MATH 6670, HOMEWORK #7

DUE THURSDAY, MARCH 21

1. (Odds and ends)

(a) We saw in class that if $f : X \to Y$ is quasicompact and $f(X)$ is closed under specialization, then $f(X)$ is closed. This example illustrates that quasicompactness is necessary. Let $Y = \mathbf{A}^1_K$ for a field K. Let X be an infinite disjoint union of closed points of $\mathbf{A}^1_K$, with f the inclusion map. Show that f is locally of finite type (to see that the failure is really due to quasicompactness, not some weird non-finite algebra phenomenon). Show that $f(X)$ is closed under specialization but not closed.

(b) Let X be a separated scheme over an affine scheme S. Show that if U and V are open affine subsets of X, then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

(c) Let $X = \operatorname{Spec} A$ where A is a discrete valuation ring with fraction field K. Describe the algebraic data needed to determine an $\mathscr{O}_X$-module on X. In terms of this algebra data, state and prove a necessary and sufficient criterion for such a sheaf to be quasi-coherent.

2. [H II.3.17 (a)–(e)] [i.e. no need to do part (f)] (Zariski Spaces)

3. (Valuation rings) An integral domain A with fraction field K is said to be a *valuation ring* if for all $x \in K^\times$, either $x \in A$ or $x^{-1} \in A$.

(a) Show that being a valuation ring is equivalent to K being equipped with a valuation, a map $v : K^\times \to \Gamma$ (for a totally ordered abelian group Γ) such that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min\{(v(x), v(y)\}$ (for $x + y \neq 0$) and that $A = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}$.

(b) Show that a valuation ring is a local ring. Give an example to show that it is not necessarily noetherian.

(c) If K is a field, we have a partial order (usually called *dominance*) on the local rings strictly contained in K: we say $A \leq B$ if $A \subset B$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$ (equivalently, if the inclusion $A \to B$ is a local homomorphism). Show that valuation rings A with given fraction field K are precisely the maximal elements under dominance among local rings strictly contained in K.

4.[EH §III.1.2] (Getting acquainted with separated schemes.)

(a) [EH III-1(a)] Let Y be the line with doubled origin over a field K (cf. [EH Exer I-44]), and let $\phi_1, \phi_2 : \mathbf{A}^1_K \to Y$ be the two obvious inclusions. Show that the locus where ϕ_1 and ϕ_2 agree (simply as continuous maps of topological spaces) is not closed.

(b) [EH III-1(b)] Now, let $X = Y \times_K Y$ and let ϕ and ψ be the two projection maps from X to Y. Show that the set of points at which ϕ and ψ agree is not closed (note that this is just the diagonal). Show that the same is the true for the set of closed points at which ϕ and ψ agree, so this is not a pathology special to schemes but occurs already in the category of varieties.

(c) [EH III-2]

(d) [EH III-3]

(e) [EH III-4]

5. [H II.4.7(a)–(e)] (Schemes over $\mathbf{R}$)