MATH 6670, HOMEWORK #9

DUE THURSDAY, APRIL 11

1. [EH §III.2.2] (Closed subschemes of Proj R)

(a) [EH Exer. III-13]
(b) [EH Exer. III.14]
(c) [EH Exer. III.15]
(d) [EH Exer. III.16]
(e) [EH Exer. III.17]

2. (Properties of properness)

(a) Show that a finite morphism is proper.
(b) (A partial converse.) Show that a proper morphism between affine varieties over a field k is finite. (Hint: Use the following important fact: if A is a subring of a field K, then the integral closure of A (in K) is the intersection of all valuation rings that contain A.)
(c) (Images of proper morphisms are proper.) Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y and that $f(Z)$ with the induced scheme-theoretic image structure is proper over S. (Hint: Factor f into the graph morphism $\Gamma_f : X \to X \times_S Y$ followed by the projection onto Y, and show that Γ_f is a closed immersion.)

3. Let X be a scheme and let $\mathscr{F}$ be a quasicoherent sheaf on X.

(a) Show that the following conditions are equivalent:
  (a) For every open $U = \operatorname{Spec}(A) \subset X$, the A-module $\Gamma(U, \mathscr{F})$ is finitely generated.
  (b) There is a covering of X by open affines $U_i = \operatorname{Spec}(A_i)$ such that each $\Gamma(U_i, \mathscr{F})$ is finitely generated as an A_i-module.

(b) Recall that a B-module is said to be *finitely presented* if it is isomorphic to the quotient of a map $B^n \to B^m$ for some integers n and m. (In particular, if B is Noetherian, a module is finitely presented if and only if it's finitely generated.) Show that the following are equivalent:
  (a) For every open affine $U = \operatorname{Spec}(A) \subset X$, the A-module $\Gamma(U, \mathscr{F})$ is finitely presented.
  (b) There exist a covering by open affines $U_i = \operatorname{Spec}(A_i)$, such that each $\Gamma(U_i, \mathscr{F})$ is finitely presented as an A_i-module.

  (Note: Quasicoherent sheaves that satisfy this condition are said to be *coherent*.)

(c) Assume that X is locally Noetherian. Show that the conditions of part (b) above are equivalent to those of part (a).
(d) Show that if $f : Y \to X$ is a morphism of schemes, then f^* sends coherent sheaves on X to coherent sheaves on Y.

4. [H II.3.18(a)–(d)] (Constructible sets.)

5. [H II.5.18(a)–(d)] (Vector bundles.)