# MATH 6670, HOMEWORK #4

DUE THURSDAY, FEBRUARY 21

**1.** (Basic properties of Spec and distinguished opens.) Let $R$ be a ring. Recall that for an ideal $I$ of $R$, the set $V(I) \subseteq \operatorname{Spec} R$ is the set of all prime ideals which contain $I$.

(a) Given two ideals $I$ and $J$ of $R$, show that

$$V(I_1) \cup V(I_2) = V(I_1 \cdot I_2) = V(I_1 \cap I_2).$$

(b) Show that $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$ (the radicals of $I$ and $J$).

(c) Show that $D(f) = \emptyset$ if and only if $f$ is nilpotent. Show that $D(f) = \operatorname{Spec}(R)$ if and only if $f$ is invertible.

(d) Show that $D(f) \supset D(g)$ if and only if $f$ is invertible as an element of $R_g$ if and only if there exists an $n$ such that $g^n \in (f)$.

(e) Is it true that $D(f) = \cup_{i \in I} D(g_i)$ if and only if there exists an $n$ such that $f^n \in \langle g_i | i \in I \rangle$?

**2.** (Maps between affine schemes) Let $\phi : A \to B$ be a ring homomorphism, and $\Phi : \operatorname{Spec} B \to \operatorname{Spec} A$ the corresponding map between affine schemes.

(a) Show that $\Phi^{-1}(D(f)) = D(\phi(f))$ and $\Phi^{-1}(V(I)) = V(B \cdot \phi(I))$.

(b) Show that the closure of $\Phi(V(I))$ (i.e. smallest closed set containing it, with respect to the Zariski topology) equals $V(\phi^{-1}(I))$.

(c) Show that the image of $\Phi$ is dense if and only if $\operatorname{Ker}(\phi)$ consists of nilpotent elements.

(d) Let $B = A_f$. Show that the map $\operatorname{Spec}(A_f) \to \operatorname{Spec}(A)$ is a homeomorphism onto $D(f)$. (We saw this kind of structure on the last homework in general.)

(e) Let $B = A/I$ for some ideal $I \subseteq A$. Show that the map $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ is homeomorphism onto $V(I)$.

**3.** (Crossing some $t$'s, dotting some $i$'s)

(a) [EH I-28] Show that to check that a sheaf of ideals (or any sheaf of modules) is quasicoherent, it is enough to check the defining property on each set $U$ of a fixed affine open cover of $X$.

(b) [EH I-30] Show that a scheme is irreducible if and only if every open subset is dense.

(c) [EH I-31] Show that an affine scheme $X = \operatorname{Spec} R$ is reduced and irreducible if and only if $R$ is a domain. Show that $X = \operatorname{Spec} R$ is irreducible if and only if $R$ has a unique minimal prime (equivalently, if the nilradical of $R$ is prime).

**4.** (Algebraic varieties) Varieties are, of course, a special case of schemes (over algebraically closed fields, they're usually considered to be reduced noetherian schemes of finite type and often assumed to be irreducible as well). In this exercise, we'll develop the main results from the theory of algebraic varieties using the scheme-theoretic perspective.

Let $k$ be an algebraically closed field, and let $A$ be a finitely generated $k$-algebra. Define $\operatorname{MaxSpec}(A)$ (sometimes called $\operatorname{Specm}(A)$) to be the subset of $\operatorname{Spec} A$ consisting of maximal ideals, endowed with the induced topology.

(a) Use Hilbert's Nullstellensatz ("$I(V(J)) = \sqrt{J}$") to identify $\operatorname{MaxSpec}(A)$ with the set of $k$-algebra homomorphisms $A \to k$.

(b) Mimic the construction of $\mathscr{O}_{\mathrm{Spec}(A)}$ to produce a sheaf of rings $\mathscr{O}_{\mathrm{MaxSpec}(A)}$. Show that this sheaf of rings can be identified with the the inverse image of $\mathscr{O}_{\mathrm{Spec}(A)}$ under the inclusion map $\iota : \mathrm{MaxSpec}(A) \to \mathrm{Spec}(A)$.
(c) Show that a map of finitely generated $k$-algebras $A \to B$ defines a morphism of locally ringed spaces $\mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$.
(d) Assume that $A$ is a reduced finitely generated $k$-algebra. Show that there is a natural embedding of the sheaf of rings $\mathscr{O}_{\mathrm{MaxSpec}(A)}$ into the sheaf $\mathscr{F}_{\mathrm{MaxSpec}(A)}$ of $k$-valued functions.
(e) Assume that $A$ and $B$ are reduced f.g. $k$-algebras. Show that we have a bijection between the following three sets:
  (i) The set of $k$-algebra homomorphisms $A \to B$.
  (ii) The set of maps of locally ringed spaces
$$(\mathrm{MaxSpec}(B), \mathscr{O}_{\mathrm{MaxSpec}(B)}) \to (\mathrm{MaxSpec}(A), \mathscr{O}_{\mathrm{MaxSpec}(A)})$$
  that respect the embedding of the field $k$ into sections of these sheaves of rings.
  (iii) The set of continuous maps $\phi : \mathrm{MaxSpec}(B) \to \mathrm{MaxSpec}(A)$ for which the induced map $\mathscr{F}_{\mathrm{MaxSpec}(A)} \to \Phi_*(\mathscr{F}_{\mathrm{MaxSpec}(B)})$ sends $\mathscr{O}_{\mathrm{MaxSpec}(A)} \hookrightarrow \mathscr{F}_{\mathrm{MaxSpec}(A)}$ to $\mathscr{O}_{\mathrm{MaxSpec}(B)} \hookrightarrow \mathscr{F}_{\mathrm{MaxSpec}(B)}$.

**5.** [Hartshorne II.2.12] (The Gluing Lemma) [This is a long exercise that requires some care to get the full and correct answer, but worth doing at least once.] Let $\{X_i\}$ be a family of schemes (possibly infinite). For each index $i \neq j$, suppose that we are given an open subset $U_{ij} \subseteq X_i$, and endow it with the induced scheme structure. Furthermore, suppose that for each $i \neq j$, we have an isomorphism of schemes
$$\phi_{ij} : U_{ij} \to U_{ji}$$
such that
(i) for each $i, j$, $\phi_{ji} = \phi_{ij}^{-1}$; and
(ii) for each $i, j, k$,
$$\phi_{ij}(U_{ij} \cap U_{ij}) = U_{ji} \cap U_{jk}$$
and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$.

Show that there is a scheme $X$ together with morphisms $\psi_i : X_i \to X$ such that
(a) $\psi_i$ is an isomorphism of $X_i$ onto an open subscheme of $X$,
(b) the $\psi_i(X_i)$ cover $X$,
(c) $\psi(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$, and
(d) $\psi_i = \psi_j \circ \phi_{ij}$ on $U_{ij}$.

We say that the scheme $X$ is obtained by **gluing the schemes $X_i$ along the isomorphisms $\phi_{ij}$**. (An interesting special case is when the family $X_i$ is arbitrary, but the $U_{ij}$ and $\phi_{ij}$ are all empty. Then the scheme $X$ is called the *disjoint union* of the $X_i$ and ix denoted by $\coprod X_i$.)