MATH 6670, HOMEWORK #5

DUE THURSDAY, FEBRUARY 28

1. (Some properties under base change)

(a) Consider the field $k = \mathbf{C}(t)$ and consider $X = \operatorname{Spec} k[x]/(x^2 - t)$. Describe X (e.g. what is the underlying topological space, is it irreducible, etc.). Now, let $k' = \mathbf{C}(\sqrt{t})$. Describe $X' = \operatorname{Spec} k'[x]/(x^2 - t)$ (this can be considered the base change of X to k').

(b) Similarly, let $k = \mathbf{F}_p(t)$ and consider $X = \operatorname{Spec} k[x]/(x^p - t)$. Describe X. Let $k' = \mathbf{F}_p(\sqrt[p]{t})$ and $X = X' \times_k \operatorname{Spec} k' = \operatorname{Spec} k'[x]/(x^p - t)$. Show that X' is not an algebraic variety.

These examples are shadows of general facts about schemes under base change. Roughly speaking, the principle is that, phenomena related to *topology* can only change under *separable* extensions, whereas phenomena related to *algebra* can only change under *inseparable* extensions.

2. (Fun with affines)

(a) Consider the rings $R = \{f \in \mathbf{R}[x] : f'(0) = 0\}$ and $S = \{f \in \mathbf{R}[x] : f(0) = f(1)\}$. Describe $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$. Draw pictures of the sets of $\mathbf{R}$-points $X(\mathbf{R})$ and $Y(\mathbf{R})$.

(b) (A strangely confusing but enlightening exercise.) For rings A and B, describe a bijection between isomorphisms $\operatorname{Spec} A \to \operatorname{Spec} B$ and ring isomorphisms $B \to A$. (Hint: This is trickier than it sounds. The most difficult part is to show that if an isomorphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$ induces an isomorphism $f^\sharp : B \to A$, which in turn induces an isomorphism $g : \operatorname{Spec} A \to B$, then $f = g$. First show on the level of topological spaces; this is the trickiest part. Then show $f = g$ as maps of topological spaces. Finally, show $f = g$ on the level of structure sheaves, using distinguished opens. Beware of circular arguments!)

3. (Projective n-space via gluing) We can construct projective n-space over a field k, denoted $\mathbf{P}^n_k$, by gluing together $n+1$ affine open sets, each isomorphic to $\mathbf{A}^n_k$. Recall that we can think of points of projective space as tuples of points $(x_0, x_1, \ldots, x_n) \in k^{n+1}$, modulo the relation $(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$ for a nonzero scalar $\lambda \in k$; we denote the corresponding equivalence class by $[x_0 : x_1 : \cdots : x_n]$ (or $[x_0, x_1, \ldots, x_n]$).

We glue together $n + 1$ open sets $U_0, U_1, \ldots, U_n$. Say that the open set U_i has coordinates $x_{0/i}, \ldots, x_{(i-1)/i}, x_{(i+1)/i}, \ldots, x_{n/i}$. It is convenient to write this as

$$U_i = \operatorname{Spec} k[x_{0/i}, x_{1/i}, \ldots, x_{n/i}]/(x_{i/i} - 1),$$

where we have introduced a "dummy variable" $x_{i/i}$, which we immediately set to 1. We glue the distinguished open set $D(x_{j/i}) \subset U_i$ to the distinguished open set $D(x_{i/j}) \subset U_j$, by identifying these two schemes by describing the identification of rings

$$\operatorname{Spec} k[x_{0/i}, x_{1/i}, \ldots, x_{n/i}, 1/x_{j/i}]/(x_{i/i} - 1) \cong \operatorname{Spec} k[x_{0/j}, x_{1/j}, \ldots, x_{n/j}, 1/x_{i/j}]/(x_{j/j} - 1)$$

via $x_{k/i} = x_{k/j}/x_{i/j}$ and $x_{k/j} = x_{k/i}/x_{j/i}$ (which implies that $x_{i/j}x_{j/i} = 1$).

(a) As painlessly as possible, check the details of the construction above and show that this glues the U_i's together to get $\mathbf{P}^n_k$. In particular, show that the gluing information agrees on triple overlaps. (Hint: The triple intersection is affine. What is the corresponding ring?)

(b) Show that the only functions on $\mathbf{P}^n_k$ are constants (i.e. we have global sections $\Gamma(\mathbf{P}^n_k, \mathscr{O}) \cong k$) and hence $\mathbf{P}^n_k$ is not affine for $n > 0$. (Hint: You only need two of your open sets to see this; not some delicate interplay of all $n + 1$ of your affines.)

4. [EH I-46] (Fiber products on affines) Prove the following facts directly from the universal property of the tensor product of algebras.

(a) For any R-algebra S, we have $R \otimes_R S = S$.

(b) If S, T are R-algebras and $I \subset S$ an ideal, then

$$(S/I) \otimes_R T = (S \otimes_R T)/(I \otimes 1)(S \otimes_R T).$$

(c) If $x_1, \dots, x_n, y_1, \dots, y_m$ are indeterminates, then

$$R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] = R[x_!, \dots, x_n, y_1, \dots, y_m].$$

Use these principles to solve the remainder of this exercise.

(d) Let m, n be integers. Compute the fibered product

$$\operatorname{Spec} \mathbf{Z}/(m) \otimes_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}/(n).$$

(e) Compute the fibered product $\operatorname{Spec} \mathbf{C} \times_{\operatorname{Spec} \mathbf{R}} \operatorname{Spec} \mathbf{C}$.

(f) Show that for any polynomial rings $R[x]$ and $R[y]$ over a ring R, we have

$$\operatorname{Spec} R[x] \times_{\operatorname{Spec} R} \operatorname{Spec} R[y] = \operatorname{Spec} R[x, y].$$

Note that in example (d), the underlying set of the fibered product is the fibered product of the underlying sets, but this not true in (e) and (f).

(g) Consider the ring homomorphism

$$R[x] \to R$$
$$x \mapsto 0$$

and

$$R[x] \to R[y]$$
$$x \mapsto y^2.$$

Show that with respect to these maps, we have

$$\operatorname{Spec} R[y] \times_{\operatorname{Spec} R[x]} \operatorname{Spec} R = \operatorname{Spec} R[y]/(y^2).$$

5. [EH I-49] Let K be a field. If X and Y are nonempty K-schemes, then the product $X \times Y = X \times_{\operatorname{Spec} K} Y$ in the category of K-schemes is nonempty. (Of course, the difficulty is to handle the non-affine case.)