MATH 6670, COMPREHENSIVE

DUE FRIDAY, MAY 17

Open book, open notes, but not "open internet" or "open friends." In particular, you *must work on these problems alone* and not discuss them with other students until all exams are turned in. You may quote results from the lectures without proof; results from the books require proper citation (e.g. [H Prop. II.3.7(c)]).

1. (Automorphisms of $\mathbf{P}^1(k)$.) Assume that k is algebraically closed.

(a) (5 points) Prove that the non-constant morphisms $\mathbf{A}^1(k) \to \mathbf{A}^1(k)$ are precisely the maps $x \mapsto f(x)$, where $f(x)$ is a nonconstant polynomial. Deduce that the only automorphisms of $\mathbf{A}^1(k)$ are of the form $x \mapsto ax + b$ where $a \in k^\times$ and $b \in k$.

(b) (5 points) For any $[a : b] \in \mathbf{P}^1(k)$, show that there is a linear change of coordinates that gives a morphism $g : \mathbf{P}^1(k) \to \mathbf{P}^1(k)$ such that $g([a : b]) = [1 : 0]$.

(c) (5 points) Show that if $\phi : \mathbf{P}^1(k) \to \mathbf{P}^1(k)$ is an automorphism with $\phi([1 : 0]) = [1 : 0]$, then the restriction of ϕ to $\mathbf{P}^1(k) - \{[1 : 0]\} = \mathbf{A}^1(k)$ gives an automorphism of $\mathbf{A}^1(k)$. Use part (a) to deduce that ϕ comes from a linear change of coordinates:

$$\phi([x : y]) = [\phi_{11}x + \phi_{12}y : \phi_{21}x + \phi_{22}y]$$

and show moreover that $\phi_{21} = 0$.

(d) (5 points) Recall that $PGL_2(k) = GL_2(k)/\sim$, where we say two invertible 2×2 matrices satisfy $A \sim B$ if $A = \lambda B$ for some $\lambda \in k^\times$. Show that any $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PGL_2(k)$ gives an automorphism ϕ of $\mathbf{P}^1(k)$ where

$$\phi([x : y]) = [ax + by : cx + dy].$$

Use parts (a)–(c) to show that any automorphism of $\mathbf{P}^1(k)$ is of this form and thus that the group of automorphisms of $\mathbf{P}^1(k)$ is $PGL_2(k)$.

2. (Scheme and set-theoretic intersections) Let k be a field and consider the subset

$$\widetilde{Z} = \{(s^3, s^2t, st^2, t^3) \mid s, t \in k\} \subset \mathbf{A}^4.$$

(a) (4 points) Show that $\widetilde{Z}$ is the set-theoretic intersection of a quadratic and a cubic hypersurface.

(b) (4 points) Show that the minimal number of generators of $I(\widetilde{Z})$, the ideal of polynomial vanishing on $\widetilde{Z}$, is 3. (Hint: Find 3 linearly independent quadratic polynomials in the standard coordinates of $\mathbf{A}^4$ that vanishing on $\widetilde{Z}$.) (This shows that $\widetilde{Z}$ is not a complete intersection as a scheme.)

(c) (4 points) Is the projection $f(x, y, z, w) = (x, w)$ restricted to $\widetilde{Z}$ a finite map from $\widetilde{Z}$ to $\mathbf{A}^2$?

(d) (4 points) In projective coordinates, prove that

$$g[s : t] = [s^3 : s^2t : st^2 : t^3]$$

is an isomorphism of $\mathbf{P}^1$ onto a closed subvariety $Z \subset \mathbf{P}^3$.

(e) (4 points) What is the relation between Z and $\widetilde{Z}$, and how should we interpret the fact that $\widetilde{Z}$ is not a complete intersection as a scheme, but that it is the intersection of two hypersurfaces?

3. (The simplest projective surfaces.)

(a) (5 points) Show that $\mathbf{P}^1 \times \mathbf{P}^1$ is a projective variety. In other words, give a closed subvariety $Z \subset \mathbf{P}^N$ for some N together with an isomorphism $\psi : \mathbf{P}^1 \times \mathbf{P}^1 \to Z$.

(b) (5 points) Show that $\mathbf{P}^1 \times \mathbf{P}^1$ is irreducible.

(c) (5 points) Show that $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ have isomorphic function fields.

(d) (5 points) Show that $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{P}^2$ are not isomorphic.

4. (Frobenius and Divisors)

(a) (5 points) Let $k = \overline{\mathbf{F}_q}$, an algebraic closure of the finite field of q elements. Let $X = \mathbf{P}^1$ and let $F_X : \mathbf{P}^1 \to \mathbf{P}^1$ be the q-Frobenius endomorphism of X (i.e. $x \mapsto x^q$). Show that F_X is bijective, but not an isomorphism.

(b) (5 points) State the definition of a prime divisor on a smooth irreducible projective curve over $\mathbf{F}_q$ and give as elementary a description of this as possible.

(c) (10 points) Describe all the prime divisors on $\mathbf{P}^1_{\mathbf{F}_q}$ and briefly justify your answer. (A detailed proof is not required for this part.)

5. Let $X \subset \mathbf{A}^3$ be the surface defined by $z^2 = xy$ and $X' = X - \{0\}$, where $0 = (0,0,0) \in \mathbf{A}^3$. Consider

$$Y = \{(t, 0, 0) \mid t \in k\} \subset X$$

and $Y' = Y - \{0\}$. Let $\mathcal{I}_Y \subset \mathscr{O}_X$ be the subsheaf of functions vanishing on Y, that is, for an open set U,

$$\mathcal{I}_Y(U) = \{f \in \mathscr{O}_X(U) : f|_{Y \cap U} = 0\}.$$

(a) (10 points) Show that $\mathcal{I}_Y|_{X'}$ is locally free of rank 1 (i.e. invertible) but $\mathcal{I}_Y$ is not locally free at 0. Conclude that $\mathscr{O}_{X,0}$ is not a unique factorization domain.

(b) (10 points) Show that $\mathcal{I}_Y|_{X'}$ is not trivial, but that its square is trivial (i.e. isomorphic to $\mathscr{O}_{X'}$).

6. (Sheaves of modules on ringed spaces) Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of $\mathscr{O}_X$-modules. Recall that the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheafification of the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$.

(a) (6 points) Prove that for any $x \in X$, we have $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x$. Conclude that the tensor product is a right exact functor in each of the two arguments.

(b) (7 points) Assume that $\mathscr{F}$ is locally free of finite rank, i.e. for all $x \in X$, there exists a neighborhood U of x such that $\mathscr{F}|_U \simeq (\mathscr{O}_X(U))^n$. Show that there is a natural isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) = \mathscr{G} \otimes \mathscr{F}^\vee,$$

where $\mathscr{H}om$ is the sheaf of homomorphisms and $\mathscr{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$ is the dual of the locally free sheaf $\mathscr{F}$.

(c) (7 points) Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Suppose that $\mathscr{F}$ is an $\mathscr{O}_X$-module and $\mathscr{G}$ is a locally free $\mathscr{O}_Y$-module of finite rank. Construct a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}) \simeq f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G}$.

7. (Quotients by finite groups) Let X be an affine variety and let G be a finite group. Assume that G acts on X algebraically, that is, for every $g \in G$, we are given a morphism $g : X \to X$ (which we denote by the same letter for simplicity of notation) such that

$$(gh)(p) = g(h(p))$$

for all $g, h \in G$ and $p \in X$.

(a) (7 points) Let $g \in G$ acts on the coordinate rings $A(X)$ via $f \mapsto f^g$, where
$$f^g(p) := f(g(p)).$$
Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all G-invariant functions on X. Show that $A(X)^G$ is a finitely generated k-algebra.

(b) (7 points) By (a), there is an affine variety Y with coordinate ring $A(X)^G$, together with a morphism $\pi : X \to Y$ corresponding to the inclusion $A(X)^G \hookrightarrow A(X)$. Show that Y can be considered as the quotient of X by G (written X/G) in the sense that if $p, q \in X$, then $\pi(p) = \pi(q)$ if and only if there is a $g \in G$ such that $g(p) = q$.

(c) (6 points) Let $\mu_n = \{\exp(2\pi i k/n) : k \in \mathbf{Z}\}$ be the group of nth roots of unity. Let μ_n act on $\mathbf{C}^m$ by multiplication in each coordinate. Describe $\mathbf{C}/\mu_n$ and $\mathbf{C}^2/\mu_n$ as affine algebraic sets.