MATH 1340 Mathematics and Politics
Spring 2017

Practice Exam #2

INSTRUCTIONS

- You have 150 minutes. If you finish within the last 15 minutes of class, please remain seated so as not to disturb your classmates.
- The exam is closed book, closed notes, no calculators/computers/etc., but you may bring a one-page "cheat sheet" (you can use both sides). *If you use such a sheet, write your name on it and turn it in with your exam.* You are free to apply any result that we covered in class, on the homeworks, or in Chapters 1–6 or 10 of the textbook, unless the problem explicitly tells you to use a certain approach.
- Mark your answers ON THE EXAM ITSELF. If you are not sure of your answer, you may wish to provide a *brief* explanation so that we can at least know what you are trying to do. All short answer sections can be successfully answered in a few sentences at most. For full credit, be sure to show your work and justify your steps. Little credit will be given for correct answers without justification.
- Questions are not given in order of difficulty. Make sure to look ahead if stuck on a particular question.

Last Name	
First Name	
Cornell NetID (e.g. bwh59)	
All the work on this exam is my own. **(please sign)**	

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Q. 1	Q. 2	Q. 3	Q. 4	Q. 5	Q.6	Q.7	Bonus	Total
/20	/20	/20	/ 20	/ 20	/ 20	/ 20	/5	/140

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1. (20 points) True/False.

If true, justify your answer. If false, give a counterexample.

(a) Only divisor methods of apportionment avoid the population paradox.

(b) Suppose that there are four voters in a yes-no voting system where a coalition is winning if and only if it has an even number of members (and is nonempty). This is a weighted voting system.

(c) A 2×2 ordinal game can have at most one Nash equilibria.

(d) Every weighted yes-no system is swap-robust.

Solution. (a) True. This is the Balinski–Young theorem.

(b) False. Winning coalitions of size 2 do not stay winning when you add a member, and all weighted voting systems are monotonic.

(c) False. The game of Chicken has two Nash equilibria. [A full-credit answer will include the game and an argument that there are two Nash equilibria there.]

(d) True. Let's show that weighted voting systems are trade-robust and thus are also swap-robust. If we have a series of trades among several winning coalitions, the total weight of all the coalitions added together is unchanged, and thus the average weight of these coalitions is also unchanged. Since we have a weighted voted system, the average weight of these coalitions must at least meet the quota. Thus, at least one of the coalitions must itself meet the quota, and so there must be at least one winning coalition. Since this argument works for any series of trades between arbitrary sets of winning coalitions, the system is trade-robust. □

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2. (20 points) Voting Systems and Manipulability. Recall that the social choice procedure "Sequential Pairwise Voting with a Fixed Agenda" pits alternatives against each other in a fixed order (the "agenda"). The surviving alternative of each one-on-one race is then pitted against the next alternative in the agenda, and so on. We will explore the importance of setting the agenda in this question.

(a) Suppose that there are four alternatives $\{a, b, c, d\}$ and three voters whose preference lists are given by

Voter 1	Voter 2	Voter 3
b	c	d
c	d	a
d	a	b
a	b	c

You want a to win. Can you design an agenda so that a wins? If so, describe the agenda and verify your answer. If not, explain why it is not possible.

(b) Now, consider the same question where the last two preferences of Voter 3 are switched, that is, we have the preference lists

Voter 1	Voter 2	Voter 3
b	c	d
c	d	a
d	a	c
a	b	b

Answer the same question as above: can you design an agenda so that a wins? If so, describe the agenda and verify your answer. If not, explain why it is not possible.

Solution. (a) According to the votes, we have $a > b, c > a, d > a, b > c, d > b$ and $c > d$. Thus, we can use the following agenda: (d, c, b, a).

Let's check that thus results in a winning. In d vs. c, c wins. In c vs b, b wins. In the last round, where b vs. a, we see that a wins.

(b) According to the votes, we have $a > b$, $c > a$, $d > a$, $c > b$, $d > b$, $c > d$.

It is impossible for a to win, there is no such agenda. Why? Because a can only defeat b, so the agenda has to end in b vs. a in order for a to win. However, b cannot survive to this stage, because it is defeated by both c and d, and so will not be able to advance to this round. □

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3. (20 points) A yes–no voting system.

Consider a mini version of the US Federal System with 9 voters: 4 members of the House, 4 Senate members, and the President. Passage in this system requires half of the House, half of the Senate, and the president; or 3 members of the House and 3 members of the Senate.

(a) Is this system swap-robust? Why or why not?

(b) How many winning coalitions can be formed such that each has exactly 3 members of the House and 3 Senators?

(c) How many winning coalitions of 5 members can be formed?

Proof. (a) False. Suppose that a, b, c, d are the four members of the House and e, f, g, h are the four Senators. Assume that P is the President. Consider the winning coalitions

$$X = \{P, a, b, e, f\}, \quad Y = \{P, b, c, g, h\}.$$

Note that a does not belong to Y and g does not belong to X. Swap these two members. We then obtain two coalitions

$$X' = \{P, g, b, e, e, f\}, \quad Y' = \{P, b, c, a, h\},$$

which are both losing. Thus, this system is not swap-robust.

(b) We can choose 3 senators in $\binom{4}{3} = 4$ ways and 3 members of theHouse in $\binom{4}{3} = 4$ ways. Thus, there are $4 \cdot 4 = 16$ winning coalitions such that each has exactly 3 members of the House and 3 members of the Senate.

(c) Note that if a coalition has 5 members and it is winning, then it must contain the President, 2 members of the House, and 2 senators. We can choose 2 senators in $\binom{4}{2} = 6$ ways and 2 members of the House in $\binom{4}{2} = 6$ ways. Thus, there are $6 \cdot 5$ winning coalitions of 5 members. $\square$

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4. (20 points) Dollar auction.

Consider the standard dollar auction with the conservative convention, where the stakes are 8 and the bankroll is at least 30. Assume that you are the player that goes first. Suppose that your last bid was 10 and that your next bid of 17 would cause the other player to pass. Explain why it would be better for you to bid 17 than for you to pass or make any higher bid.

Proof. We have $s = 8$ and $b \geq 30$. There are 3 cases to consider.

Case 1: You pass. Then you will be the second-highest bidder and will lose 10.

Case 2: You bid 17 and player 2 passes. In this case, you are the highest bidder and you will gain $-17+9 = -9$, or more accurately, you will lose 9.

Case 3: You bid 18 or higher. If you are the highest bidder, and your gain will be at most $-18 + 8 = -10$. If your are the second highest bidder, then you will gain at most -18.

Thus, it would be better for you to bid 17 than to pass or make any higher bid. $\square$

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5. (20 points) A Fair Division Procedure and its Properties.

Consider the following fair division procedure for 4 players (Alice, Bob, Cassie, David). For simplicity, assume that we are dividing a cake.

Step 1. Alice cuts a piece of the cake that she values at 1/4.

Step 2. Here we have two possibilities:

(a) If Bob thinks this piece is $> 1/4$ (according to his valuation), then he trims it so that it's 1/4 in his valuation, and adds the trimmings to the remaining big piece of the cake, then passes the trimmed slice to Cassie.

(b) If Bob thinks this piece is $\leq 1/4$ of the cake, then he passes the slice to Cassie without any modification.

Step 3. Cassie repeats what Bob did in step 3, except she passes it to David.

Step 4. David repeats what Bob and Cassie did, and then passes it to the last person who trimmed it (either Bob or Cassie). Then the person who receives the cake exits.

We are now left with 3 people who all think that there is at least 3/4 of the cake left, according to their own valuations. Repeat the process above, where a random person (say Carol) gets to cut a piece she thinks is 1/4 the size of the original cake, and the other two get to trim or not trim.

When we are left with two people, those two people do a divide-and-choose.

(a) Is this fair division procedure proportional? Prove it or provide a case in which it is not.

(b) Is this procedure envy-free? Prove it or provide a case in which it is not.

(c) Is this procedure efficient? Prove it or provide a case in which it is not.

Proof. (a) This fair division procedure is proportional. We need to show that everybody gets what they feels is more than 1/4 of the cake. In the first cycle, the person who exits is either Alice, who receives an untrimmed cake which she valued at 1/4, or the last person who trimmed it, who trimmed it so that it was 1/4 in their own valuation. In the second cycle, the person who exists is either the divider (Carol) who slices a piece of cake that she considers to be 1/4, or the last person who trimmed it, who trimmed it so that it was 1/4 in their own valuation. In the last cycle, we apply the divide-and-choose method to two people who both believe there is more than 1/2 of the cake left, and the divide and choose method is proportional, thus the last two get at least 1/2 of what remains, that is, 1/4 of the total cake with respect to their valuation.

(b) No. Consider the case in which nobody trims and so Alice exists, but then feels like the later slices are larger than hers. For example, where all the parties exclusively like a different 1/4 of the cake. In this case, Alice with get only 1/4 of the part of the cake she wants, and unless the others happen to always include 1/4 of the part she likes in all of their cuts, somebody will get more of the cake that Alice likes.

(c) No. Consider the same situation in which all parties exclusively like a different 1/4 of the cake. To be efficient, everybody would get precisely their favorite 1/4 of the cake, so everybody feels like they got 100% of the cake according to their own valuation. However, with this procedure, Alice would only get 1/4 of the slice of cake that she likes, even though nobody else gives any value to the other 3/4 of the slice that Alice likes. □

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6. (20 points) What happens when you give the powerless power?

Consider the weighted voting system $[6 : 4, 1, 3, 3]$ where the voters are named A, B, C, and D.

(a) Show that voter B has a Banzhaf index of zero.

(b) Now suppose that we modify this voting system by adding a clause that gives B veto power. Show that this is a weighted voting system.

(c) In the new voting system, calculate the Banzhaf index of B.

Solution. (a) The winning coalitions are $\{A, C\}$, $\{A, D\}$, $\{C, D\}$, $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, $\{B, C, D\}$, and $\{A, B, C, D\}$. Note that voter B is not critical in any coalition and so $TBP(B) = 0$ and so B has Banzhaf index 0.

(b) If B has veto power, then the possible winning coalitions are $\{A, B, C\}$, $\{A, B, D\}$, $\{B, C, D\}$, and $\{A, B, C, D\}$. Let $r = 4 + 1 + 3 + 3 = 11$ and $q = 6 + r = 17$. Then $[17 : 4, 12, 3, 3]$ is a new yes-no voting system in which B has veto power.

In this system, the coalition $\{B, C, D\}$ has a total weight of $12 + 3 + 3 = 18 > 17$ and so it is winning. While the coalition $\{A, C, D\}$ has a total weight of $4 + 3 + 3 = 10 < 17$ and so is losing. A similar analysis can be done for the remaining coalitions to show that these weights work.

(c) This will likely depend on the weights used, but assume that it is $[17 : 4, 12, 3, 3]$. In this case, once you do all the calculations, the TBPs will be (4,8,4,4) respectively. Thus, the Banzhaf index of B will be $8/20 = 2/5$.

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7. (20 points) Analyzing a game

Consider the following two-person game (a 2×3 non-zero-sum game):

	A	B	C
X	10, 10	5, 0	-10, 15
Y	-10, 0	15, -10	-5, -5

So the pure strategies available to Row are X and Y, and the pure strategies available to Column are $A, B,$, and C.

Suppose that Row is using the mixed strategy $P = (1/2, 1/2)$ and Column is using the mixed strategy $Q = (1/3, 1/6, 1/2)$.

(a) Draw a 2-by-3 matrix (2 rows, 3 columns) indicating the probabilities of each of the six possible outcomes. (e.g. what is the probability that outcome (A, X) will occur, if the strategies P and Q are adopted by Row and Column, respectively? This would go in the top left entry of this matrix.)

(b) What is the expected payoff for Row?

(c) What is the expected payoff for Column?

(d) Is P Row's best response to Column's Q? If not, find the best response.

Solutions. (a)

1/6	1/12	1/4
1/6	1/12	1/4

(b) The expected payoff for Row is

$$\begin{aligned} E(R, P, Q) &= 10(1/6) + 5(1/12) + (-10)(1/4) + (-10)(1/6) + (15)(1/12) + (-5)(1/4) \\ &= 20/12 + 5/12 - 30/12 - 20/12 + 15/12 - 15/12 \\ &= -25/12. \end{aligned}$$

(c) The expected payoff for Column is

$$\begin{aligned} E(C, P, Q) &= 10(1/6) + 0(1/12) + 15(1/4) + 0(1/6) + (-10)(1/12) + (-5)(1/4) \\ &= 20/12 + 45/12 - 10/12 - 15/12 \\ &= 40/12 = 10/3. \end{aligned}$$

(d) No. We can see this by looking Row's pure strategy expected payoffs against Q. Let P_X and P_Y denote the respective pure strategies:

$$\begin{aligned} E(R, P_X, Q) &= 10(1/3) + 5(1/6) + (-10)(1/2) = 20/6 + 5/6 - 30/6 = -5/6 \\ E(R, P_Y, Q) &= (-10)(1/3) + 15(1/6) + (-5)(1/2) = -20/6 + 15/6 - 15/6 = -20/6. \end{aligned}$$

Thus, P is not the best response, since the expected value from adopting P_X is better ($-5/6 >= -25/12$).

It also turns out that P_X is Row's best response to Q. This is because if S is any mixed strategy (say, which chooses X with probability s, and so chooses Y with probability $1 - s$), then its expected payoff is

$$E(S) = sE(R, P_X, Q) + (1 - s)E(R, P_X, Q) = s \cdot (-5/6) + (1 - s)(-20/6),$$

which is largest when $s = 1$. □

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