MATH 6670, HOMEWORK #2

DUE THURSDAY, FEBRUARY 7

1. [EH I-7] Recall the two two-pointed spaces from HW 1, Exercise 4 (EH I-5). What are stalks of the sheaves at each of the points?

2. [EH I-8] (The sheaf of germs construction.) Topologize the disjoint union $\overline{\mathscr{F}} = \bigcup \mathscr{F}_x$ by taking as a base for the open sets of $\overline{\mathscr{F}}$ all sets of the form

$$\mathscr{V}(U, s) := \{(x, s_x) : x \in U\},$$

where U is an open set and s is a fixed section over U.

(a) Show that the natural map $\pi : \overline{\mathscr{F}} \to X$ is continuous, and that, for U and $s \in \mathscr{F}(U)$, the map $\sigma : x \mapsto s_x$ from U to $\overline{\mathscr{F}}$ is a continuous section of π over U (that is, it is continuous and $\pi \circ \sigma$ is the identity on U).

(b) Conversely, show that any continuous map $\sigma : U \to \overline{\mathscr{F}}$ such that $\pi \circ \sigma$ is the identity on U arises in this way. (*Hint:* Take $x \in U$ and a basic open set $\mathscr{V}(V, t)$ containing $\sigma(tx)$, where $V \subset U$. What relation does t have to σ?)

3. If $f : X \to Y$ is a continuous map and $\mathscr{F}$ is a presheaf on X, the **pushforward $f_*\mathscr{F}$ of $\mathscr{F}$ (along f)** is the presheaf on Y defined by

$$f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$$

for all open sets $U \subset Y$. Show that if $\mathscr{F}$ is a sheaf, then $f_*\mathscr{F}$ is also a sheaf.

4. [EH I-10] Recall that we showed in class that if $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective (resp. bijective) for all open sets $U \subset X$ if and only if ϕ_x is injective (resp. bijective) for all points $x \in X$. In this exercise, you will show that this is *false* if you replace "injective" by "surjective" by checking that in each of the following examples, the maps induced by ϕ on stalks are surjective, but for some open set U, the map $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

(a) Let X be the topological space $\mathbf{C}\setminus\{0\}$, let $\mathscr{F} = \mathscr{G}$ be the sheaf of nowhere-zero, continuous, complex-valued functions, and let ϕ be the map sending a function f to f^2.

(b) Let X be the Riemann sphere $\mathbf{CP}^1 = \mathbf{C} \cup \{\infty\}$ and let $\mathscr{G}$ be the sheaf of analytic functions. Let $\mathscr{F}_1$ be the sheaf of analytic functions vanishing at 0; that is, $\mathscr{F}_1(U)$ is the set of analytic functions on U that vanish at 0 if $0 \in U$, and the set of all analytic functions on U if $0 \notin U$. Similarly, let $\mathscr{F}_2$ be the sheaf of analytic functions vanishing at ∞. Let $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$, and let $\phi : \mathscr{F} \to \mathscr{G}$ be the addition map.

(c) Find an example of this phenomenon in which the set X consists of three points.

5. [EH I-13] ($\mathscr{B}$-sheaves) A situation that often occurs in scheme theory is that given a base $\mathscr{B}$ for the open sets of a topological space (i.e. every open can be written as a union of elements in $\mathscr{B}$), you want to specify a sheaf by just defining $\mathscr{F}(U)$ and restriction maps for $U, V \in \mathscr{B}$ instead of for *every* open set in a topological space. The following result shows that this can be done.

We say a collection of groups $\mathscr{F}(U)$ for open sets $U \in \mathscr{B}$ and restriction maps $\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ for opens $V \subset U$ form a **$\mathscr{B}$-sheaf** if they satisfy the sheaf axiom with respect to inclusions of basic open sets into basic open sets and coverings of basic open sets by basic open sets. (The condition in the definition that sections of $U_a, U_b \in \mathscr{B}$ agrees on $U_a \cap U_b$ must be replaced by the condition that they agree on any basic open $V \in \mathscr{B}$ such that $V \subset U_a \cap U_b$.

Let $\mathscr{B}$ be a base of open sets for X. Prove the following properties.

(a) Every $\mathscr{B}$-sheaf on X extends uniquely to a sheaf on X.
(b) Given sheaves $\mathscr{F}$ and $\mathscr{G}$ on X and a collection of maps

$$\widetilde{\phi}(U) : \mathscr{F}(U) \to \mathscr{G}(U) \quad \text{for all } U \in \mathscr{B}$$

commuting with restrictions, there is a unique morphism $\phi : \mathscr{F} \to \mathscr{G}$ of sheaves such that $\phi(U) = \widetilde{\phi}(U)$ for all $U \in \mathscr{B}$.

(*Hint:* This is Prop I-12 in [EH], and one may want to follow the beginning of the proof that is sketched there.)