MATH 6670, HOMEWORK #6

DUE THURSDAY, MARCH 7

1. (A nontrivial 2-dimensional scheme: $\operatorname{Spec}\mathbf{Z}[x]$, [EH §II.4.3])

(a) [EH II.37] Classify the prime ideals of $\mathbf{Z}[x]$.

(b) [EH II.38] (Look at the first figure in §II.4.3) What is the point marked with a ? in the picture above? Why are the closures of the points $(4x+1)$ and $(x-2)$ indicated by curves meeting tangentially at the point $(3, x-2)$, while they are both transverse to the closure of (3)? Why is the closure of the point $(4x+1)$ drawn as having a vertical asymptote over the point $(2) \in \operatorname{Spec}\mathbf{Z}$?

(c) [EH II.39] (Look at the second figure in §II.4.3) Identify the three unlabeled points in the diagram.

2. (Zariski tangent space) [II.2.8] Let X be scheme. For any point $x \in X$, the *Zariski tangent space* T_x of X at x is the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. (If we do not take the dual, we get the Zariski *cotangent* space.)

Assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of $\operatorname{Spec} k[\epsilon]/\epsilon^2$ to X is equivalent to giving a point of $x \in X$, rational over k (i.e. such that $k(x) = k$), and an element of T_x.

3. (Finiteness properties for morphisms) [H, II.3.3, II.3.7] A morphism of $f : X \to Y$ of schemes is *locally of finite type* if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that for all i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. The morphism f is *of finite type* if in addition each $f^{-1}(V_i)$ can be covered by a finite number of the U_{ij}.

(a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

(c) Show also if f is of finite type, then for every open affine subet $V = \operatorname{Spec} B \subseteq Y$ and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

A morphism $f : X \to Y$ is a *finite* morphism if there exists a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.

(d) Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

4. [H, II.3.8] A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec}\widetilde{A}$.

(a) Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the *normalization* of X.

(b) Show that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant (i.e. dense image) morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$.

(c) If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism.

5. (Closed points in schemes)

(a) Show that if X is a nonempty quasicompact scheme, then it has a closed point. (Warning: there exist nonempty schemes with no closed points, so you need to use the quasicompact hypothesis!)

(b) If X is a scheme of finite type over a field, show that the closed points of X are dense.

(c) Give an example of a scheme in which the closed points are not dense.