Research Statement

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1 My interests

I am interested in the representation theory of algebraic groups and the algebraic geometry of their homogeneous spaces, with particular emphasis on the connections to combinatorics. My work involves turning interesting problems in geometry or representation theory into combinatorics, and studying them through some combinatorial gadget. Some historical examples where this approach has been highly successful are the convexity theorem for moment maps of Atiyah [Ati82], and independently, of Guillemin-Sternberg [GS82] and Kashiwara's crystal basis [Kas91], also independently discovered by Lusztig [Lus90] as the canonical basis of a representation. These results involve turning a geometric/representation theoretic object into a combinatorial one. In particular, the moment map image of a projective variety with a torus action is a convex polytope, and the crystal of a representation of a semisimple Lie algebra is a colored directed graph. Both of these results share the amazing feature that the combinatorial gadget retains most of the information about the original object. For example, one can compute the equivariant cohomology ring of a toric variety by looking at the polytope, and one can compute tensor product decompositions and branching rules of representations by looking at the crystals.

2 Current projects

Many of my curent projects involve crystals in some form, so we recall some background on the subject

2.1 Some background on crystals

What is a good basis for a vector space? In a linear algebra class, we tell students that they should not think in terms of a basis, but often the right choice of a basis makes a problem easier. This is especially true if the vector space has some additional stucture. For example, for a $T = \left(\mathbb{C}^\times\right)^k$-representation V we can always choose a basis of V such that T acts diagonally. Equivalently, this means that V is a direct sum of 1-dimensional subspaces, where T acts on each summand by a character $T \to \mathbb{C}^\times$, so we can choose a basis such that T acts by rescaling each vector.

For a semisimple Lie algebra $\mathfrak{g}$, the situation is more complicated. Consider, for example, any finite-dimensional representation V of $\mathfrak{sl}_2(\mathbb{C})$. Then if $\dim V = n+1$, the representation V can be identified with the vector space of homogeneous polynomials of degree n in x and y. This has a basis of monomials of the form $x^k y^{n-k}$, and the matrices $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $F =$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ act as differential operators $y\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$, respectively, essentially permuting the basis vectors. Also, the tensor product decompositions of $\mathfrak{sl}_2(\mathbb{C})$-representations can be completely understood in terms of the tensor products of a good choice of a basis (this is not trivial, as the tensor product does not immediately decompose in a way that is compatible with the basis we mentioned above). One would like to define a similar basis for a representation of an arbitrary semisimple Lie algebra, but this naive idea does not generalize to other Lie algebras, in fact it is already impossible to choose such a basis for the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$.

This is where quantum groups come to the rescue. Quantized enveloping algebras $U_q(\mathfrak{g})$ were introduced by Drinfeld and Jimbo in the mid 1980s. Originally motivated by physics and integrable systems, they have rapidly found applications in many other fields of mathematics, for example to knot theory and representation theory. In [Kas91], Kashiwara interpreted the deformation parameter q as a parameter for "temperature" in physics, and stated that his motivation was the belief that the representation theory of $U_q(\mathfrak{g})$ ought to be simple at "absolute zero". The work of Date, Jimbo and Miwa [DJMM90] also suggested this. Kashiwara then introduced **crystal bases** for $U_q(\mathfrak{g})$-representations, and proved that (due to the fact that the representation theory of $U_q(\mathfrak{g})$ is very similar to that of $U(\mathfrak{g})$) these fundamentally combinatorial objects can be used to compute tensor product decompositions and branching rules for representations of $\mathfrak{g}$.

2.2 Quiver variety components, heaps and minuscule combinatorics

In [KS97], Kashiwara and Saito defined a crystal structure on the set of irreducible components of Lusztig quiver varieties. In [Sav06], Savage showed that in types A and D this coincides with the combinatorial definition of crystals, establishing a link between the geometry of quiver varieties and the combinatorics of crystals.

In a similar flavor, joint with Anne Dranowski, Joel Kamnitzer, Tanny Libman and Calder Morton-Ferguson, I connected certain modules of the preprojective algebra of an ADE type Dynkin quiver to crystals [DEKM22].

Let W denote the Weyl group and let P be a maximal parabolic subgroup corresponding to a minuscule fundamental weight ω. Let $w = w_0^P$ denote the minimal length representative of $w_0 W_P$ where W_P is P's Weyl group. Our construction works in a slightly more general setting, the technical assumption we need on w is **dominant minuscule** [Ste01], but we'll focus on the w_0^P case for simplicity.

Consider the **heap** $H(w)$ ([Ste96]) of w. This is a partially ordered set with a map to the Dynkin quiver. Let k be a positive integer. Using $H(w)$ one can construct a combinatorial model $RPP(w, k)$ for the crystal $B(k\omega)$ (elements of this model are called **reverse plane partitions** and have been defined before, see [GPT18], for example) and we construct space of modules $L(k\omega)$ for the preprojective algebra of the Dynkin quiver. We use Nakajima's tensor product varieties ([Nak01]) to establish an explicit crystal isomorphism between $RPP(w, k)$ and the irreducible components of the preprojective algebra modules.

2.3 Schubert Varieties

Let v and w be permutations in S_n, and let $X_\circ^w = BwB/B$ be a Bruhat cell and $X_v = \overline{B^- vB/B}$ an opposite Schubert variety. In [KL79], Kazhdan and Lusztig introduced **Kazhdan-Lusztig varieties** $\mathcal{X}_{v,\circ}^w = X_\circ^w \cap X_v$. They have since found numerous applications, for example, they can be used to study the geometry of Schubert varieties in the neighborhood of a torus-fixed point. Since the Bruhat cell $X_\circ^w$ is just an affine space $\mathbb{C}^{l(w)}$, it is natural to study the **Kazhdan-Lusztig ideal**, the ideal defining $X_{v,\circ}^w$ inside $X_\circ^w$. In [WY12], Woo and Yong gave a Gröbner basis for Kazhdan-Lusztig ideals for the type A flag variety. Having a Gröbner basis of any interesting ideal is desirable because it can simplify computation, but even better, for Kazhdan-Lusztig ideals, the Gröbner basis has square-free leading terms, and therefore can be used to degenerate the Kazhdan-Lusztig variety into a reduced union of coordinate subspaces, and this union is described by a **subword complex** of Knutson and Miller [KM05].

Joint with Daoji Huang, I extended Woo and Yong's results to affine type A. Building on previous work of Huang [Hua19], we present a Gröbner basis for Kazhdan-Lusztig ideals. We develop some tools that are well known in the finite type A case, including a generalization of Fulton's essential set [Ful92] and a linear parametrization for the Bruhat cell $X_\circ^w$. We then adapt the proof in [WY12] to the affine case.

We hope that our results can be used to practically compute with equations defining Kazhdan-Lusztig varieties in affine type A using software, we have used `Macaulay2` extensively. We also hope that our result can be useful in studying singularities of Schubert varieties in affine type A.

2.4 Toggle groups and Cactus groups

The Bender-Knuth moves t_i are certain involutions defined on semistandard tableaux that act on the i-s and $i+1$-s in the filling. They can also be interpreted as certain sequences of **toggles** on the corresponding **Gelfand-Tsetlin patterns** (these are combinatorial objects in bijection with seminstandard tableaux). Berenstein and Kirillov [BK96] studied the relations satisfied by the t_is and subsequently introduced the Berenstein-Kirillov group (or $\mathcal{BK}$) as the the free group generated by all the $t_i, i \in \mathbb{N}$, modulo the relations satisfied by the t_is when acting on all semistandard tableaux of all possible shapes. One feature of this definition is that while the group is well defined, and comes with an explicit generating set, but without a specific set of relations. As of now, there is no known presentation of $\mathcal{BK}$ (see Remark 1.9. in [CGP16]).

The **cactus group** $\mathrm{Cact}_\mathfrak{g}$ of $\mathfrak{g}$, studied by Henriques and Kamnitzer [HK06] is a group closely related to the braid group and Weyl group of $\mathfrak{g}$. It is generated by elements ξ_J corresponding to subsets I of the simple roots of $\mathfrak{g}$, and there is a surjection $\mathrm{Cact}_\mathfrak{g} \twoheadrightarrow W_\mathfrak{g}$ mapping $\xi_I \mapsto w_0^{P_I}$ where $w_0^{P_I}$ is the longest element in the Weyl group of the parabolic subgroup P_I. Halacheva [Hal16] showed that the cactus group acts on crystals of representations of $\mathfrak{g}$.

Chmutov, Glick, and Pylyavksyy [CGP16] realized $\mathcal{BK}$ as a quotient of the type A cactus group Cact_n, using growth diagram computations. They used this to derive some previously unknown relations of $\mathcal{BK}$. They also pointed out that the only relation that does not follow from the relations in Cact_n is $(t_1t_2)^6 = e$. This relation corresponds to the result of Kashiwara [Kas94] that the Cactus elements $\xi_{\{i\}}$ corresponding to one-element subsets of the Dynkin diagram define a W-action on the crystal. Note that there is no relation $(\xi_{\{i\}}\xi_{\{i+1\}})^6 = e$ in Cact_n.

In other types, there is in general no clear analogue for the Gelfand-Tsetlin patterns. However, for a choice of a minuscule fundamental weight ω, as in section 2.2, one can use reverse plane partitions as a model for crystals $B(k\omega)$. In this case, one can define the sequences of toggles that play the role of the Bender-Knuth moves in type A and define a group (called the **toggle group**) generated by them. I conjecture, based on solid computational evidence, that, as in type A, the cactus action factors through the toggle group. More precisely, for every minuscule fundamental weight ω_p and each t_i defined on the corresponding minuscule reverse plane partition, there is a cactus group element $c(t_i) \in \mathrm{Cact}_\mathfrak{g}$ which acts on all the crystals $B(k\omega_p)$

the same way and moreover that these elements generate the action of the cactus group (in general, the toggle group will depend on the choice of ω_p, unlike in type A).

One interesting thing to point out that while in type A_n, there are n toggles and the type A_n cactus group has a generating set of size n, in type D_n, there are still n toggles, but the cactus group does not have a generating set of size n. In particular, our conjecture would imply that the cactus action is substantially simpler on the crystals $B(l\omega_k)$.

2.5 Kirillov-Reshetikhin crystals and Cactus groups

For $\tilde{\mathfrak{g}}$ an affine type Lie algebra, Hatayama et al. introduced **Kirillov-Reshetikhin (KR) modules** for the quantized loop algebra $U'_q(\tilde{\mathfrak{g}})$ [HKO+02]. These are finite-dimensional modules that are defined for a choice of a node in the Dynkin diagram of the finite type algebra $\mathfrak{g}$ and a positive integer. The existence of crystal bases for these modules has only been settled recently; see, for example [FOS09]. When the node corresponds to a cominuscule fundamental weight ω, the KR module is irreducible even for the subalgebra $U_q(\mathfrak{g})$. In this case, the crystal of the representation $V_{k\omega}$ of $\mathfrak{g}$ has an additional lowering operator f_0 corresponding to the affine root. So far this operator has only been defined case by case [FOS09], [HN06]. Combinatorially this was first done for type A KR crystals by Shimozono [Shi02] by using Schützenberger's promotion operator (see [Sta09] for a survey on promotion) on Young tableaux, and in other types by finding some analogue of the promotion operator.

In [Kwo13], Kwon used the Robinson-Schensted-Knuth correspondence to give a combinatorial model for classically irreducible KR crystals in types A, C, D. In section 6, Kwon points out that the lowering operator f_0 can be defined in terms of the cactus action (he does not use the language of cactus groups) as $f_0 = \xi_{[n]\setminus\{k\}} e_k \xi_{[n]\setminus\{k\}}$ (where $[n] = \{1, \ldots n\}$).

I conjecture, based on solid computational evidence, that the formula $\xi_{[n]\setminus\{k\}} e_k \xi_{[n]\setminus\{k\}}$ defines the f_0 operator for classically irreducible crystals of all types. By the uniqueness of crystals of representations, it suffices to show that f_0, as defined by the cactus action, interacts with the other crystal operations the expected way. Using Stembridge's [Ste03a] characterization of crystals of representations, this comes down to checking some commutation relations between f_0 and other lowering operators already defined.

To connect this project to the one described in section 2.2, it would be interesting to see how the cactus groups and toggle groups (defined in Section 2.4) act on the set of components $L(k\omega)$. Recently, in [GPT18], Garver, Patrias and Thomas used a similar approach to show that the order of promotion (which is a particular element of both the toggle and Cactus groups) equals to the Coxeter number.

3 Previous work

3.1 My thesis work

Motivated by the combinatorics of the positroid stratifications of Grassmannians in [KLS13] and the corresponding geometry in [Sni10], He Knutson and Lu introduced Bruhat atlases in [HKL]. A **Bruhat atlas** on a stratified manifold $(M, \mathcal{Y})$ is a way to model the stratification $\mathcal{Y}$ on the Bruhat stratification. Specifically it is an open cover of M by Schubert cells X_o^w in a flag manifold H/B_H (of a Kac-Moody group H) compatible with the stratification $\mathcal{Y}$ and with the opposite Schubert stratifications of the X_o^w's. In [KLS13], Bruhat atlases are described on the wonderful compactifications $\overline{G}$ of DeConcini-Procesi [DCP83], and on partial flag manifolds

G/P of semisimple algebraic groups. The existence of Bruhat atlases on these two interesting families of varieties leads one to ask the question: Can one classify manifolds with Bruhat atlases?

I worked on the classification of toric surfaces with Bruhat atlases compatible with the torus action, with complete results in the simply-laced case. This project is in the intersection of combinatorics, representation theory and algebraic geometry, and the proofs involved a broad range of methods. In [Ele16], I proved that the only smooth toric surfaces with Bruhat atlases are $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$, but there are more varieties with a so-called **Kazhdan-Lusztig atlas**, defined also in [KLS13]. I classified smooth toric surfaces with simply-laced Kazhdan-Lusztig atlases in my thesis ([Ele16]). This involved a number of steps, with surprisingly different techniques for each one.

- Classifying the moment polygons of smooth Richardson surfaces in arbitrary Kac-Moody groups. There are 10 of these, plus an infinite family.
- Assembling these into a polygon of the toric surface. I reduced this to a finite problem by considering the abelianization map from $SL(2,\mathbb{Z})$ to $\mathbb{Z}/12\mathbb{Z}$, then used Sage ([Dev15]) to obtain the 20 possible configurations, which I called pizzas.
- Finding a Kac-Moody group H such that a given pizza appears in H/B_H. I found these by building a root system from necessary conditions for the equivariant cohomology classes of the 1-skeleton of the pizza, which came from the structure of Bruhat intervals of height 2 and 3.

Using the work of Dyer [Dye91], I also proved that there are at most 7543 non-simply laced pizzas with Kazhdan-Lusztig atlases, each coming from a subdivision of a polygon with at most 12 vertices.

3.2 Finite type multiple flag varieties

In [MWZ00] and [MWZ99] Magyar, Weyman and Zelevinsky consider G-actions on multiple flag varieties $G/P_1 \times \ldots \times G/P_k$ (for $P_1, \ldots, P_k$ parabolic subgroups of $GL(n)$ or $Sp(n)$) and answer the question of when such a G-action has finitely many orbits. Several special cases of this have been considered, for instance, the spherical case (i.e. when $P_1 = B$) in [AP13], [Lit94], [Ste03b], and the case $P = P_1 = P_2 = P_3$ in [Dev14], [Pop07].

Joint with with Dan Barbasch, Sergio Da Silva and Gautam Gopal Krishnan, I extended the above results to exceptional groups. In [MWZ00], the authors exploit the description of GL_n/P as the variety of partial flags in $\mathbb{C}^n$ to transform the question to quiver representations, and in [MWZ99], they reduce the type C cases to the type A case. This approach does not obviously generalize to exceptional groups, so we took a more direct approach. The G_2 case is trivial, as by a crude dimension count one notices that G can have finitely many orbits only when $k = 2$, in which case the question is completely answered by the Bruhat stratification on $G/P_2 = \bigsqcup_{w \in P_1 \backslash W/P_2} P_1 w P_2/P_2$. We finished the F_4 case in [BDSEK17] and are currently working on the type E cases.

3.3 The standard Poisson structure on Bott-Samelson varieties

Let G be a semisimple Lie Group over $\mathbb{C}$ and let $Q = (s_1, \ldots, s_k)$ be a word in the simple reflections of W. Let $P_{s_i} = B \cup Bs_iB$. We then define the **Bott-Samelson variety** BS^Q as the quotient of the product manifold $P_{s_1} \times \ldots \times P_{s_k}$ by the right action of B^n by

$$(p_1, p_2, \ldots, p_n) \cdot (b_1, b_2, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n).$$

Then BS^Q is an iterated $\mathbb{P}^1$-bundle with a canonical map m to G/B, given by multiplying the p_i's. Bott-Samelson varieties have been used as effective tools in answering important questions in representation theory and the geometry of Schubert varieties, see, for example [Bri05] and [EW14].

A choice of a pair $T \subseteq B$ of a maximal torus T, a Borel subgroup B and a symmetric non-degenerate biliear form gives rise to a multiplicative holomorphic Poisson structure π_{st} on G, making (G, π_{st}) into a Poisson-Lie group that is the semi-classical limit of the quantum group associated to G [CP94]. Since B is a Poisson Lie subgroup of G, this projects to a well-defined Poisson structure $\pi_{G/B}$ on G/B. Similarly, since all the P_{s_i}'s are Poisson subgroups, there is a standard Poisson structure π_Q on BS^Q and the map $m : BS^Q \to G/B$ is Poisson.

Since BS^Q is an iterated $\mathbb{P}^1$-bundle, it has many natural coordinate charts. Joint with Jiang-Hua Lu, in [EL19] I explicitly computed the standard poisson structure in each of these charts. The formulas are given in terms of root strings in the root system of G, and are entirely combinatorial, and I implemented their computation in GAP [GAP19].

We showed that he Poisson structure in each of the charts is a Poisson polynomial algebra, and that in one of the charts, it is a **symmetric Poisson CGL extension**, studied by Goodearl and Yakimov [GY14]. Symmetric Poisson CGL extensions have deep connections to the theory of cluster algebras, see, for example [GSV10]. In particular, in [GY16], Goodearl and Yakimov prove the Berenstein-Zelevinsky conjectures on the equality of the cluster algebras and upper cluster algebras associated to double Bruhat cells using Poisson CGL extensions.

Our results in [EL19] also have applications to quantum groups, integrable systems, total positivity and toric degenerations of some Poisson varieties associated to G. See also the introduction of [EL19] for some directions for future research.

3.4 Promotion and cyclic sieving on δ-semistandard tableaux

Let X be a finite set with an action of the cyclic group $C_l = \langle c \rangle$, ζ a primitive lth root of unity and $f(q)$ a polynomial. Then the triple $(X, \langle c \rangle, f(q)$ **exhibits the cyclic sieving phenomenon (CSP)** (introduced by Reiner, Stanton and White in [RSW04]) if $f(\zeta^d) = |X^{c^d}|$ for all integers $d \geq 0$ where X^{c^d} denotes the fixed-point set of c^d. One can show that such a polynomial always exists, but the interest in CSP is that often the polynomial $f(q)$ is a q-deformation of a formula for enumerating $|X|$. For example, in [RSW04], Reiner, Stanton and White show that if X is the set of all subsets of $\{1, 2, \ldots, n\}$ of size k, and C_n acts on X via the long cycle $(1, 2, \ldots, n) \in S_n$, then $\left(X, C_n, \begin{bmatrix} n \\ k \end{bmatrix}_q \right)$ exhibits the CSP, where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-analogue of the binomial coefficient. Note that in this case $|X| = \binom{n}{k}$, the ordinary binomial coefficient.

In [Rho10], Rhoades proves several CSP results about rectangular Young tableaux. It is well-known that on the set of Young tableaux with fixed rectangular shape with entries up to n, promotion has order n. The cyclic sieving polynomials are again q-deformations of natural counting formulae. In particular, Theorem 1.5. in [Rho10] establishes a cyclic sieving result on rectangular column-strict tableaux on fixed shape and content, where the cyclic sieving polynomial is the Kostka-Foulkes polynomial. In [FK14], Fontaine and Kamnitzer interpreted the vector space $\mathbb{C}X$ spanned by these tableaux as the space of invariants in a tensor product of minuscule representations of SL_n, where the promotion on tableaux corresponds to cyclic rotation of the tensor factors. Using the geometric Satake correspondence and Nakajima quiver varieties, they give a geometric proof and a generalization of Rhoades' cyclic sieving result.

Joint with Tair Akhmejanov, in [AE20], I extended Fontaine and Kamnitzer's results to a case where the geometric action is on an invariant space in a tensor product

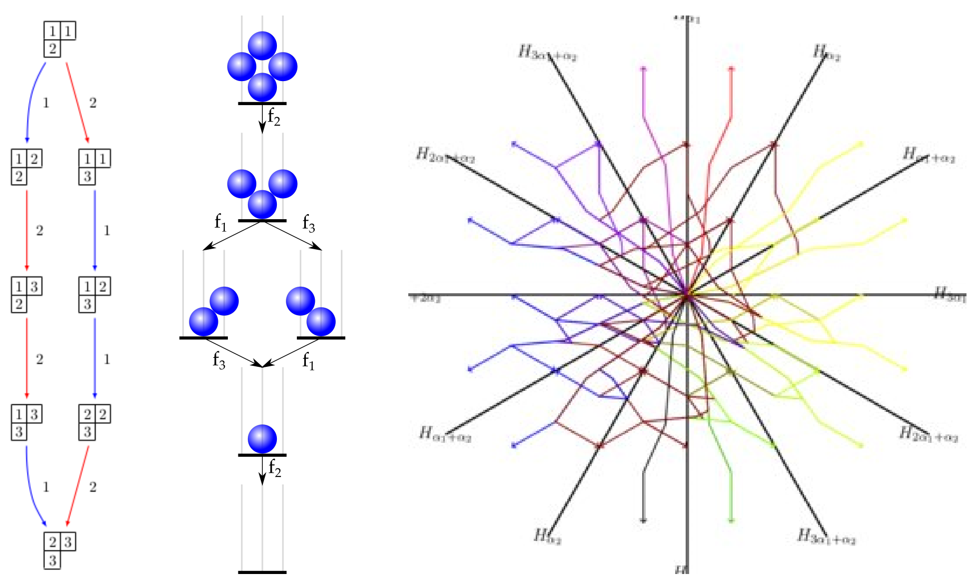
$$V(\vec{\lambda}) = V(\lambda_1) \otimes V(\lambda_2) \otimes \ldots \otimes V(\lambda_n),$$

where each $V(\lambda_i)$ is either a symmetric or alternating power of the defining representation $\mathbb{C}^n$ (in this context, the result of Fontain and Kamnitzer is where all the representations are $\bigwedge^k \mathbb{C}^n$s).

We define a new class of tableaux, called **δ-semistandard tableaux**, to naturally index the basis for the invariant space $V(\vec{\lambda})^G$, where $\delta = (\delta_1, \ldots, \delta_n)$ is a binary string with δ_i specifying whether the entries equal to i of the tableau appear as a horizontal or vertical strip. Using the methods developed in Akhmejanov's thesis [Akh18], we are able to prove that the δ-analogue of the promotion operator still has order n, and we adapt the techniques of Fontaine and Kamnitzer to establish a CSP with the polynomial equal to the **generalized Kostka polynomials** introduced in [SW99] and [KS02].

4 Potential undergraduate research projects

I believe that my area of research is particularly well suited for undergraduate research. For example, Kashiwara crystals are concrete and tangible objects that one can draw in many interesting ways:



Students can very quickly start computing with these objects without needing to internalize all of the theoretical background. I have a large number of conjectures that are easy to state about the symmetries that the crystals themselves satisfy (I mention these in Sections 2.5 and 2.4). I suspect that many of these could be proved by direct combinatorial methods.

Once students are engaged with a problem, they can use this to learn about the beautiful structures that underlie the theory of crystals, and this can serve as a great entry point for learning more about quantum groups and canonical bases, or representation theory in general.

I found that this approach worked remarkably well when I led a reading course on quiver varieties in Fall 2019 at the University of Toronto for an undergraduate student. Through the concrete computations and bite-sized projects I assigned to him (some of which can be found on the notes section of his website), he could put many of the more abstract notions in perspective, and he finished the semester with a good understanding of aspects of for example, homological algebra and category theory. He then participated in the research project I describe in Section 2.2 and went on to graduate school in UCSB.

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