldots : x_n + ty_n] | t \in \mathbb{C}\}.$$

It is the image of a 2-space (minus the origin) in $\mathbb{P}^n$. And in general, when we mention a k-space of $\mathbb{P}^n$, we mean the image of a $k+1$-subspace of $\mathbb{C}^{n+1}$ (minus the origin) in $\mathbb{P}^n$. We will, however, carry along linear algebra terminology.

Now we want to represent a line in $\mathbb{P}^3$ as a unique point (somewhere). So let's secretly pick a basis $\{x, y\}$ for the 2-space it's coming from, then try to forget it ($L \mapsto [x \wedge y] \in \mathbb{P}(\wedge^2 \mathbb{C}^4)$). Form the matrix

$$\begin{pmatrix} x^T \\ y^T \end{pmatrix} \tag{1}$$

(we are taking transposes so it won't topple over) and let p_{ij} denote the 2×2 minor involving columns i and j. Since x and y are LI, there is a nonzero minor, so we can map this to $\mathbb{P}^5$ (Plücker embedding).

$$P : L \mapsto [p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}].$$

To see it is well-defined, note that if $\{w, z\}$ is another basis for the 2-space of L, then $\exists C \in GL(2, \mathbb{C})$ s.t.

$$C \begin{pmatrix} x^T \\ y^T \end{pmatrix} = \begin{pmatrix} z^T \\ w^T \end{pmatrix}$$

so

$$p_{ij} \begin{pmatrix} z^T \\ w^T \end{pmatrix} = \det C \begin{pmatrix} x^T \\ y^T \end{pmatrix},$$

so it was a good idea to go to projective space.

However, P is not surjective. Note that for any 2×4 matrix,

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Proof. Exercise.

Q.E.D.

Conversely, if a point in $\mathbb{P}^5$ satisfies the Plücker relation, then it is coming from a line in $\mathbb{P}^{\nVdash}$. To see this, assume wlog that $p_{01} \neq 0$. then

$$p_{23} = \frac{p_{02}p_{13} - p_{03}p_{12}}{p_{01}}$$

Now let our matrix be

$$\begin{pmatrix} 1 & 0 & -\frac{p_{12}}{p_{01}} & -\frac{p_{13}}{p_{01}} \\ 0 & 1 & \frac{p_{02}}{p_{01}} & \frac{p_{03}}{p_{01}} \end{pmatrix}$$

and we have our line.

Theorem 2.1. *There is a bijective correspondence between k-planes in $\mathbb{P}^n$ and points of $\mathbb{P}^{\binom{n+1}{k+1}-1}$ satisfying the Plücker relations, which are, for any sequences $0 \leq j_l, m_l \leq n$:*

$$\sum_{i=0}^{k+1} (-1)^i p_{j_0 \cdots j_{k-1} m_i} p_{m_0 \cdots \hat{m_i} \cdots m_{k+1}} = 0.$$

And we will refer to both the k-planes in $\mathbb{P}^n$ and its image in P^N as $\mathrm{Gr}(k, n)$. Also note that we have equipped $\mathrm{Gr}(k, n)$ with an atlas (the open set $p_{i_0 \cdots i_k} \neq 0$ is isomorphic to $\mathbb{C}^{(k+1)(n-k)}$).

3 Schubert Varieties

Goal: to express the condition that our line in $\mathbb{P}^3$ intersects some subspaces in a certain way in the Plücker coordinates. By a (partial) flag, we mean a strictly increasing sequence of linear subspaces of $\mathbb{P}^3$ (i.e. images of vector subspaces of $\mathbb{C}^4$ in $\mathbb{P}^3$).

$$A_0 \subset A_1$$

We will say that the line L is in the *Schubert variety* $X(A_0 A_1)$ if

$$\dim A_0 \cap L \geq 0 \quad \text{and} \quad \dim A_1 \cap L \geq 1.$$

For example, if we let $A_0 = L'$ be a line and $A_1 = \mathbb{P}^3$, then $X(A_0 A_1)$ is just the set of lines which intersect a given line L'. Let

$$\mathcal{F} = \left(\mathcal{F}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x \end{bmatrix} \subset \mathcal{F}_1 = \begin{bmatrix} 0 \\ 0 \\ y \\ x \end{bmatrix} \subset \mathcal{F}_2 = \begin{bmatrix} 0 \\ z \\ y \\ x \end{bmatrix} \subset \mathcal{F}_3 = \mathbb{P}^3 \right)$$

be the standard flag.

Proposition 3.1. *The line L is in the Schubert variety $X(\mathcal{F}_i \mathcal{F}_j)$ if and only if $p_{kl} = 0$ whenever $l < 3 - i$ or $k < 3 - j$.*

Proof. Pick a basis $\{x, y\}$ for L, and wlog put the matrix to rref. Now this should be obvious.

Q.E.D.

It turns out that the Schubert varieties $X(\mathcal{F}_i \mathcal{F}_j)$ associated to our base flag are sufficient for everything we want to do, because

Proposition 3.2. *Let $A_0 \subset A_1$ and $B_0 \subset B_1$ be two flags satisfying $\dim(A_i) = \dim(B_i)$ for $i = 1, 2$. Then there is an invertible linear transformation of $\mathbb{P}^5$ into itself which preserves $\mathrm{Gr}(1,3)$ and sends $X(A_0 A_1)$ to $X(B_0 B_1)$.*

Proof. First note that there is certainly an invertible linear transformation T such that

$$T(A_0) = B_0, T(A_1) = B_1.$$

Also, T clearly sends lines to lines (it preserves $\mathrm{Gr}(1,3)$). Pick a basis x, y for L, and note that if L is represented by the matrix (as in (1))

$$\begin{pmatrix} x^T \\ y^T \end{pmatrix}$$

then $T(L)$ is represented by

$$\begin{pmatrix} x^T \\ y^T \end{pmatrix} M = \begin{pmatrix} x^T M \\ y^T M \end{pmatrix}$$

where M is the matrix of T, and we see that the 2×2 minors of this matrix can be expressed as linear combinations of that of the matrix $\begin{pmatrix} x^T \\ y^T \end{pmatrix}$.

Q.E.D.

Corollary 3.3. *Let $A_0 \subset A_1$ be as above. Then $X(A_0A_1)$ consists of those points of* $\mathrm{Gr}(1,3)$ *whose Plücker coordinates satisfy certain linear equations, i.e. the intersection of* $\mathrm{Gr}(1,3)$ *and a certain linear space in $\mathbb{P}^5$. Also, this space is a hyperplane if and only if* $\dim(A_0) = 1$ *and* $\dim(A_1) = 3$.

Proof. Let T be an invertible linear transformation mapping $A_0 \subset A_1$ to $\mathcal{F}_i \subset \mathcal{F}_j$ (the base flag). A line L is in $X(A_0A_1)$ if and only if $T(L)$ satisfies the conditions in Proposition 3.1. We see that the Plücker coordinates of L must satisfy certain linear equations. The last statement is true for dimensional reasons.

Q.E.D.

4 The same problem again

We come to the first way to solve the enumerative problem mentioned in the beginning in a more rigorous way. We know $\mathrm{Gr}(1,3)$ is given (as a subset of $\mathbb{P}^5$) by the single equation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \tag{2}$$

Now, for a line L to intersect a fixed line L_1, L must be contained in the Schubert variety $X(L_1\mathbb{P}^3)$, and this condition is represented by intersecting $\mathrm{Gr}(1,3) \subset \mathbb{P}^5$ with a certain hyperplane H_1. Then the set of lines intersecting L_1, L_2, L_3, L_4 is represented by the intersection

$$\mathrm{Gr}(1,3) \cap H_1 \cap H_2 \cap H_3 \cap H_4.$$

Now, if the H_i are linearly independent, $\bigcap_{i=1}^4 H_i$ is a line, which we can parametrize, and then use the relation (2) to obtain a quadratic equation in one variable, we see that the number of solutions (two) matches with Schubert's prediction.

5 Schubert Calculus

Hilbert's 15th problem was finding a rigorous foundation for Schubert Calculus, and mathematicians of the 20th century have done this via cohomology. Recall that

$$H^*\left(\mathrm{Gr}(k,n), \mathbb{Z}\right) = \bigoplus H^i\left(\mathrm{Gr}(k,n), \mathbb{Z}\right)$$

is a graded ring (it is just a ring where multiplication describes how subvarieties intersect, so there is nothing scary about it). There are a couple important properties of the cohomology ring of $\mathrm{Gr}(1,3)$ which we'll take for granted:

1. $H^N(\mathrm{Gr}(k,n), \mathbb{Z}) \cong \mathbb{Z}$, where $N = \dim(\mathrm{Gr}(k,n)) = \binom{n-k}{k+1}$.

2. Homotopic subvarieties are assigned the same cohomology class.

3. If a set of subvarieties Y_α intersect as you'd expext (i.e. transversally) and $\bigcap_\alpha Y_\alpha = \{n \text{ points}\}$, then in cohomology, $\prod_\alpha [Y_\alpha] = n$ (as an element of H^{top}).

Now there are a couple theorems that we'll need to do Schubert calculus, but first, notice that since varieties in a continuous system should be assigned the same class in cohomology, the standard Schubert varieties $X(\mathcal{F}_0\mathcal{F}_1 \ldots \mathcal{F}_k)$ coming from the base flag are sufficient to describe the classes of all the Schubert varieties $X(A_0A_1 \ldots A_k)$, since from $t \in [0,1]$

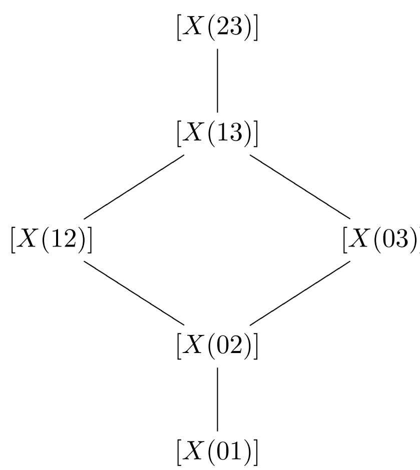
$$\begin{pmatrix} tc_{11} + (1-t) & tc_{12} & tc_{13} & \ldots & tc_{1n} \\ tc_{21} & tc_{22} + (1-t) & tc_{23} & \ldots & tc_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ tc_{n1} & tc_{n2} & tc_{n3} & \ldots & tc_{nn} + (1-t) \end{pmatrix}$$

puts them in a continuous family with one of the standard ones. Since the standard ones are uniquely determined by their dimension sequence, we will use the notation

$$X(i_0 i_1 \cdots i_k) = X(\mathcal{F}_{i_0}\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}).$$

Theorem 5.1. *(The basis theorem) Each odd dimensional cohomology group of* $\mathrm{Gr}(k,n)$ *is zero, and* $H^{2p}(\mathrm{Gr}(k,n))$ *is free abelian, and generated by the classes of the Schubert varieties* $X(i_0 i_1 \cdots i_k)$ *with* $(k+1)(n-k) - \sum_{i=0}^{k} a_i - i = p$*. Moreover, the bases* $\{\cdots, X(i_0 i_1 \cdots i_k), \cdots\}$ *and* $\{\cdots, X((n-i_k)(n-i_{k-1})\cdots(n-i_0)), \cdots\}$ *of* $H^N(\mathrm{Gr}(k,n),\mathbb{Z})$ *and* $H^{N-2p}(\mathrm{Gr}(k,n),\mathbb{Z})$ *respectively are dual.*

This means that the cohomology of $\mathrm{Gr}(1,3)$ looks like (H^0 on top, H^8 on the bottom):



But it turns out we don't even need all the standard Schubert varieties to generate $H^*(\mathrm{Gr}(1,3))$ *as a* $\mathbb{Z}$*-algebra*. For $i \le n-k$, let $x(i) = X((i)(n-k+1)(n-k+2)\cdots(n))$.

Theorem 5.2. *(The determinental formula) In* $H^*(\mathrm{Gr}(k,n))$*, we have the following formula*

$$X(i_0 i_1 \cdots i_k) = \det \begin{pmatrix} x(i_0) & x(i_0 - 1) & \cdots & x(a_0 - k) \\ \vdots & \vdots & \ddots & \vdots \\ x(i_k) & x(i_k - 1) & \cdots & x(a_k - k) \end{pmatrix}$$

where $x(i) = 0$ *for* $h \notin [0, n-k]$.

So now we only need to know how to multiply the special Schubert varieties to Schubert varieties to describe the whole ring structure.

Theorem 5.3. *(Pieri's formula) In* $H^*(\mathrm{Gr}(k,n))$*, we have the following formula*

$$x(l)X(i_0 \cdots i_k) = \sum X(j_0 \cdots j_k),$$

where the sum ranges over all sequences $0 \le j_0 < j_1 < \cdots < j_k \le n$ *satisfying* $0 \le j_0 \le i_0 < j_1 \le i_1 < \cdots < j_k \le i_k$ *and* $\sum_{i=0}^{k} b_i = \left(\sum_{i=0}^{k} a_i\right) - (n-k-l)$.

6 And the same problem again

Now we are ready to solve the problem of counting the number of lines in $\mathbb{P}^3$ intersecting 4 given lines. We want to find

$$X(13)^4$$

in $H^*(\mathrm{Gr}(1,3))$. First we compute, using Pieri's formula,

$$X(13)^2 = x(1)X(13) = X(03) + X(12).$$

Notice that this says that the set of lines intersecting 2 given lines has the same cohomology class as the set of lines going through a fixed point ($X(03)$) or contained in a plane ($X(12)$). One can see this using Schubert's specialization by insisting that the two lines L_1, L_2 intersect at a point P. Then clearly any line which intersects these two lines either contains P, or else it is contained in the plane determined by L_1 and L_2. Now using the basis theorem, we see that

$$X(13)^4 = (X(03) + X(12))^2 = X(03)^2 + 2X(03)X(12) + X(12)^2 = 2X(01)$$

and since $X(01)$ is the class of a single point, we see that our answer is (again) 2.

References

[1] Kleiman, S.L., Laksov, D., *Schubert Calculus, The American Mathematical Monthly, Vol. 79, No. 10 (Dec., 1972), pp. 1061-1082*

[2] Knutson, A., Tao, T. *Puzzles and (equivariant) cohomology of Grassmannians, Duke Math. J. Volume 119, Number 2 (2003), 221-260.*