MATH 6670, HOMEWORK #1

DUE THURSDAY, JANUARY 31

In this course, all rings are commutative rings with unity.

1. (From first lecture)

(a) Show that if $f : B \to A$ is a ring homomorphism, then for every prime ideal $\mathfrak{p}$ of A, the set $f^{-1}(\mathfrak{p})$ is a prime ideal of B.

(b) Give an example where this is not true if you replace "prime" by "maximal."

2. Let X be a topological space. Consider the category $\mathbf{Open}(X)$, whose objects are open sets of X, and whose morphisms are

$$\mathrm{Mor}(U_1, U_2) = \begin{cases} \{*\}, & U_1 \subseteq U_2 \\ \emptyset, & \text{otherwise.} \end{cases}$$

Show that a presheaf of sets (or abelian groups, vector spaces, rings) is the same as a contravariant functor from $\mathbf{Open}(X)$ to $\mathbf{Sets}$ (or $\mathbf{Ab}$, $\mathbf{Vect}_k$, $\mathbf{Rings}$).

3. (Category theory review) Assume all categories are abelian and functors between them are additive.

(a) Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories such that F is left adjoint to G (equivalently, G is right adjoint to F). Show that F is right exact and G is left exact.

(b) Let $i : \mathcal{C}_0 \to \mathcal{C}$ be the embedding of a full subcategory. Suppose that i admits a left (resp. right) adjoint s. Let L (resp. R) denote the composition $i \circ s$. Show that $X \in \mathcal{C}_0$ if and only if the canonical map $X \mapsto L(X)$ (resp. $R(X) \mapsto X$) is an isomorphism.

4. [EH I-5]

(a) Let X be the two-element set $\{0, 1\}$, and make X into a topological space by taking each of the four subsets to be open. A sheaf on X is thus a collection of four sets with certain maps between them; describe the relations among these objects.

(b) Do the same in the case where the topology of $X = \{0, 1\}$ has as open sets only $\emptyset$, $\{0\}$, and $\{0, 1\}$.

(c) (Optional) The spaces in (a) and (b) can be realized as $\operatorname{Spec} R$ from some ring R; can you find one for each?

5. (The constant sheaf) Let A be a nontrivial Abelian group. Let X be a topological space. Define the constant presheaf of groups $A_X(U) = A$ for every nonempty open set U, where the restriction maps $\rho_V^U(U) = \mathrm{id}_A$ for all nonempty $V \subseteq U$.

(a) Show that A_X is not generally a sheaf. (Given an explicit example of X where the sheaf condition fails.)

(b) Show that the sheaf associated to A_X (i.e. its sheafification) is such that if $U \subseteq X$ is a nonempty open whose connected components are open (which is always true on a locally connected topological space), then $A_X(U)$ is a direct product of copies of A, one for each connected component of U.