MATH 6670, HOMEWORK #3

DUE THURSDAY, FEBRUARY 14

1. [Hartshorne II.2.13] A topological space is *quasicompact* if every open cover has a finite subcover. (We say "quasi" because the spaces are not necessarily Hausdorff.)

A topological space is *noetherian* if it satisfies the *descending chain condition* for closed subsets: for any sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets, there exists an integer r such that $Y_r = Y_{r+1} = \cdots$. (Note that this is dual to the condition to be a noetherian ring, which satisfies the *ascending* chain condition on ideals.)

Prove the following results about the relations between quasicompactness and noetherianity.

(a) A topological space is noetherian if and only if every open subset is quasicompact.
(b) Given a scheme $X = (X, \mathscr{O}_X)$ (a locally ringed space), we denote the underlying topological space X by $\mathrm{sp}(X)$ ("space of X"). If X is an affine scheme, show that $\mathrm{sp}(X)$ is quasicompact. (We say a scheme $X = (X, \mathscr{O}_X)$ is *quasicompact* if $\mathrm{sp}(X)$ is.) Give an example of a ring R such that $\mathrm{sp}(\operatorname{Spec} R)$ is not noetherian.
(c) If A is a noetherian ring, show that $\mathrm{sp}(\operatorname{Spec} A)$ is a noetherian topological space.
(d) Give an example to show that $\mathrm{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

2. [EH I-20, I-26] Describe the points and the sheaf of functions of each of the following schemes.

(a) $X_1 = \operatorname{Spec} \mathbf{C}[x]/(x^2)$.
(b) $X_2 = \operatorname{Spec} \mathbf{C}[x]/(x^2 - x)$.
(c) $X_3 = \operatorname{Spec} \mathbf{C}[x]/(x^3 - x^2)$.
(d) $X_4 = \operatorname{Spec} \mathbf{R}[x]/(x^2 + 1)$.

The schemes X_1, X_2, and X_3 here can all be viewed as closed subschemes of $\operatorname{Spec} \mathbf{C}[x]$. (Recall a closed subscheme of an *affine* scheme $\operatorname{Spec} R$ is one that is the spectrum of a quotient ring of R; hence there is a one-to-one correspondence between closed subschemes of an affine scheme and ideals of R.) We say that a closed subscheme $Y = \operatorname{Spec} R/I$ of an affine scheme $\operatorname{Spec} R$ *contains* the closed subscheme $Z = R/J$ if we have the inclusion $J \supset I$ of ideals. (Of course, this algebraic condition implies that Z is a closed subscheme of Y.)

(e) Show that
$$X_1 \subset X_3 \quad \text{and } X_2 \subset X_3,$$
but no other inclusions $X_i \subset X_j$ hold, even though the underlying sets of X_2 and X_3 coincide and the underlying set of X_1 is contained in the underlying set of X_2.

3. [EH I-25] (The smallest non-affine scheme: "The germ of the doubled point.") Let X be a topological space with three points p, q_1, and q_2. Topologize X by making $X_1 := \{p, q_1\}$ and $X_2 := \{p, q_2\}$ open sets (so that, in addition, $\emptyset$, $\{p\}$, and X itself are open). Define a presheaf $\mathscr{O}$ of rings on X by setting

$$\mathscr{O}(X) = \mathscr{O}(X_1) = \mathscr{O}(X_2) = K[x]_{(x)}, \quad \mathscr{O}(\{p\}) = K(x)$$

with restriction maps $\mathscr{O}(X) \to \mathscr{O}(X_i)$ the identity and $\mathscr{O}(X_i) \to \mathscr{O}(\{p\})$ the obvious inclusion map.

(a) Check that this presheaf is a sheaf and that $(X, \mathscr{O})$ is a scheme.
(b) Show that $(X, \mathscr{O})$ is not an affine scheme.

4. [EH I-21] (Ring of rational functions) The schemes that are analogous to compact manifolds (the "proper" schemes) tend to not have non-constant regular functions, in stark contrast to the case of, say, smooth or complex manifolds. Thus, in algebraic geometry, partially defined functions on a scheme X (namely, elements $\mathscr{O}_X(U)$ for a dense open set U of X) play an important role. Such functions are called *rational functions* on X despite the fact that they are not generally "functions on X" at all, as they are not defined on the subset $X \backslash U$. This turns out to be all right, because in many important cases, every nonempty open subset is dense in X, so the behavior of rational functions reflect the properties of X. In this exercise, we will try and understand why they are given this name.

(a) Let $X = \operatorname{Spec} R$ where R is an integral domain. Let $U = X_f := D(f)$ (Note that X_f is E-H's notation for distinguished opens). Show that the elements of $\mathscr{O}_X(X_f) = R_f$, so are given by ratios of elements in R.

(b) Let $\mathfrak{U}$ be the set of open and dense sets in X. Consider the *ring of rational functions*

$$\operatorname{colim}_{U \in \mathfrak{U}} \mathscr{O}_X(U) := \left(\coprod_{U \in \mathfrak{U}} \mathscr{O}_X(U) \right) / \sim$$

where $\sigma \in \mathscr{O}_X(U)$ is equivalent ("$\sim$") to $\tau \in \mathscr{O}_X(V)$ (for $U, V \in \mathfrak{U}$) if there exists a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$ and $\sigma|_W = \tau|_W$. Compute $\operatorname{colim}_{U \in \mathfrak{U}} \mathscr{O}_X(U)$ when $X = \operatorname{Spec} R$, first when R is a domain, and then for R an arbitrary Noetherian ring.

5. [Hartshorne II.2.1-2] (Open subschemes)

(a) Let A be a ring and $X = \operatorname{Spec} A$. Given $f \in A$, let $D(f) \subseteq X$ be the open complement of $V((f))$ in X (a "distinguished open"). Show that the locally ringed space $(D(f), \mathscr{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$. (This generalizes and goes beyond part (a) of the previous exercise.)

(b) Let $(X, \mathscr{O}_X)$ be a scheme and let $U \subseteq X$ be any open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme. (We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathscr{O}_X|_U)$ as an *open subscheme* of X.)