COMPUTING THE STANDARD POISSON STRUCTURE ON BOTT-SAMELSON VARIETIES IN COORDINATES

by

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B.Sc. *HKU*

A thesis submitted in partial fulfillment of the requirements for
the degree of Master of Philosophy
at The University of Hong Kong.

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Abstract of thesis entitled

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Bott-Samelson varieties associated to reductive algebraic groups are much studied in representation theory and algebraic geometry. They not only provide resolutions of singularities for Schubert varieties but also have interesting geometric properties of their own. A distinguished feature of Bott-Samelson varieties is that they admit natural affine coordinate charts, which allow explicit computations of geometric quantities in coordinates.

Poisson geometry dates back to 19th century mechanics, and the more recent theory of quantum groups provides a large class of Poisson structures associated to reductive algebraic groups. A holomorphic Poisson structure Π on Bott-Samelson varieties associated to complex semisimple Lie groups, referred to as the *standard* Poisson structure on Bott-Samelson varieties in this thesis, was introduced and studied by J. H. Lu. In particular, it was shown by Lu that the Poisson structure Π was algebraic and gave rise to an iterated Poisson polynomial algebra associated to each affine chart of the Bott-Samelson variety. The formula by Lu, however, was in terms of certain holomorphic vector fields on the Bott-Samelson variety, and it is much desirable to have explicit formulas for these vector fields in coordinates.

In this thesis, the holomorphic vector fields in Lu's formula for the Poisson structure Π were computed explicitly in coordinates in every affine chart of the Bott-Samelson variety, resulting in an explicit formula for the Poisson structure Π in coordinates. The formula revealed the explicit relations between the Poisson structure and the root system and the structure constants of the underlying Lie algebra in any basis. Using a Chevalley basis, it was

shown that the Poisson structure restricted to every affine chart of the Bott-Samelson variety was defined over the integers. Consequently, one obtained a large class of iterated Poisson polynomial algebras over any field, and in particular, over fields of positive characteristic. Concrete examples were given at the end of the thesis.

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Declaration

This thesis contains the work done by the author during the period from September 2010 to June 2012 for the degree of Master of Philosophy at The University of Hong Kong under the supervision of Prof. Lu Jiang-Hua. The work submitted has not been previously included in any thesis, dissertation or report submitted to this or any other institution for a degree, diploma or other qualifications.

Elek Balázs

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Contents

1 Introduction

Let G be a connected complex semisimple Lie group with a fixed Borel subgroup B and a maximal torus T contained in B. Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{h}$ be the Lie algebras of G and T, respectively. The choice of (B,T) determines a root system $\Delta \subseteq \mathfrak{h}^*$ and a set of simple roots $\Gamma \subseteq \Delta$. Let $W = N_G(T)/T$ be the Weyl group of G, where $N_G(T)$ is the normalizer of T in G. It is a Coxeter group generated by the simple reflections $\{s_\alpha | \alpha \in \Gamma\}$.

Let $\mathbf{w} = (s_1, s_2, \cdots, s_n)$ be any sequence of simple reflections in W, and for $1 \leq k \leq n$, let $P_k = B \cup Bs_kB$ be the parabolic subgroup associated s_k. Consider $P_1 \times P_2 \times \ldots \times P_n$ with the right action of B^n (the n-fold product of B) by

$$(p_1, p_2 \ldots p_n).(b_1, \ldots, b_n) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, b_{n-1}^{-1}p_nb_n).$$

The quotient space $Z_\mathbf{w} = P_1 \times_B P_2 \times_B \ldots \times_B P_n/B$ is the Bott-Samelson variety associated to $\mathbf{w}$. For $(p_1, \ldots, p_n) \in P_1 \times \ldots \times P_n$, let $[p_1, \ldots, p_n] \in Z_\mathbf{w}$ denote the equivalence class of $(p_1, \ldots p_n)$ under the B^n-action. Multiplication in the group G gives a well-defined map

$$\Xi : Z_\mathbf{w} \to G/B : \ \Xi([p_1, \ldots, p_n]) = p_1 \cdots p_n/B.$$

When $\mathbf{w}$ is a reduced word, Ξ is a resolution of singularities of the Schubert variety

$$\overline{Bs_1 \cdots s_nB/B} \subseteq G/B$$

in the weak sense.

Bott-Samelson varieties have been studied extensively in the literature and play an important role in geometric representation theory. See, for example, [2, 3] and the references therein.

It is well known (see Section 1.5 of [6], for example) that the choice of the pair (B,T) gives rise to a multiplicative holomorphic Poisson structure on G (see Section 3.2 for details) with respect to which each parabolic subgroup of G containing B is a Poisson submanifold. The Poisson-Lie group (G, π_G) is the semi-classical limit of the much studied quantum groups associated to G (see [6]). It is shown in [12] that the restriction of the n-fold product Poisson structure $\pi_G \times \ldots \times \pi_G$ to $P_1 \times \ldots \times P_n$ projects to a well-defined Poisson structure Π on $Z_\mathbf{w}$. We refer to Π as the standard Poisson structure on $Z_\mathbf{w}$.

The geometry of Π, such as the T-orbits of its symplectic leaves and Poisson divisors, is studied in [12]. It is also shown in [12] that Π gives rise to an iterated Poisson polynomial algebra for each of the 2^n affine charts on $Z_\mathbf{w}$ (see Section 4.1 for details). Iterated Poisson polynomial algebras have been studied in the context of Dixmier-Moeglin equivalences of Poisson algebras and the theory of quantum groups (see [4], [8], [9], [14] for example). The standard Poisson structure Π on Bott-Samelson varieties thus provides a rich source of examples for such algebras.

The Borel subgroup B acts on the Bott-Samelson variety $Z_\mathbf{w}$ by

$$b.[p_1, p_2, \ldots, p_n] = [bp_1, p_2, \ldots, p_n], \qquad b \in B, \ (p_1, p_2, \ldots, p_n) \in P_1 \times P_2 \times \cdots \times P_n.$$

For $x \in \mathfrak{b}$, let e_x be the holomorphic vector field on $Z_\mathbf{w}$ generating the action of B in the direction of x. In [12], the Poisson structure Π in each affine chart on $Z_\mathbf{w}$ is expressed in terms of the vector fields e_x for certain root vectors in $\mathfrak{b}$ (see Lemma 4.1 for detail). While for low dimensional examples such vector fields can be computed directly using the definition, it is clearly desirable to have a general formula for them in each affine coordinate chart of $Z_\mathbf{w}$.

The main result in this thesis, which we think is of interest irrespective of the Poisson structure, is an explicit formula for the vector field e_x, where $x \in \mathfrak{b}$ is any root vector, in every affine coordinate chart on $Z_\mathbf{w}$. The formula, as stated in Theorem 4.5, is in terms of the combinatorics of the root system and the structure constants of $\mathfrak{g}$ in any given basis consisting of root vectors. Applying Theorem 4.5 to the Poisson structure Π, one sees the explicit relations between the Poisson structure Π and the root system and the structure constants of the Lie algebra $\mathfrak{g}$. In particular, using a Chevalley basis of $\mathfrak{g}$, the Poisson structure Π is shown to be defined over $\mathbb{Z}$ in every affine chart (see Corollary 4.7). As a result, associated to each affine chart, one obtains an iterated Poisson polynomial algebra $k[z_1, \ldots, z_n]$ for any field k.

The thesis is organized as follows: In Section 3.2, we recall from [12] the definition of the Poisson structure Π on $Z_{\mathbf{w}}$. Section 4.3 contains the main result on the explicit formula for the vector fields e_x in every coordinate chart on $Z_{\mathbf{w}}$, where $x \in \mathfrak{b}$ is any root vector. Examples are presented in Section 5.

2 Notation and some facts on root strings

2.1 Notation

Let G be a connected complex semisimple Lie group as in Section 1, and let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$. For $\alpha \in \Delta$, let H_α be the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. Recall that $\Gamma \subseteq \Delta$ determines a decomposition of $\Delta = \Delta^+ \cup \Delta^-$ into positive and negative roots. For $\alpha \in \Delta^+$, let $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be such that $[E_\alpha, E_{-\alpha}] = H_\alpha$. Let $\langle \cdot, \cdot \rangle$ be a fixed multiple of the Killing form of $\mathfrak{g}$ such that the induced bilinear form on $\mathfrak{h}^*$, still denoted by $\langle \cdot, \cdot \rangle$, satisfies $\frac{\langle \beta,\beta\rangle}{2} \in \mathbb{Z}_{>0}$ for any root β.

Let $\beta \in \Delta^+$. Let $\theta_\beta : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ be the Lie algebra homomorphism defined by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto H_\beta, \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto E_\beta, \qquad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto E_{-\beta}.$$

The induced Lie group homomorphism is denoted by $\sigma_\beta : \mathrm{SL}(2, \mathbb{C}) \to G$. For $t \in \mathbb{C}^*$, let

$$\beta^\vee(t) = \sigma_\beta\left(\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}\right).$$

For $z \in \mathbb{C}$, let

$$u_\beta(z) = \sigma_\beta\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \exp(zE_\beta), \quad u_{-\beta}(z) = \sigma_\beta\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}\right) = \exp(zE_{-\beta}).$$

Let $\dot{s}_\beta = u_\beta(-1)u_{-\beta}(1)\, u_\beta(-1) \in N_G(T)$.

Lemma 2.1. *For $\alpha \in \Gamma$, one has*

$$u_\alpha(t)u_\alpha(z)\dot{s}_\alpha = u_\alpha(t+z)\dot{s}_\alpha, \qquad t, z \in \mathbb{C}, \tag{1}$$

$$u_\alpha(t)u_{-\alpha}(z) = u_{-\alpha}\left(\frac{z}{1+tz}\right) u_\alpha(t(1+tz))\alpha^\vee(1+tz), \qquad t, z \in \mathbb{C}, 1+tz \neq 0, \tag{2}$$

$$u_{-\alpha}(w) = u_\alpha\left(\frac{1}{w}\right) \dot{s}_\alpha u_\alpha(w)\alpha^\vee(w), \qquad w \in \mathbb{C}^*. \tag{3}$$

For $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, one has .

$$u_\beta(w)\beta^\vee(w)u_{-\alpha}(z) = u_{-\alpha}\left(w^{\frac{-2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}} z\right) u_\beta(w)\beta^\vee(w), \qquad w \in \mathbb{C}^*,\, z \in \mathbb{C}. \tag{4}$$

Proof. Identities (2) and (3) follow from computations in $\mathrm{SL}(2, \mathbb{C})$, and (4) follows from the fact that the two root subgroups corresponding to $-\alpha$ and β commute.

Q.E.D.

2.2 A lemma on root strings

Let the basis $\{H_\alpha, E_\beta \mid \alpha \in \Gamma, \beta \in \Delta\}$ for $\mathfrak{g}$ be chosen as in Section 2.1. For α, $\beta \in \Delta$ such that $\alpha+\beta \in \Delta$, let $N_{\alpha,\beta} \neq 0$ be such that $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\beta+\alpha}$.

Let α and β be two linearly independent roots and let $\{\beta + j\alpha : -p \leq j \leq q\}$, where p and q are non-negative integers, be the α-string through β. For $0 \leq i \leq p+q-1$, let

$$\varepsilon_i = \frac{i+1}{N_{\alpha,\beta-(p-i)\alpha}}. \tag{5}$$

Then subspace $L = \sum_{j=-p}^{q} \mathfrak{g}_{\beta+j\alpha}$ of $\mathfrak{g}$ becomes an $SL(2,\mathbb{C})$-module via the group homomorphism $\sigma_\alpha : SL(2,\mathbb{C}) \to G$, and the adjoint representation of G on $\mathfrak{g}$. On the other hand, let V^{p+q} be the vector space of homogeneous polynomials in (x, y) of degree $p+q$ with the (left) action of $\mathrm{SL}(2,\mathbb{C})$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f\right)(x,y) = f\left((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = f\left((ax+cy)^i (bx+dy)^{p+q-i}\right).$$

Consider the basis $\{u_0, \ldots, u_{p+q}\}$ for V^{p+q} given by

$$u_i = \varepsilon_0 \varepsilon_1 \cdots \varepsilon_{i-1} \binom{p+q}{i} x^i y^{p+q-i}, \qquad i = 0, 1, \ldots, p+q,$$

where ϵ_j, for $0 \leq j \leq p+q-1$, is defined in (5), and it is understood that $\varepsilon_0 \varepsilon_1 \cdots \varepsilon_{i-1} = 1$ when $i = 0$.

Lemma 2.2. *With the notation as above, the linear map*

$$\chi : \ L \longrightarrow V^{p+q} : \ \chi(E_{\beta+j\alpha}) = u_{p+j}, \quad -p \leq j \leq q, \tag{6}$$

is an $\mathrm{SL}(2,\mathbb{C})$*-equivariant isomorphism.*

Proof. The two irreducible representations of $SL(2,\mathbb{C})$ on L and on V^{p+q}, being of the same dimension, must be isomorphic, and by Schur's lemma, there is a unique $SL(2,\mathbb{C})$-equivariant isomorphism $\chi : L \to V^{p+q}$ such that $\chi(E_{\beta-p\alpha}) = u_0$. Straightforward calculations show that χ must be given as in (6). See also Lemma 6.2.2 of [5].

Q.E.D.

The following Lemma 2.3 will be used in Section 4.3 to prove the main results of the thesis.

Lemma 2.3. *Let α, β be two linearly independent roots and let $\{\beta + j\alpha : -p \leq j \leq q\}$ be the α-string through β. Then for any $t \in \mathbb{C}$, one has*

$$\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(E_\beta) = \sum_{j=0}^{q} (-1)^p \frac{\varepsilon_0 \cdots \varepsilon_{p-1}}{\varepsilon_0 \cdots \varepsilon_{q-j-1}} \binom{p+j}{j} t^j E_{\beta+(q-p-j)\alpha}, \tag{7}$$

$$\mathrm{Ad}_{(u_{-\alpha}(t))^{-1}}(E_\beta) = \sum_{j=0}^{p} (-1)^j \varepsilon_{p-1} \varepsilon_{p-2} \cdots \varepsilon_{p-j} \binom{q+j}{j} t^j E_{\beta-j\alpha}. \tag{8}$$

Proof. By Lemma 2.2, one has

$$\begin{aligned}
\chi\left(\mathrm{Ad}_{(u_\alpha(t)\dot{s}_\alpha)^{-1}}(E_\beta)\right) &= \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix}.u_p \\
&= \varepsilon_0\cdots\varepsilon_{p-1}\binom{p+q}{p}(-y)^p(x+ty)^q \\
&= \varepsilon_0\cdots\varepsilon_{p-1}\binom{p+q}{p}(-y)^p\left(\sum_{j=0}^{q}\binom{q}{j}t^jy^jx^{q-j}\right) \\
&= \sum_{j=0}^{q}(-1)^p\frac{\varepsilon_0\cdots\varepsilon_{p-1}}{\varepsilon_0\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}u_{q-j}t^j,
\end{aligned}$$

from which (7) follows. One proves (8) similarly (see also Lemma 6.2.1 in [5]).

Q.E.D.

Let α, β be as in Lemma 2.3. For notational simplicity, denote the constants appearing in (7) and (8) by

$$\begin{aligned}
c_{\alpha,\beta}^{\sigma,j} &= (-1)^p\frac{\varepsilon_0\cdots\varepsilon_{p-1}}{\varepsilon_0\cdots\varepsilon_{q-j-1}}\binom{p+j}{j}, && j=0,\ldots,q \text{ and } \sigma = s_\alpha, \\
c_{\alpha,\beta}^{\sigma,j} &= (-1)^j\varepsilon_{p-1}\varepsilon_{p-2}\cdots\varepsilon_{p-j}\binom{q+j}{j}, && j=0,\ldots,p \text{ and } \sigma = e. \qquad (9)
\end{aligned}$$

We also set $c_{\alpha,\alpha}^{e,j} = 1$.

Remark 2.4. Recall that $\{H_\alpha, E_\beta \mid \alpha \in \Gamma, \beta \in \Delta\}$ is said to be a Chevalley basis if for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, one has $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. If $\{H_\alpha, E_\beta \mid \alpha \in \Gamma, \beta \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$, then, by Theorem 4.1.2 of [5], $N_{\alpha,\beta} = \pm(p+1)$ for any roots α and β such that $\alpha+\beta \in \Delta$, where p is the largest non-negative integer such that $\beta - p\alpha \in \Delta$. Thus, for α and β as in Lemma 2.3 and for every $0 \leq i \leq p+q-1$, one has $\varepsilon_i = \pm 1$, and consequently all the coefficients $c_{\alpha,\beta}^{\sigma,j}$'s appearing in (7) and (8) are integers.

3 The Poisson structure Π on the Bott-Samelson variety

3.1 Poisson-Lie groups and Poisson actions

In this section, we review from [12] the definition of the Poisson structure Π on $Z_{\mathbf{w}}$. Recall that a Poisson structure π_P on a manifold P is a bivector field $\pi_P \in \Gamma(\wedge^2 TP)$ such that the induced bracket

$$\{\cdot,\cdot\} : C^\infty(P)\times C^\infty(P) \to C^\infty(P) : \{f,g\} = (df \wedge dg)(\pi_P), \qquad f,g \in C^\infty(P),$$

satisfies the Jacobi identity

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \qquad f,g,h \in C^\infty(P).$$

A differentiable map $F : (P,\pi_P) \to (Q,\pi_Q)$ between Poisson manifolds is said to be Poisson if $F_*(\pi_P)(q) = \pi_Q(q)$ for all $q \in Q$, where F_* denotes the pushforward of tangent vectors by F.

Recall from [6] that if G is a Lie group, a Poisson bivector field π_G on G is said to be multiplicative if the map $(G\times G, \pi_G\times\pi_G) \to (G,\pi_G) : (g_1,g_2) \mapsto g_1g_2$ is Poisson. A Poisson

Lie group is a pair (G, π_G), where G is a Lie group and π_G is a multiplicative Poisson bivector field on G. A Lie subgroup G_1 of a Poisson Lie group (G, π_G) is called a Poisson Lie subgroup if G_1 is a Poisson submanifold with respect to π_G. In this case, $(G_1, \pi_G|_{G_1})$ is a Poisson Lie group. Let (G, π_G) be a Poisson-Lie group, and (P, π_P) a Poisson manifold. A smooth (right) action of G on P is said to be a Poisson action if the action map

$$\Theta : (P \times G, \pi_P \times \pi_G) \longrightarrow (P, \pi_P)$$

is a Poisson map. Left Poisson actions are defined similarly.

Let (B, π) be a Poisson-Lie group and assume that (P, π_P) and (Q, π_Q) are two Poisson manifolds with a right Poisson action of (B, π) on (P, π_P) and a left Poisson action of (B, π) on (Q, π_Q). Assume that the quotient space P/B is a smooth manifold and let $\rho : P \to P/B$ be the natural projection. Let B act on $P \times Q$ by

$$(p, q) \cdot b = (pb, b^{-1}q), \qquad p \in P, q \in Q, b \in B,$$

let $P \times_B Q = (P \times Q)/B$, and let $\omega : P \times Q \to P \times_B Q$ be the natural projection. For $(p, q) \in P \times Q$, let $[p, q] = \omega(p, q) \in P \times_B Q$. The proof of the following Lemma 3.1 can be found in [12].

Lemma 3.1. 1) *The projection $\rho(\pi_P)$ is a well-defined Poisson structure on P/B;*
2) *The projection $\omega(\pi_P \times \pi_Q)$ is a well-defined Poisson structure on $P \times_B Q$.*

3.2 The definition of the Poisson structure Π on $Z_{\mathbf{w}}$

Let G be a complex connected semisimple Lie group as in Section 1 and let the notation be as in Section 2.1. Let

$$\Lambda = \sum_{\alpha \in \Delta^+} \frac{\langle \alpha, \alpha \rangle}{2} E_\alpha \wedge E_{-\alpha} \in \wedge^2 \mathfrak{g},$$

and let π_G be the bivector field on G given by

$$\pi_G = \Lambda^l - \Lambda^r,$$

where Λ^l and Λ^r are respectively the left-invariant and right-invariant bivector fields on G with value Λ at the identity. By [6], (G, π_G) is a Poisson-Lie group. Every parabolic subgroup P containing B, being a union of Bruhat cells BwB, is a Poisson-Lie subgroup (see [10]).

Let $\mathbf{w} = (s_1, \ldots, s_n)$ be any sequence of simple reflections and consider the Bott-Samelson variety $Z_{\mathbf{w}} = P_1 \times_B P_2 \times_B \cdots \times_B P_n/B$. By repeatedly using Lemma 3.1, one obtains a holomorphic Poisson structure Π on the Bott-Samelson variety $Z_{\mathbf{w}}$ such that the natural projection

$$(P_1 \times P_2 \times \ldots \times P_n, \pi_G|_P \times \ldots \times \pi_G|_P) \to (Z_{\mathbf{w}}, \Pi)$$

is a Poisson map. The Poisson structure Π on $Z_{\mathbf{w}}$ was first introduced in [12] and we will refer to Π as the standard Poisson structure on $Z_{\mathbf{w}}$.

4 The Poisson structure Π in coordinate charts

4.1 Coordinate charts on $Z_{\mathbf{w}}$

Let G be as in Section 1 and let the notation be as in Section 2.1. Let $\mathbf{w} = (s_1, \ldots, s_n)$ be any sequence of simple reflections in W, and let $Z_{\mathbf{w}}$ be the Bott-Samelson variety associated to $\mathbf{w}$. For $1 \leq i \leq n$, let $\alpha_i \in \Gamma$ be such that $s_i = s_{\alpha_i}$. Let

$$\Upsilon_{\mathbf{w}} = \{e, s_1\} \times \{e, s_2\} \times \cdots \{e, s_n\},$$

where e denotes the identity element of W. Elements in $\Upsilon_{\mathbf{w}}$ will be called *subexpressions* of $\mathbf{w}$. For $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \Upsilon_{\mathbf{w}}$, let $\gamma^j = \gamma_1\gamma_2\cdots\gamma_j \in W$ for $1 \leq j \leq n$.

The maximal torus T of G acts on $Z_{\mathbf{w}}$ by

$$t.[p_1, \ldots, p_n] = [tp_1, \ldots, p_n], \qquad t \in T,\ (p_1, \ldots, p_n) \in P_1 \times \cdots \times P_n.$$

The fixed points of this T-action are the points of the form $[\dot{\gamma}_1, \ldots, \dot{\gamma}_n]$, where $(\gamma_1, \ldots, \gamma_n) \in \Upsilon_{\mathbf{w}}$, and $\dot{e} = e$. Around each T-fixed point $[\dot{\gamma}_1, \ldots, \dot{\gamma}_n]$, one has a natural affine chart $\mathcal{O}^\gamma$, defined as the image of the map

$$\Phi_\gamma : \mathbb{C}^n \longrightarrow Z_{\mathbf{w}} : \quad \Phi_\gamma(z_1, \ldots, z_n) = [u_{-\gamma_1(\alpha_1)}(z_1)\dot{\gamma}_1, u_{-\gamma_2(\alpha_2)}(z_2)\dot{\gamma}_2, \ldots, u_{-\gamma_n(\alpha_n)}(z_n)\dot{\gamma}_n].$$

Note that the parametrization of $\mathcal{O}^\gamma$ by $\mathbb{C}^n$ depends on the choice of the root vectors for the simple roots α_i for $1 \leq i \leq n$, but different choices of such root vectors only result in re-scalings of the coordinate functions. In particular, the affine chart $\mathcal{O}^\gamma$ is canonically defined. It is also easy to see that each $\mathcal{O}^\gamma$ is T-invariant with

$$t.\Phi_\gamma(z_1, z_2, \ldots, z_n) = \Phi_\gamma(t^{-\gamma^1(\alpha_1)}z_1,\ t^{-\gamma^2(\alpha_2)}z_2, \ldots, t^{-\gamma^n(\alpha_n)}z_n], \tag{10}$$

where $t \in T$ and $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$. Note also that $\bigcup_{\gamma \in \Upsilon_{\mathbf{w}}} \mathcal{O}^\gamma = Z_{\mathbf{w}}$, i.e., $Z_{\mathbf{w}}$ is covered by the 2^n T-invariant affine charts $\mathcal{O}^\gamma$, $\gamma \in \Upsilon_{\mathbf{w}}$.

4.2 A formula for Π from [12]

Let the notation be as in Section 4.1. We recall from [12] a formula for the Poisson structure Π in each the coordinate charts $\mathcal{O}^\gamma$ on $Z_{\mathbf{w}}$.

For $1 \leq i \leq n-1$, let e_i be the holomorphic vector field on the Bott-Samelson variety $Z_{(s_{i+1}, \ldots, s_n)}$ given by

$$e_i(\varphi)(p) = \frac{d}{dt}|_{t=0}\varphi\left((\exp tE_{\alpha_i})p\right), \tag{11}$$

where $p \in Z_{(s_{i+1}, \ldots, s_n)}$ and φ is any local holomorphic function on $Z_{(s_{i+1}, \ldots, s_n)}$.

Lemma 4.1. *([12]) Let $\gamma \in \Upsilon_{\mathbf{w}}$. In the coordinates $(z_1, \ldots z_n)$ on the affine chart $\mathcal{O}^\gamma$, the Poisson structure Π is given by,*

$$\{z_i, z_j\} = \begin{cases} \langle \gamma^i(\alpha_i),\ \gamma^j(\alpha_j)\rangle z_i z_j, & \text{if } \gamma_i = e, \\ -\langle \gamma^i(\alpha_i),\ \gamma^j(\alpha_j)\rangle z_i z_j - \langle \alpha_i, \alpha_i\rangle e_i(z_j) & \text{if } \gamma_i = s_i, \end{cases} \quad 1 \leq i < j \leq n,$$

where for $1 \leq i < j \leq n$, $e_i(z_j)$ denotes the action of the vector field e_i on the coordinate function z_j.

Remark 4.2. Recall that a Poisson algebra is a commutative $\mathbb{C}$-algebra A equipped with a Lie bracket $\{\cdot, \cdot\}$ satisfying the Leibniz rule $\{f, gh\} = \{f, g\}h + \{f, h\}g$ for all $f, g, h \in A$. Recall from [9] that an iterated Poisson polynomial algebra over $\mathbb{C}$ is the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ with a Poisson bracket $\{\cdot, \cdot\}$ satisfying

$$\{z_i, \mathbb{C}[z_{i+1}, \ldots, z_n]\} \subseteq z_i\mathbb{C}[z_{i+1}, \ldots, z_n] + \mathbb{C}[z_{i+1}, \ldots, z_n], \quad \forall\, 1 \leq i \leq n-1.$$

By Lemma 4.1, for each $\gamma \in \Upsilon_{\mathbf{w}}$, the Poisson structure Π makes $\mathbb{C}[z_1, \ldots, z_n]$ into an iterated Poisson polynomial algebra via the identification $\mathcal{O}^\gamma \cong \mathbb{C}^n$.

To derive an explicit formula for the Poisson structure Π in the coordinates $(z_1, z_2, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$, one thus needs to compute the vector fields e_i in Lemma 4.1 for each $1 \leq i \leq n-1$. To this end, for any positive root β, let e_β be the vector field on $Z_\mathbf{w}$ given by

$$e_\beta(\varphi)(p) = \frac{d}{dt}|_{t=0}\varphi\left((\exp tE_\beta)p\right), \qquad p \in Z_\mathbf{w}, \tag{12}$$

where φ is any local holomorphic function on $Z_\mathbf{w}$. In the following Section 4.3, which contains the main result of the thesis, we give an explicit formula for the vector field e_β in the coordinates $(z_1, z_2, \ldots, z_n)$ in the affine chart $\mathcal{O}^\gamma$ for every $\gamma \in \Upsilon_\mathbf{w}$. We believe that our formula for e_β, as will be stated in Theorem 4.5, is of independent interest in the study of Bott-Samelson varieties.

4.3 An explicit formula for the vector field e_β

Let $\beta \in \Delta^+$. In this section, we compute the vector field e_β on $Z_\mathbf{w}$ given in (12) in the affine chart $\mathcal{O}^\gamma$ for every $\gamma \in \Upsilon_\mathbf{w}$. We will first give a recursive formula for e_β in Lemma 4.3 and then a closed formula for e_β in Theorem 4.5.

To derive the recursive formula for e_β, we need to consider vector fields on different Bott-Samelson varieties. To this end, introduce, for a positive root β and any sequence $\mathbf{w}'$ of simple roots in W, two vector fields on the Bott-Samelson variety $Z_{\mathbf{w}'}$ by:

$$h_\beta^{\mathbf{w}'}(\varphi)(p) = \frac{d}{dt}|_{t=0}\varphi\left((\exp tH_\beta)p\right),$$
$$e_\beta^{\mathbf{w}'}(\varphi)(p) = \frac{d}{dt}|_{t=0}\varphi\left((\exp tE_\beta)p\right),$$

where $p \in Z_{\mathbf{w}'}$ and φ is any local holomorphic function on $Z_{\mathbf{w}'}$. Note that $e_\beta = e_\beta^\mathbf{w}$ for e_β defined in (12).

Fix $\gamma \in \Upsilon_\mathbf{w}$ and keep the notation from the previous sections. In particular, recall from Section 4.1 the coordinates $(z_1, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$. Assume that $n \geq 2$ and for $2 \leq k \leq n$, let $e_\beta^{(k)}$ denote the restriction of the vector field $e_\beta^{(s_k,\ldots,s_n)}$ on the Bott-Samelson variety $Z_{(s_k,\ldots,s_n)}$ to the affine chart $\mathcal{O}^{(\gamma_k,\ldots,\gamma_n)}$ of $Z_{(s_k,\ldots,s_n)}$, and we regard $e_\beta^{(k)}$ as a derivation on the polynomial algebra $\mathbb{C}[z_k, \ldots, z_n]$. Similarly, one has the derivation $h_\beta^{(k)}$ on $\mathbb{C}[z_k, \ldots, z_n]$.

Lemma 4.3. *Let $\beta \in \Delta^+$. Let $\gamma \in \Upsilon_\mathbf{w}$ and let $(z_1, z_2, \ldots, z_n)$ be the coordinates on the affine chart $\mathcal{O}^\gamma$.*

Case 1.1. *$\beta = \alpha_1$ and $\gamma_1 = s_1$. In this case, $e_\beta(z_1) = 1$ and $e_\beta(z_k) = 0$ for all $k \geq 2$;*

Case 1.2. *$\beta = \alpha_1$ and $\gamma_1 = e$. In this case, $e_\beta(z_1) = -z_1^2$ and for $k \geq 2$,*

$$e_\beta(z_k) = e_\beta^{(2)}(z_k) + z_1 h_\beta^{(2)}(z_k).$$

Case 2. *$\beta \neq \alpha_1$. In this case, $e_\beta(z_1) = 0$ and for $k \geq 2$,*

$$e_\beta(z_k) = \sum_{\substack{j \geq 0 \\ \gamma_1(\beta) - j\alpha_1 \in \Delta}} c_{\alpha_1,\beta}^{\gamma_1,j} z_1^j e_{\gamma_1(\beta)-j\alpha_1}^{(2)}(z_k)$$

Proof. Cases 1.1., 1.2., and 2. follow from (1), (2), and Lemma 2.3, respectively.

Q.E.D.

To obtain a closed formula for the vector field e_β, we introduce more notation. Let $\mathbb{N}$ denote the set of non-negative integers.

Notation 4.4. For $1 \le l \le n$ and $(j_1, \ldots, j_l) \in \mathbb{N}^l$, let

$$\beta_{(j_1,\ldots,j_l)} = \gamma_l\gamma_{l-1}\cdots\gamma_2\gamma_1(\beta) - j_1\gamma_l\gamma_{l-1}\cdots\gamma_2(\alpha_1) - \ldots - j_{l-1}\gamma_l(\alpha_{l-1}) - j_l\alpha_l \in \mathfrak{h}^*. \quad (13)$$

For $2 \le k \le n$, define

$$N_k = \left\{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} | \beta_{(n_1,\ldots,n_{k-1})} = \alpha_k, \beta_{(n_1,\ldots,n_l)} \in \Delta^+, \forall\, 1 \le l \le k-2\right\}, \quad (14)$$

and for $(j_1, \ldots, j_{k-1}) \in N_k$, let

$$c_{j_1,\ldots,j_{k-1}} = c^{\gamma_1,j_1}_{\alpha_1,\beta}\cdots c^{\gamma_{k-1},j_{k-1}}_{\alpha_{k-1},\beta_{(j_1,\ldots,j_{k-2})}} \quad (15)$$

(see notation in Section 2.1). To each $1 \le k \le n$, introduce two functions $\phi_\beta(z_1, \ldots, z_{k-1})$ and $\psi_\beta(z_1, \ldots, z_{k-1}) \in \mathbb{C}[z_1, \ldots, z_{k-1}]$ as follows: for $k = 1$, let

$$\phi_\beta(z_1, \ldots, z_{k-1}) = \begin{cases} 1 & \text{if } \beta = \alpha_1, \\ 0 & \text{if } \beta \ne \alpha_1, \end{cases} \quad \text{and} \quad \psi_\beta(z_1, \ldots, z_{k-1}) = 0,$$

and for $2 \le k \le n$, let

$$\phi_\beta(z_1, \ldots, z_{k-1}) = \sum_{(j_1,\ldots,j_{k-1})\in N_k} c_{j_1,\ldots,j_{k-1}} z_1^{j_1} z_2^{j_2}\cdots z_{k-1}^{j_{k-1}}, \quad (16)$$

$$\psi_\beta(z_1, \ldots, z_{k-1}) = -\sum_{\substack{\gamma_j=e \\ 1\le j\le k-1}} \frac{2\langle\gamma^j(\alpha_j), \gamma^k(\alpha_k)\rangle}{\langle\gamma^j(\alpha_j), \gamma^j(\alpha_j)\rangle} z_j\phi_\beta(z_1, \ldots z_{j-1}), \quad (17)$$

where recall that $\gamma^j = \gamma_1\gamma_2\cdots\gamma_j$ for $1 \le j \le n$. Note that the functions $\phi_\beta(z_1, \ldots, z_{k-1})$ and $\psi_\beta(z_1, \ldots, z_{k-1})$ depend on γ.

The following Theorem 4.5, which is the first main result of the thesis, gives a purely combinatorial formula for the vector field e_β.

Theorem 4.5. *Let $\beta \in \Delta^+$ and let $\gamma \in \Upsilon_{\mathbf{w}}$. The vector field e_β acts on the coordinate function $\{z_k : 1 \le k \le n\}$ on affine chart $\mathcal{O}^\gamma$ as follows:*

$$e_\beta(z_k) = \begin{cases} \phi_\beta(z_1, \ldots, z_{k-1}) + \psi_\beta(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = s_k, \\ -\phi_\beta(z_1, \ldots, z_{k-1})z_k^2 + \psi_\beta(z_1, \ldots, z_{k-1})z_k, & \text{if } \gamma_k = e. \end{cases}$$

Proof. If $k = 1$, Theorem 4.5 holds by definition in this case. Assume that $2 \le k \le n$. Let $N'_k = \{(j_1, \ldots, j_{k-1}) \in \mathbb{N}^{k-1} \mid \beta_{(j_1,\ldots,j_{k-1})} \in \Delta^+, \forall\, 1 \le i \le k-1\}$, and for $(j_1, \ldots, j_{k-1}) \in N'_k$, let the (nonzero) constants $c_{j_1,\ldots,j_{k-1}}$ be as in Notation 15. For $k = 2$, one has

$$N'_2 = \begin{cases} \emptyset & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = s_1, \\ \{0\} & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ \{j_1 \mid j_1 \in \mathbb{N}, \gamma_1(\beta) - j_1\alpha_1 \in \Delta^+\} & \text{if } \beta \ne \alpha_1, \end{cases}$$

where, by definition, $c^{e,0}_{\alpha_1,\alpha_1} = 1$ when $\beta = \alpha_1$ and $\gamma_1 = e$. Therefore, one can combine the cases in Lemma 4.3 to get

$$e_\beta(z_k) = \sum_{j_1\in N'_2} c^{\gamma_1,j_1}_{\alpha_1,\beta} z_1^{j_1} e^{(2)}_{\beta_{(j_1)}}(z_k) + \begin{cases} z_1 h^{(2)}_{\alpha_1}(z_k), & \text{if } \beta = \alpha_1 \text{ and } \gamma_1 = e, \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

By repeatedly using (18), one has

$$e_\beta(z_k) = \sum_{(j_1,\ldots,j_{k-1})\in N'_k} c_{j_1,\ldots,j_{k-1}} z_1^{j_1}\cdots z_{k-1}^{j_{k-1}} e^{(k)}_{\beta_{(j_1,\ldots,j_{k-1})}}(z_k) + \sum_{\substack{\gamma_j=e\\1\le j\le k-1}} \phi_\beta(z_1,\ldots,z_{j-1}) z_j h^{(j+1)}_{\alpha_j}(z_k).$$

Let $z'_k = 1$ if $\gamma_k = s_k$ and $z'_k = -z_k^2$ if $\gamma_k = e$. By Lemma 4.3, for $(j_1,\ldots,j_{k-1}) \in N'_k$, one has $e^{(k)}_{(j_1,\ldots,j_{k-1})}(z_k) = 0$ unless $\beta_{(j_1,\ldots,j_{k-1})} = \alpha_k$, in which case $e^{(k)}_{(j_1,\ldots,j_{k-1})}(z_k) = z'_k$. Thus

$$\begin{aligned} e_\beta(z_k) &= \sum_{(j_1,\ldots,j_{k-1})\in N_k} c_{j_1,\ldots,j_{k-1}} z_1^{j_1}\cdots z_{k-1}^{j_{k-1}} z'_k + \sum_{\substack{1\le j\le k-1\\ \gamma_j=e}} \phi_\beta(z_1,\ldots,z_{j-1}) z_j h^{(j+1)}_{\alpha_j}(z_k) \\ &= \phi_\beta(z_1,\ldots,z_{k-1}) z'_k + \sum_{\substack{1\le j\le k-1\\ \gamma_j=e}} \phi_\beta(z_1,\ldots,z_{j-1}) z_j h^{(j+1)}_{\alpha_j}(z_k). \end{aligned}$$

On the other hand, for each $1 \le j \le k-1$ with $\gamma_j = e$,

$$h^{(j+1)}_{\alpha_j}(z_k) = -\frac{2\langle \alpha_j, \gamma_{j+1}\cdots\gamma_k(\alpha_k)\rangle}{\langle \alpha_j, \alpha_j\rangle} z_k = -\frac{2\langle \gamma^j(\alpha_j), \gamma^k(\alpha_k)\rangle}{\langle \gamma^j(\alpha_j), \gamma^j(\alpha_j)\rangle} z_k.$$

It follows that

$$e_\beta(z_k) = \phi_\beta(z_1,\ldots,z_{k-1}) z'_k + \psi_\beta(z_1,\ldots,z_{k-1}) z_k.$$

Q.E.D.

Remark 4.6. Recall from Section 4.1 that the while the affine chart $\mathcal{O}^\gamma$ is canonically defined, its parametrization by $\mathbb{C}^n$ depends on the choice of the basis $\{H_\alpha,\ E_\beta \mid \alpha \in \Gamma, \beta \in \Delta\}$ of $\mathfrak{g}$. Assume that $\{H_\alpha,\ E_\beta \mid \alpha \in \Gamma, \beta \in \Delta\}$ is a Chevalley basis of $\mathfrak{g}$. Then by Remark 2.4, for each positive root β and for every $1 \le k \le n$, the polynomials $\phi_\beta(z_1,\ldots,z_{k-1})$ and $\phi_\beta(z_1,\ldots,z_{k-1})$ have integral coefficients. Consequently, the polynomials $e_\beta(z_k)$ for $1 \le k \le n$ all have integral coefficients.

4.4 The Poisson structure Π is defined over $\mathbb{Z}$ in any chart

We now return to the holomorphic Poisson structure Π on the Bott-Samelson variety $Z_{\mathbf{w}}$ associated to a sequence $\mathbf{w} = (s_1, s_2, \ldots, s_n)$ of simple reflections. The following Corollary 4.7, which is the second main result of the thesis, says that the Poisson structure Π is defined over $\mathbb{Z}$ in every affine chart $\mathcal{O}^\gamma$.

Corollary 4.7. *. Let $\gamma \in \Upsilon_{\mathbf{w}}$ and let the coordinates $(z_1, z_2, \ldots, z_n)$ on the affine chart $\mathcal{O}^\gamma$ be defined by a Chevalley basis of $\mathfrak{g}$. Then for every $1 \le i < j \le n$, the Poisson bracket $\{z_i, z_j\}$ is a polynomial in the variables $(z_i, z_{i+1}, \ldots, z_j)$ with integral coefficients.*

Proof. Recall from Section 2.1 that the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ that is used to define the Poisson structure π_G on G is such that $\langle\alpha,\alpha\rangle/2$ is an integer for every root α. It follows that for any pair of roots α and β, $\langle\alpha,\beta\rangle = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\cdot\frac{\langle\alpha,\alpha\rangle}{2}$ is an integer. Corollary 4.7 now follows from Lemma 4.1 and Remark 4.6.

Q.E.D.

5 Examples

5.1 The vector field e_β in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$

Let the notation be as in Section 4.3 and assume that β is a simple root. In this section, we compute the vector field e_β defined in (12) on the affine chart $\mathcal{O}^\gamma$ for $\gamma = (e, e, \ldots, e)$.

Recall from Section 4.1 that one has the coordinates $(z_1, z_2, \ldots, z_n)$ on $\mathcal{O}^{(e,e,\ldots,e)}$ via

$$(z_1, z_2, \ldots, z_n) \longmapsto [u_{-\alpha_1}(z_1), u_{-\alpha_2}(z_2), \ldots, u_{-\alpha_n}(z_n)] \in \mathcal{O}^{(e,e,\ldots,e)}.$$

Lemma 5.1. *In the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$ the vector field e_β for a simple root β is given by*

$$e_\beta(z_k) = -\frac{2\langle\beta, \alpha_k\rangle}{\langle\beta, \beta\rangle}\left(\sum_{1\le j\le k-1,\, \alpha_j=\beta} z_j\right) z_k + \begin{cases} 0, & \text{if } \alpha_k \neq \beta, \\ -z_k^2, & \text{if } \alpha_k = \beta, \end{cases} \qquad 1 \le k \le n.$$

Proof. Let $1 \le k \le n$. By Theorem 4.5, one has,

$$e_\beta(z_k) = -\phi_\beta(z_1, \ldots, z_{k-1})z_k^2 + \psi_\beta(z_1, \ldots, z_{k-1})z_k.$$

Using the assumption that β is a simple root, one sees easily from the definition of ϕ_β that $\phi_\beta(z_1, \ldots, z_{k-1}) = 1$ if $\alpha_k = \beta$ and $\phi_\beta(z_1, \ldots, z_{k-1}) = 0$ if $\alpha_k \neq \beta$. It follows from the definition of ψ_β that

$$\psi_\beta(z_1, \ldots, z_{k-1}) = -\frac{2\langle\beta, \alpha_k\rangle}{\langle\beta, \beta\rangle}\left(\sum_{1\le j\le k-1,\, \alpha_j=\beta} z_j\right).$$

This proves Lemma 5.1.

Q.E.D.

Consider now the subexpression $\gamma = (s_1, e, \ldots, e)$ of $\mathbf{w}$ and the affine chart $\mathcal{O}^{(s_1,e,\ldots,e)}$ with the parametrization

$$\mathbb{C}^n \ni (z_1, z_2, \ldots, z_n) \longmapsto [u_{\alpha_1}(z_1)\dot{s}_1,\, u_{-\alpha_2}(z_2), \ldots, u_{-\alpha_n}(z_n)] \in \mathcal{O}^{(s_1,e,\ldots,e)}.$$

Corollary 5.2. *In the affine chart $\mathcal{O}^{(s_1,e,\ldots,e)}$, the Poisson structure Π is given by*

$$\{z_i, z_j\} = \langle\alpha_i, \alpha_j\rangle z_i z_j, \qquad \text{if } 2 \le i < j \le n,$$

$$\{z_1, z_j\} = \begin{cases} -\langle\alpha_1, \alpha_j\rangle\left(z_1 - 2\sum_{2\le i\le j-1,\, \alpha_i=\alpha_1} z_i\right) z_j, & \text{if } 2 \le j \le n \text{ and } \alpha_j \neq \alpha_1, \\ -\langle\alpha_1, \alpha_1\rangle\left(z_1 - 2\sum_{2\le i\le j-1,\, \alpha_i=\alpha_1} z_i - z_j\right) z_j, & \text{if } 2 \le j \le n \text{ and } \alpha_j = \alpha_1. \end{cases}$$

Proof. Corollary 5.2 follows directly from Corollary 4.7 and Lemma 5.1.

Q.E.D.

Remark 5.3. The coordinates on $\mathcal{O}^{(s_1,e,\ldots,e)}$ used in Lemma 5.2 are defined by any basis of $\mathfrak{g}$ as specified in Section 2.1 which is not necessarily a Chevalley basis. Indeed, since the Poisson structure Π is quadratic, it is invariant under re-scalings of the coordinates.

5.2 An example for $G = G_2$

Let G be the connected complex semisimple Lie group with Lie algebra of type G_2. Let α and β be the two simple roots, where $\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle} = 3$. We fix a Chevalley basis of $\mathfrak{g}_2$ with the following system of structure constants N_{α_1,α_2} for roots α_1 and α_2:

	α	β	$\alpha+\beta$	$2\alpha+\beta$	$3\alpha+\beta$	$3\alpha+2\beta$
α	0	-1	-2	-3	0	0
β	1	0	0	0	-1	0
$\alpha+\beta$	2	0	0	3	0	0
$2\alpha+\beta$	3	0	-3	0	0	0
$3\alpha+\beta$	0	1	0	0	0	0
$3\alpha+2\beta$	0	0	0	0	0	0
$-\alpha$	0	0	-3	-2	-1	0
$-\beta$	0	0	1	0	0	-1
$-\alpha-\beta$	-3	1	0	2	0	1
$-2\alpha-\beta$	-2	0	2	0	1	-1
$-3\alpha-\beta$	-1	0	0	1	0	1
$-3\alpha-2\beta$	0	-1	1	-1	1	0

Using $N_{\alpha_1,\alpha_2} = -N_{-\alpha_1,-\alpha_2}$, we can easily find the remaining structure constants.

Consider the Bott-Samelson variety associated to the word $\mathbf{w_0} = (s_1, s_2, s_1, s_2, s_1, s_2)$, where $s_1 = s_\alpha$ and $s_2 = s_\beta$. Let $\gamma = (s_1, s_2, s_1, s_2, s_1, s_2)$. The Poisson structure in the chart $\mathcal{O}^\gamma$ in the coordinates $(z_1, \ldots z_6)$ was computed using a computer programme developed by the author in the GAP [7] language.

$$
\begin{aligned}
\{z_1, z_2\} &= -3z_1z_2\\
\{z_1, z_3\} &= -z_1z_3 - 2z_2\\
\{z_1, z_4\} &= -6z_3^2\\
\{z_1, z_5\} &= z_1z_5 - 4z_3\\
\{z_1, z_6\} &= 3z_1z_6 - 6z_5\\
\{z_2, z_3\} &= -3z_2z_3\\
\{z_2, z_4\} &= -6z_3^3 - 3z_2z_4\\
\{z_2, z_5\} &= -6z_3^2\\
\{z_2, z_6\} &= 3z_2z_6 - 18z_3z_5 + 6z_4\\
\{z_3, z_4\} &= -3z_3z_4\\
\{z_3, z_5\} &= -z_3z_5 - 2z_4\\
\{z_3, z_6\} &= -6z_5^2\\
\{z_4, z_5\} &= -3z_4z_5\\
\{z_4, z_6\} &= -6z_5^3 - 3z_4z_6\\
\{z_5, z_6\} &= -3z_5z_6
\end{aligned}
$$

A Another proof of Corollary 5.2 by change of coordinates

Lemma 4.1 expresses the Poisson structure Π on the Bott-Samelson variety $Z_{\mathbf{w}}$ in different affine coordinate charts. In particular, in the affine chart $\mathcal{O}^{(e,e,\ldots,e)}$ with coordinates

$$(\xi_1, \xi_2, \ldots, \xi_n) \mapsto [u_{-\alpha_1}(\xi_1), u_{-\alpha_2}(\xi_2), \ldots, u_{-\alpha_n}(\xi_n)],$$

the Poisson structure is especially simple, namely, $\{\xi_i, \xi_j\} = \langle \alpha_i, \alpha_j \rangle \xi_i \xi_j$ for all $1 \leq i < j \leq n$. In Corollary 5.2, we have also given the explicit formulas for the Poisson structure Π in the affine chart $\mathcal{O}^{(s_1,e,\ldots,e)}$ with coordinates

$$(z_1, z_2 \ldots, z_n) \mapsto [u_{\alpha_1}(z_1)\dot{s_1}, u_{-\alpha_2}(z_2), \ldots, u_{-\alpha_n}(z_n)].$$

In this appendix, we first write down the transition functions between the coordinates $(\xi_1, \xi_2, \ldots, \xi_n)$ and the coordinates $(z_1, z_2, \ldots, z_n)$ on the intersection $\mathcal{O}^{(e,e,\ldots,e)} \cap \mathcal{O}^{(s_1,e,\ldots,e)}$. We then use the change of coordinates and the simple formulas for the Poisson structure in $\mathcal{O}^{(e,e,\ldots,e)}$ to give another proof of Corollary 5.2.

Lemma A.1. *On the intersection $\mathcal{O}^{(e,e,\ldots,e)} \cap \mathcal{O}^{(s_1,e,\ldots,e)}$, one has $z_1 = 1/\xi_1$, and for $2 \leq j \leq n$, one has*

$$z_j = \begin{cases} \left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq j-1}} \xi_i\right)^{\frac{-2\langle\alpha_1,\alpha_2\rangle}{\langle\alpha_1,\alpha_1\rangle}} & \text{if } \alpha_j \neq \alpha_1, \\[2ex] \dfrac{\xi_j}{\left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq j-1}} \xi_i\right)\left(\sum_{\substack{\alpha_i=\alpha_1 \\ 1\leq i\leq j}} \xi_i\right)} & \text{if } \alpha_j = \alpha_1. \end{cases}$$

Proof. By (3), and the fact that $u_\beta(w)\beta^\vee(w) \in B$ for any $w \in \mathbb{C}^*$, one has

$$[u_{-\alpha_1}(\xi_1), u_{-\alpha_2}(\xi_2), \ldots, u_{-\alpha_n}(\xi_n)] = \left[u_{\alpha_1}\left(\frac{1}{\xi_1}\right)\dot{s_1}, u_{\alpha_1}(\xi_1)\alpha_1^\vee(\xi_1)u_{-\alpha_2}(\xi_2), \ldots, u_{-\alpha_n}(\xi_n)\right],$$

and the result follows from (4).

Q.E.D.

We introduce some notation to simplify the formulas in Lemma A.1. For $2 \leq j \leq n$, let

$$n_{j,1} = \frac{-\langle \alpha_j, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle}, \quad \Sigma^1_{<j} = \left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k\leq j-1}} \xi_k\right)^{n_{j,1}}, \quad \Sigma^1_{\leq j} = \left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k\leq j}} \xi_k\right)^{n_{j,1}}.$$

Then $z_j = \Sigma^1_{<j}\Sigma^1_{\leq j}\xi_j$ for all $j \geq 2$.

First let $2 \leq i < j \leq n$. We now prove that $\{z_i, z_j\} = \langle \alpha_i, \alpha_j \rangle z_i z_j$. Using $\{\xi_i, \xi_j\} = \langle \alpha_i, \alpha_j \rangle \xi_i \xi_j$ and the Leinbiz rule, one has

$$\begin{aligned} \{z_i, z_j\} = {} & \Sigma^1_{<i}\Sigma^1_{\leq i}\Sigma^1_{<j}\Sigma^1_{\leq j}\{\xi_i, \xi_j\} + \Sigma^1_{<i}\Sigma^1_{\leq i}\xi_j\{\xi_i, \Sigma^1_{<j}\Sigma^1_{\leq j}\} \\ & + \xi_i\Sigma^1_{<j}\Sigma^1_{\leq j}\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \xi_j\} + \xi_i\xi_j\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \Sigma^1_{<j}\Sigma^1_{\leq j}\}. \end{aligned}$$

It is clear that $\Sigma^1_{<i}\Sigma^1_{\leq i}\Sigma^1_{<j}\Sigma^1_{\leq j}\{\xi_i, \xi_j\} = \langle \alpha_i, \alpha_j \rangle z_i z_j$. It thus remains to prove that

$$\Sigma^1_{<i}\Sigma^1_{\leq i}\xi_j\{\xi_i, \Sigma^1_{<j}\Sigma^1_{\leq j}\} + \xi_i\Sigma^1_{<j}\Sigma^1_{\leq j}\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \xi_j\} + \xi_i\xi_j\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \Sigma^1_{<j}\Sigma^1_{\leq j}\} = 0. \quad (19)$$

Since $\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \xi_j\} = \Sigma^1_{<i}\{\Sigma^1_{\leq i}, \xi_j\} + \Sigma^1_{\leq i}\{\Sigma^1_{<i}, \xi_j\} = 2n_{i,1}\langle \alpha_1, \alpha_j\rangle \Sigma^1_{<i}\Sigma^1_{\leq i}\xi_j$, one has

$$\xi_i \Sigma^1_{<j}\Sigma^1_{\leq j}\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \xi_j\} = 2n_{i,1}\langle \alpha_1, \alpha_j\rangle z_i z_j. \tag{20}$$

On the other hand,

$$\{\xi_i, \Sigma^1_{<j}\Sigma^1_{\leq j}\} = \Sigma^1_{<j}\{\xi_i, \Sigma^1_{\leq j}\} + \Sigma^1_{\leq j}\{\xi_i, \Sigma^1_{<j}\} = -2n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{<j}\Sigma^1_{\leq j}\xi_i + O,$$

where, if $\alpha_i \neq \alpha_1$,

$$\begin{aligned} O = {} & 2n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{<j}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k\leq j}} \xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ i\leq k\leq j}} \xi_k\xi_i\right) \\ & + 2n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{\leq j}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k<j}} \xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ i\leq k<j}} \xi_k\xi_i\right), \end{aligned}$$

and if $\alpha_i = \alpha_1$,

$$\begin{aligned} O = {} & 2n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{<j}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k\leq j}} \xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ i\leq k\leq j}} \xi_k\xi_i\right) \\ & + 2n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{\leq j}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k<j}} \xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ i\leq k<j}} \xi_k\xi_i\right) \\ & - n_{j,1}\langle \alpha_1, \alpha_i\rangle \left(\Sigma^1_{<j}\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k\leq j}} \xi_k\right)^{n_{j,1}-1}\xi_i^2 - n_{j,1}\langle \alpha_1, \alpha_i\rangle \Sigma^1_{\leq j}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k<j}} \xi_k\right)^{n_{j,1}-1}\xi_i^2. \end{aligned}$$

Therefore,

$$\xi_j \Sigma^1_{<i}\Sigma^1_{\leq i}\{\xi_i, \Sigma^1_{<j}\Sigma^1_{\leq j}\} = -2n_{j,1}\langle \alpha_1, \alpha_i\rangle (z_i z_j) + O\xi_j\Sigma^1_{<i}\Sigma^1_{\leq i}. \tag{21}$$

Observe that $n_{j,1}\langle \alpha_1, \alpha_i\rangle = n_{i,1}\langle \alpha_1, \alpha_j\rangle$, so we need to show that

$$\xi_i\xi_j\{\Sigma^1_{<i}\Sigma^1_{\leq i}, \Sigma^1_{<j}\Sigma^1_{\leq j}\} + O\xi_j\Sigma^1_{<i}\Sigma^1_{\leq i} = 0.$$

Now,

$$\begin{aligned} \{\Sigma^1_{<i}\Sigma^1_{\leq i}, \Sigma^1_{<j}\Sigma^1_{\leq j}\} = {} & \Sigma^1_{<i}\Sigma^1_{<j}\{\Sigma^1_{\leq i}, \Sigma^1_{\leq j}\} + \Sigma^1_{<i}\Sigma^1_{\leq j}\{\Sigma^1_{\leq i}, \Sigma^1_{<j}\} \\ & + \Sigma^1_{\leq i}\Sigma^1_{<j}\{\Sigma^1_{<i}, \Sigma^1_{\leq j}\} + \Sigma^1_{\leq i}\Sigma^1_{\leq j}\{\Sigma^1_{<i}, \Sigma^1_{<j}\}. \end{aligned}$$

For the first term, one has

$$\{\Sigma^1_{\leq i}, \Sigma^1_{\leq j}\} = \langle \alpha_1, \alpha_1\rangle n_{j,1}n_{i,1}\Sigma^1_{\leq i}\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k\leq j}} \xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_l=\alpha_1 \\ i<l\leq j}} \xi_l\right).$$

Hence

$$\Sigma^1_{<i}\Sigma^1_{<j}\{\Sigma^1_{\leq i},\Sigma^1_{\leq j}\} = \langle\alpha_1,\alpha_1\rangle n_{j,1}n_{i,1}\Sigma^1_{<i}\Sigma^1_{<j}\Sigma^1_{\leq i}\left(\sum_{\substack{\alpha_k=\alpha_1\\1<k\leq j}}\xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_l=\alpha_1\\i<l\leq j}}\xi_l\right). \tag{22}$$

Similarly, we have

$$\Sigma^1_{<i}\Sigma^1_{\leq j}\{\Sigma^1_{\leq i},\Sigma^1_{<j}\} = \langle\alpha_1,\alpha_1\rangle n_{j,1}n_{i,1}\Sigma^1_{<i}\Sigma^1_{\leq j}\Sigma^1_{\leq i}\left(\sum_{\substack{\alpha_k=\alpha_1\\1<k<j}}\xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_l=\alpha_1\\i<l<j}}\xi_l\right), \tag{23}$$

$$\Sigma^1_{\leq i}\Sigma^1_{<j}\{\Sigma^1_{<i},\Sigma^1_{\leq j}\} = \langle\alpha_1,\alpha_1\rangle n_{j,1}n_{i,1}\Sigma^1_{<i}\Sigma^1_{<j}\Sigma^1_{\leq i}\left(\sum_{\substack{\alpha_k=\alpha_1\\1<k\leq j}}\xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_l=\alpha_1\\i\leq l\leq j}}\xi_l\right), \tag{24}$$

$$\Sigma^1_{\leq i}\Sigma^1_{\leq j}\{\Sigma^1_{<i},\Sigma^1_{<j}\} = \langle\alpha_1,\alpha_1\rangle n_{j,1}n_{i,1}\Sigma^1_{<i}\Sigma^1_{\leq j}\Sigma^1_{\leq i}\left(\sum_{\substack{\alpha_k=\alpha_1\\1<k<j}}\xi_k\right)^{n_{j,1}-1}\left(\sum_{\substack{\alpha_l=\alpha_1\\i\leq l<j}}\xi_l\right). \tag{25}$$

Noting that $\langle\alpha_1,\alpha_1\rangle n_{j,1}n_{i,1} = -\langle\alpha_1,\alpha_i\rangle n_{j,1}$, the sum of all the terms from (20) to (25), is equal to 0. Hence $\{z_i,z_j\} = \langle\alpha_i,\alpha_j\rangle z_iz_j$.

We now compute $\{z_1,z_j\}$ for $2\leq j\leq n$. One has

$$\begin{aligned}
\{z_1,z_j\} &= \left\{\frac{1}{\xi_1},\Sigma^1_{<j}\Sigma^1_{\leq j}\xi_j\right\}\\
&= -\frac{1}{\xi_1^2}\left(\Sigma^1_{<j}\Sigma^1_{\leq j}\left\{\xi_1,\xi_j\right\} + \Sigma^1_{<j}\xi_j\{\xi_1,\Sigma^1_{\leq j}\} + \Sigma^1_{\leq j}\xi_j\left\{\xi_1,\Sigma^1_{<j}\right\}\right)\\
&= -\frac{1}{\xi_1^2}\Sigma^1_{<j}\Sigma^1_{\leq j}\langle\alpha_1,\alpha_j\rangle\xi_1\xi_j + \Sigma^1_{<j}\xi_j n_{j,1}\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k\leq j}}\xi_k\right)^{n_{j,1}-1}\left\{\xi_1,\sum_{\substack{\alpha_k=\alpha_1\\1<k\leq j}}\xi_k\right\}\\
&- \frac{1}{\xi_1^2}\Sigma^1_{\leq j}\xi_j n_{j,1}\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k<j}}\xi_k\right)^{n_{j,1}-1}\left\{\xi_1,\sum_{\substack{\alpha_k=\alpha_1\\1<k<j}}\xi_k\right\}\\
&= -\langle\alpha_1,\alpha_j\rangle z_1z_j - \frac{1}{\xi_1^2}\left(\Sigma^1_{<j}\xi_j n_{j,1}\Sigma^1_{\leq j}\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k\leq j}}\xi_k\right)^{-1}\sum_{\substack{\alpha_k=\alpha_1\\1<k\leq j}}\langle\alpha_1,\alpha_1\rangle\xi_1\xi_k\right)\\
&- \frac{1}{\xi_1^2}\left(\Sigma^1_{\leq j}\xi_j n_{j,1}\Sigma^1_{<j}\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k<j}}\xi_k\right)^{-1}\sum_{\substack{\alpha_k=\alpha_1\\1<k<j}}\langle\alpha_1,\alpha_1\rangle\xi_1\xi_k\right)\\
&= -\langle\alpha_1,\alpha_j\rangle z_1z_j + \langle\alpha_1,\alpha_j\rangle z_j\left(\frac{\sum_{\substack{\alpha_k=\alpha_1\\1<k\leq j}}\xi_k}{\xi_1\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k\leq j}}\xi_k\right)} + \frac{\sum_{\substack{\alpha_k=\alpha_1\\1<k<j}}\xi_k}{\xi_1\left(\sum_{\substack{\alpha_k=\alpha_1\\1\leq k<j}}\xi_k\right)}\right).
\end{aligned}$$

Recall that for $2 \leq m \leq n$ and $\alpha_m = \alpha_1$,

$$z_m = \frac{\xi_m}{\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k<m}} \xi_k\right)\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k\leq m}} \xi_k\right)} = \frac{1}{\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k<m}} \xi_k} - \frac{1}{\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k\leq m}} \xi_k}.$$

Therefore

$$\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k<j}} z_k = \frac{1}{\xi_1} - \frac{1}{\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k\leq j}} \xi_k\right)} = \frac{\sum_{\substack{\alpha_k=\alpha_1 \\ 1<k<j}} \xi_k}{\xi_1\left(\sum_{\substack{\alpha_k=\alpha_1 \\ 1\leq k<j}} \xi_k\right)}.$$

Substituting this to the last line of the equation for $\{z_1, z_j\}$, one has

$$\{z_1, z_j\} = \begin{cases} -\langle\alpha_1, \alpha_j\rangle \left(z_1 - 2\sum_{2\leq i\leq j-1,\, \alpha_i=\alpha_1} z_i\right) z_j, & \text{if } 2 \leq j \leq n \text{ and } \alpha_j \neq \alpha_1, \\ -\langle\alpha_1, \alpha_1\rangle \left(z_1 - 2\sum_{2\leq i\leq j-1,\, \alpha_i=\alpha_1} z_i - z_j\right) z_j, & \text{if } 2 \leq j \leq n \text{ and } \alpha_j = \alpha_1, \end{cases}$$

as desired.

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