SURREAL NUMBERS

The surreal numbers are a vast extension of the notion of a real number and were discovered around 1970. Almost all definitions in the topic are constructed inductively, but like the real numbers, the surreals are "continuous" in a certain sense. In a way, it gives a method to "construct the continuous discretely."

We'll use it as a domain to practice arguments about induction and cardinality, which are probably the trickiest topics of our class.

Definition 1. A *surreal number* x is denoted by a pair

$$x = \{L|R\} = \{a, b, c, \dots |d, e, f, \dots\}$$

where $L = \{a, b, c, \dots\}$ and $R = \{d, e, f, \dots\}$ are subsets of (previously constructed) surreal numbers with the property:

(O): all the members of L are strictly less than ("$<$") all the members of R.

We'll define the ordering $\leq$ shortly, but it agrees with the usual one for real numbers.

We write x^L for an arbitrary member of L, and x^R for an arbitrary member of R, so a common shorthand is to write $x = \{x^L|x^R\}$. So property (O) can be summarized by "$x^L < x^R$" or "$\neg(x^L \geq x^R)$."

Roughly speaking, $x = \{L|R\}$ represents the "simplest" number in between the elements of L and R.

The surreal numbers, which are usually denoted by **No**, are constructed in stages called "days." The day at which a surreal number is first constructed is called its "birthday." In the beginning (day zero), we have our origin

$$0 := \{\emptyset|\emptyset\} = \{|\}.$$

We can now put $0 = \{|\}$ into L and R, so on day one (the first stage), we can define

$$\{0|\} = 1, \quad \{|0\} = -1.$$

Now we have three potential numbers we can put in our subsets L and R. So on day two, we have

$$\{1|\} = 2, \{| - 1\} = -2, \{0|1\} = \frac{1}{2}, \{-1|0\} = -\frac{1}{2}, \{-1|1\} = 0.$$

but we can also construct

$$\{-1, 0|1\} = \frac{1}{2}, \{-1|0, 1\} = -\frac{1}{2}.$$

On day three, we can incorporate any of the numbers $\{2, -2, 1/2, -1/2, 0, 1, -1\}$ for our subsets x_L and x_R, and so on. Also, note that just like real numbers (or rational numbers), there is more than one way to express a given number.

Question 2. *Let S_n denote the numbers constructed by (the end of) Day n. (e.g. $S_0 = \{0\}$, $S_1 = \{0, \pm 1\}$, etc.) We have $S_2 = \{\pm 2, \pm 1, \pm 0, \pm \frac{1}{2}\}$. Using our tentative definition of $x = \{L|R\}$ as the "simplest number" between L and R, what are all the different ways to express each number in S_2 with elements in S_1 allowed for L and R? (Including $\{|\}, \{0|\}, and\{|0\}$, there are 20 different possible pairs, which only correspond to the $|S_2| = 7$ different numbers.) For example, $-1 = \{|0\} = \{|0, 1\}$.*

Like the real numbers, the surreals admit an ordering. This is also defined inductively.

Definition 3. We say $x \geq y$ if

(1) there is no x^R such that $x^R \leq y$ (every element in the right part of the bigger number is bigger than the smaller number); and
(2) there is no y^L such that $x \leq y^L$ (every element in the left part of the smaller number is smaller than the bigger number).

Thus, to check if $\geq$ holds, we just need to check the "extremes": the "smallest part" (L) of the smaller number and the "biggest part" (R) of the larger number. For example, $\frac{1}{2} = \{0|1\} \leq 1 = \{0|\}$ because $0 \leq 1$ and since $1^R \in \emptyset$, condition (1) holds vacuously. Now that we have $\leq$, we can define equality ("$x = y$") as $x \leq y$ and $x \geq y$. Similarly, strict inequality ("$x < y$") is given by $x \leq y$ and $x \neq y$.

Question 4. *To make sure you understand $\geq$, check for yourself that numbers in S_2 satisfy the inequalities you expect from the corresponding real numbers. (Don't forget $a \geq a$!)*

Proposition 5. *Show that $x \geq x$ for all $x \in \mathbf{No}$.*

Addition in **No** is also defined inductively:

$$x + y = \{x^L + y, x + y^L | x^R + y, x + y^R\}$$

Question 6. *Given $x = \{L|R\} = \{x^L|x^R\}$, how can you define $-x$? Show that your definition implies that $x + (-x) = 0$.*

Question 7. *Show that $\frac{1}{2} + \frac{1}{2} = 1$.*

Multiplication on **No** is also defined inductively:

$$xy = \{x^L y + xy^L - x^L y^L, x^R y + xy^R - x^R y^R | x^L y + xy^R - x^L y^R, x^R y + xy^L - x^R y^L\}.$$

Question 8. *Why does defining the multiplication by*

$$xy = \{x^L y, xy^L | x^R y, xy^R\}$$

not work?

Remark 1. You can also define division by any nonzero number. If you're up for a challenge, you can try and find the division operation on **No**, but it is quite tricky and we won't use it.

Let's practice using the above definitions.

Question 9. *Show that $\{1|2\} = \frac{3}{2}$. What is $\{0|\frac{1}{2}\}$? What is $\{\frac{1}{2}|1\}$?*

The heuristic that $x = \{L|R\}$ is the "simplest number" between L and R requires care. For example, it is not generally the mean, as the following proposition will show.

Question 10. *Show that $x = \{-1|2\} = 0$ (i.e. $x \geq 0$ and $x \leq 0$). Generalize the argument to show that if every $x^L < 0$ and every $x^R > 0$, then $x = 0$.*

Question 11. *Show that a surreal number $x = \{L|R\} = y$, where y is the surreal number with the earliest birthday such that $x^L < y$ and $y < x^R$.*

How can we determine the value of a number created on day n?

Question 12. *For a given $n \geq 0$, can you characterize the numbers that lie in S_n (e.g. as a subset of the real numbers)? What can you say about the size of $|S_n|$ (e.g. upper or lower bounds)?*

Consider the union $S_* = \bigcup_{n \in \mathbb{N}} S_n$ is it countable or uncountable? Can you characterize the numbers that lie in S_*?

So far we haven't gotten to any non-real numbers. But with what we've done so far, the fireworks can begin.

Our process of constructing the surreals doesn't stop on Day n for $n \in \mathbb{N}$. After we have constructed S_n for $n \in \mathbb{N}$, the "next" day, we call Day ω ("omega"[1]). The largest number constructed on this day is

$$\omega := \{0, 1, 2, 3, \dots |\},$$

and we can write it in many ways, e.g. $\omega = \{1, 2, 4, 8, 16, \cdots |\}$. This is the first surreal number we've constructed that is not a real number. It is larger than any natural number and is the simplest (but not the smallest!) number with that property.

Question 13. *What is the most negative number born on Day ω? What is the smallest positive number born on Day ω?*

Aside from these extremes, lots of familiar numbers are constructed on Day ω.

Question 14. *Show that $x = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \cdots | \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \dots\} = \frac{1}{3}$, that is, $x + x + x = 1$. What other fractions can you defined like this? (*Hint: *Recall geometric series from calculus.)*

Question 15. *Write S_ω for all the numbers constructed on Day ω or earlier (on Day n for any $n \in \mathbb{N}$). Is $|S_\omega|$ countable or uncountably infinite? (*Hint*: What L and R are possible?)*

Question 16. *Show that there exists a sum of dyadic rationals (i.e. rationals with denominator 2^k for some integer k) that converges to any rational number r. Show that such a sum also converges to any real number r. What does this imply about S_ω or those of later days?*

After Day ω, since we have constructed all rational numbers by this point, we get things like the following expression for π:

$$\pi = \{3, 3.1, 3.14, 3.141 \dots |4, 3.2, 3.15, 3.142, \dots\}$$

where L is a set of (monotonically) increasing rational numbers converging to π and R is set of (monotonically) decreasing rational numbers converging to π. Of course, such an expression is far from unique.

Remark 2. After this stage, it's often difficult to determine on what day a surreal number is created, so we'll ignore that question going forward.

As a surreal number, we can manipulate ω like any other number. For example, for any natural number n, we have

$$\omega - n = \{0, 1, 2, 3, \dots |\omega, \omega - 1, \omega - 2, \dots, \omega - (n-1)\}.$$

Question 17. *Show that $x = \{0, 1, 2, 3, \dots |\omega, \omega - 1, \omega - 2, \dots\}$ is equal to $\omega/2$. How would you define $\omega/4$ or $\omega/8$ and beyond? How would you define, say, $\omega/3$? What is*

$$y = \{0, 1, 2, 3, \dots |\omega, \omega/2, \omega/4, \omega/8, \dots\}$$

Question 18. *Show that $\{0|\frac{1}{\omega}\} = \frac{1}{2\omega}$. What is $\{\frac{1}{\omega}|1, \frac{1}{2}, \frac{1}{4}, \dots\}$? What is $\{0|\frac{1}{\omega}, \frac{1}{2\omega}, \frac{1}{4\omega}, \dots\}$?*

We can also try and go backwards and see what surreal numbers correspond to familiar arithmetic operations.

Question 19. *What is $\sqrt[3]{\omega}$? What is $\omega^{1/\omega}$? What is $\omega + \pi$? What is $\frac{1}{\omega+1}$?*

[1]The ω comes from the o for "ordinal," which is a similar to the notion of "cardinal" as in cardinality. An ordinal is a set that can be "ordered," i.e. a set that has a first, second, third; in contrast to a cardinal, which is something that can be "counted." *Finite* ordinal and cardinals coincide, but what is nonintuitive (and fascinating) is that *infinite* cardinals and ordinals are different! Ordinals are "finer," in some sense. For example, there are many different ordinals that have the same infinite cardinality.

Formally, an ordinal is an equivalence class of sets with the well-ordering property, where we say $S \sim T$ for *ordered* sets S and T if there is a bijection $f : S \to T$ that preserves the ordering, i.e. $s_1 \leq s_2$ if and only if $f(s_1) \leq f(s_2)$.