MATH 6670, HOMEWORK #10

DUE THURSDAY, APRIL 25

1. (A classical consequence) (If it makes you more comfortable, you can assume that you are working with varieties over $\mathbf{C}$; but these arguments hold in general.)

(a) Show that the set of curves of degree d in $\mathbf{P}^2$ is naturally parameterized by $\mathbf{P}^N$ where $N = \binom{d+2}{2} - 1$. Let $H \subseteq \mathbf{P}^N \times \mathbf{P}^2$ be the "tautological family" whose fiber over a point of $\mathbf{P}^N$ is the curve parameterized by that point. Note that if f is a polynomial that is separately homogeneous in the projective coordinates on $\mathbf{P}^N$ and $\mathbf{P}^2$, then the zero locus $V(f) \subseteq \mathbf{P}^N \times \mathbf{P}^2$ makes sense and forms a closed subvariety. Find such an f such that $H = V(f)$.

(b) For $d = 2$, let Y be the locus in $\mathbf{P}^5$ which parameterizes quadratic curves (a.k.a. conics) that degenerate to two lines or a double line. Show that Y is a closed subvariety of $\mathbf{P}^5$ and find the equations.

(c) Repeat (b) for degree $d = 3$, where $Y \subseteq \mathbf{P}^9$ parametrizes cubic curves that degenerate to a line and a conic, or three lines, or a line and a double line, or a triple line.

(d) (Challenge) Show for general d that the degenerate (i.e. not reduced and irreducible) curves are parametrized by a closed subvariety $Y \subseteq \mathbf{P}^N$. (Hint: This is a corollary of the fact that every projective scheme is proper. How does it apply here?)

Remark. In the classical setting of varieties, the fact that projective schemes are proper is often called "the main theorem of elimination theory."

2. [EH §III.2.5] (Morphisms to Projective Space)

(a) [EH III-38]
(b) [EH III-41]
(c) [EH III-43(a)–(b)]
(d) [EH III-45(a)–(b)]

3. [H II.4.10 (a)–(d)] (Chow's Lemma.) (Key: What is needed in order to pass from the version for varieties that we did in class to one for general schemes?)

4. [H II.5.7 (a)–(c)] (Some basic properties of coherent sheaves.)

5. [H II.5.15 (a)–(e)] (Extension of Coherent Sheaves)