HOW TO APPORTION FAIRLY

Part 1: Representing, electing and ranking

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Lecture 3: Why blatant political gerrymandering is unavoidable in today's system . . . *and what to do about it*.

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- 50 Senators represent 16% of the population,
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- 51 Democratic Senators represent 58% of the population,
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- In fact, as I will argue, a minority of voters can elect a majority of the U.S. House of Representatives (and probably has).

The sorry state of representation elsewhere

In the United Kingdom

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The "winners" of the last six British elections:

	1983	1987	1992	1997	2001	2005
Votes	42.4%	42.2%	41.9%	43.2%	40.7%	35.2%
Seats	61.1%	57.8%	51.6%	63.4%	62.5%	55.1%

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The "winners" of the last six British elections:

	1983	1987	1992	1997	2001	2005
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Seats	61.1%	57.8%	51.6%	63.4%	62.5%	55.1%

2005 election:

	Votes	Seats
Labour	35.2%	55.1%
Conservatives	32.3%	30.7%
Liberals	22.0%	9.6%

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2^{nd} Lozère	2^{nd} Val d'Oise	1^{st} Var	6^{th} Var
34,374	188,200	73,946	180,153

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2. Why Proportionality?
3. Fair Representation
4. Allocating Kidneys

United States Constitution

Article I, section 2:

> *Representatives and direct Taxes shall be apportioned among the several States . . . according to their respective Numbers . . . The actual Enumeration shall be made . . . every subsequent Term of ten years . . . The Number of Representatives shall not exceed one for every thirty thousand, but each State shall have at least one Representative.*

1791-1792: House proposal, 112 seats, Jefferson's method

State	Population	÷30,000	Bill	
Virginia	630,560	21.019	21	
Massachusetts	475,327	15.844	15	
Pennsylvania	432,879	14.429	14	
North Carolina	353,523	11.784	11	
New York	331,589	11.053	11	
Maryland	278,514	9.284	9	
Connecticut	236,841	7.895	7	
South Carolina	206,236	6.875	6	
New Jersey	179,570	5.986	5	
New Hampshire	141,822	4.727	4	
Vermont	85,533	2.851	2	
Georgia	70,835	2.361	2	
Kentucky	68,705	2.290	2	
Rhode Island	68,446	2.282	2	
Delaware	55,540	1.851	1	

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Massachusetts	475,327	15.844	15	14.723
Pennsylvania	432,879	14.429	14	13.408
North Carolina	353,523	11.784	11	10.950
New York	331,589	11.053	11	**10.271**
Maryland	278,514	9.284	9	8.627
Connecticut	236,841	7.895	7	7.336
South Carolina	206,236	6.875	6	6.388
New Jersey	179,570	5.986	5	5.562
New Hampshire	141,822	4.727	4	4.393
Vermont	85,533	2.851	2	2.649
Georgia	70,835	2.361	2	2.194
Kentucky	68,705	2.290	2	2.128
Rhode Island	68,446	2.282	2	2.120
Delaware	55,540	1.851	1	**1.720**

1791-1792: Hamilton's retort: Congress's bill, 120 seats

State	Quota of 120	Bill
Virginia	20.926*	21
Massachusetts	15.774*	16
Pennsylvania	14.366	14
North Carolina	11.732*	12
New York	11.004	11
Maryland	9.243	9
Connecticut	7.860*	8
South Carolina	6.844*	7
New Jersey	5.959*	6
New Hampshire	4.707*	5
Vermont	2.839*	3
GA	2.351	2
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(2) *assign 9 left-over seats to largest remainders* (*).

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"*The secret of the business is that by these different rules the relative number of East.n & South.n members is varied. The number 120 is made out by applying 1 for 30,000 . . . and allowing to* ***fractions*** *of certain amount an additional member.*"

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Richard Henry Lee (letter to *his* father):
"*Six Eastern States have one apiece more than they ought, Jersey and Delaware the same,'. . . if the plain constitutional mode had been pursued of dividing the number of people in* ***each State Respectively*** *by the agreed ration of 30,000. But by a certain arithmetico-political sophistry an arrangement of six to two against the South has been made . . .*"

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II. *This number is apportioned among the several states by the following rule—As the aggregate numbers of the **United States** are to the **total number** of representatives found as above, so are the **particular numbers** of **each state** to the numbers of each state to the number of representatives of such state.* But

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III. *As this second process leaves a residue of Eight out of the 120 members unapportioned, these are distributed among those states which upon that second process have the largest fractions or remainders."*

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I answer, then, that taxes must be divided ***exactly*** *and representatives* ***as nearly*** *is the* ***nearest***, ***ratio*** *will admit; and that fractions must be neglected because the Constitution . . . has left them unprovided for.*"

1791-1792: Jefferson's method

"The bill does not say that it has given the residuary representatives ***to the greatest fractions****; though in fact it has done so. It seems to have avoided establishing that into a rule, lest it might not suit on another occasion. Perhaps it may be found the next time more convenient to distribute them* ***among the smaller States****; at another time* ***among the larger States****; at other times according to any other crochet which ingenuity may invent, and the combinations of the day give strength to carry; or they may do it arbitrarily by open bargains and cabals.*

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120 being once found ... We must take the nearest common divisor, ... that divisor which applied to every State, gives to them such numbers as, added together, come nearest to 120."

1791-1792: the Virginians' decision

Jefferson's account of April 5:

"[Washington] observed that the vote for & against the bill was perfectly geographical, a northern agt a southern vote, & he feared he should be thought to be taking side with a southern party. I admitted this motive of delicacy, but that it would not induce him to do wrong . . . He here expressed his fear that there would ere long, be a separation of the Union . . . He went home, sent for Randolph . . . desired him to get Mr. Madison . . . They came. Our minds had been before made up. We drew the instrument."

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James Madison and Edmond Randolph, attorney general, were, of course, fellow Virginians.

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"A few of the hottest friends of the bill expressed passion, but the majority was satisfied, and both in and out of doors it gave pleasure to have, at length, an instance of the negative being exercised."

1791-1792: the outcome, Jefferson's method

State	Quota of 105	Bill: Jefferson	
Virginia	18.310	19	
Massachusetts	13.803	14	
Pennsylvania	<u>12.570</u>	13	
North Carolina	10.266	10	
New York	9.629	10	
Maryland	8.088	8	
Connecticut	6.877	7	
South Carolina	5.989	6	
New Jersey	5.214	5	
New Hampshire	4.118	4	
Vermont	2.484	2	
Georgia	2.057	2	
Kentucky	1.995	2	
Rhode Island	1.988	2	
Delaware	1.613	1	

1791-1792: the outcome, Jefferson's method

State	Quota of 105	Bill: Jefferson	Hamilton
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Candidate	Popular vote	Electoral College
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- Hill's method—still used!—definitely replaced Webster's in 1940: **1** seat shifted (Arkansas was safely Democratic):

State	Population	Quota	Hill	Webster
Arkansas	1,949,387	6.473	7	6
Michigan	5,256,106	17.453	17	18

The problem

Who was right: Jefferson or Hamilton?

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Who was right: Jefferson or Hamilton?

Or someone else?

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Why?

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3 Fair Representation

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Proportionality

Aristotle's eloquence triumphs despite its tautological aspects:

"This, then is what the just is—the proportional; the unjust is what violates proportion . . . [The] justice which distributes common possessions is always in accordance with the kind of proportion mentioned above; . . . and the injustice opposed to this kind of justice is that which violates proportion."

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"The just . . . is a species of the proportionate . . . For proportion is equality of ratios, and involves four terms at least. . . ; and the ratio between one pair is the same as that between another pair; for there is a similar distinction between the persons and the things. As the term A, then, is to B, so will C be to D, and therefore, alternatively, as A is to C, B will be to D . . . "

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That is why proportionality seems fair!

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If one says, "It is all mine," and the other says, "Half of it is mine," he who says "It is all mine," shall swear that his share in it is not less than three-quarters, and he who says, "Half of it is mine," shall swear that his share in it is not less than a quarter. The former then receives three-quarter and the latter receives one-quarter.

The contested garment

Claimants	Claims		CG rule	Proportional rule
A	1		3/4	2/3
B	1/2		1/4	1/3
	Total claim: $1\frac{1}{2}$	Estate:	1	1

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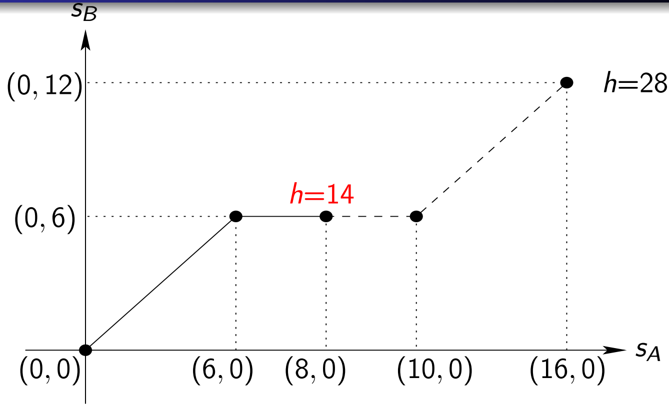
Explanation 2: Equal losses. *I.e.*, A loses $1-\frac{3}{4}=\frac{1}{4}$ and B loses $\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$.

The contested garment: all allocations for claims $(16, 12)$



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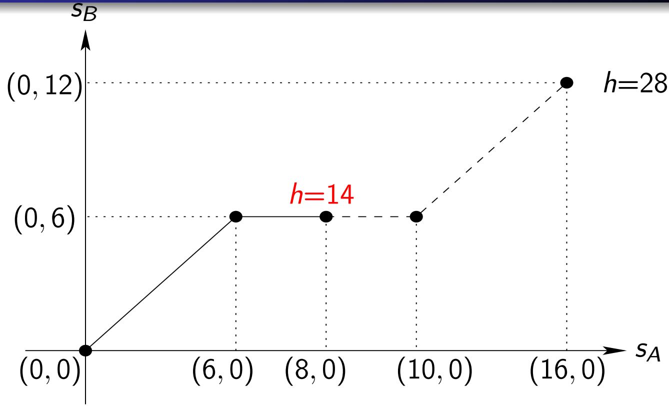

For estates that are half the total claim, $0 \leq h \leq 14$:

$$(s_A, s_B) = (\min\{\lambda, 8\}, \min\{\lambda, 6\}),$$

where λ is chosen so that $s_A + s_B = h$.

The contested garment rule: all allocations for claims $(16, 12)$

Since the CG-rule allocates losses exactly as it does awards:

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If $h = 18 = 28 - 10$, then

$$(s_A, s_B) = (16 - 5, 12 - 5) = (11, 7).$$

The marriage contract

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"If a man who was married to three wives died, and the Kethubah was a maneh [100 zuz], of the other two hundred zuz, and of the third three hundred zuz and the estate [was worth a hundred zuz, the sum] is divided equally.

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If the estate [was worth] two hundred zuz [the claimant] of the maneh [100 zuz] receives fifty zuz and [the other claimants receive each] three gold denarii [75 zuz].

The marriage contract

If the estate [was worth] three hundred zuz, [the claimant] of the maneh [100 zuz] receives fifty zuz and the [the claimant] of two hundred zuz [receives] a maneh [100 zuz] while [the claimant] of the three hundred zuz [receives] six gold denarii [150 zuz]. Similarly, if three persons contributed to a joint fund and they had made a loss or a profit they share in the same manner."

The marriage contract

Claimants	Claims		Case 1	Case 2	Case 3
A	100		$33\frac{1}{3}$	50	50
B	200		$33\frac{1}{3}$	75	100
C	300		$33\frac{1}{3}$	75	150
	Total claim: 600	Estate:	100	200	300

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Case 1: when the estate is worth 100, equal division,

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Case 3: when the estate is worth 300, proportional division,

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Case 1: when the estate is worth 100, equal division,

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Case 2: when the estate is worth 200, . . . why these shares?

What **rule of fair apportionment** did the *Kethuboth* have in mind? This question was not answered until 1985 (*via* very sophisticated concepts of game theory).

The *Kethuboth* rule

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Specifically, **coherence with the rule of contested garment:**

Every pair of claimants must share what they receive together in accordance with the rule of the contested garment.

Case 1: each pair receives $66\frac{2}{3}$ together, each claimant claims all, so the CG-rule divides the amount equally.

Claimants	Claims	*Kethuboth*
A	100	$33\frac{1}{3}$
B	200	$33\frac{1}{3}$
C	300	$33\frac{1}{3}$
		Estate: 100

The *Kethuboth* rule

Case 2: A and B receive 125 together, A concedes 25 to B, B claims all, so the CG-rule gives 25 to B and divides what is left equally.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	75
C	300	75
		Estate: 200

The *Kethuboth* rule

Case 2: A and B receive 125 together, A concedes 25 to B, B claims all, so the CG-rule gives 25 to B and divides what is left equally.

The same is true for A and C.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	75
C	300	75
		Estate: 200

The *Kethuboth* rule

Case 2: A and B receive 125 together, A concedes 25 to B, B claims all, so the CG-rule gives 25 to B and divides what is left equally.

The same is true for A and C.

B and C receive 150 together, each claims all, so the CG-rule gives them equal shares.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	75
C	300	75
		Estate: 200

The *Kethuboth* rule

Case 3: A and B receive 150 together, A concedes 50 to B, B claims all, so the CG-rule gives 50 to B and divides the 100 that is left equally.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	100
C	300	150
		Estate: 300

The *Kethuboth* rule

Case 3: A and B receive 150 together, A concedes 50 to B, B claims all, so the CG-rule gives 50 to B and divides the 100 that is left equally.

A and C receive 200 together, A concedes 100 to C, C claims all, so the CG-rule gives 100 to C and divides the 100 that is left equally.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	100
C	300	150
		Estate: 300

The *Kethuboth* rule

Case 3: A and B receive 150 together, A concedes 50 to B, B claims all, so the CG-rule gives 50 to B and divides the 100 that is left equally.

A and C receive 200 together, A concedes 100 to C, C claims all, so the CG-rule gives 100 to C and divides the 100 that is left equally.

B and C receive 250 together, B concedes 50 to C, C claims all, so the CG-rule gives 50 to C and divides the 200 that is left equally.

Claimants	Claims	*Kethuboth*
A	100	50
B	200	100
C	300	150
		Estate: 300

The proportional rule

Consider the problem with claims $(100, 200, 300)$ and all possible estates $0 \leq h \leq 600$.

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The **proportional rule** for an estate h is:

$$(s_A, s_B, s_C) = (100\lambda, 200\lambda, 300\lambda)$$

where λ is chosen so that $s_A + s_B + s_C = h$.

The *Kethuboth* rule: a formula

The ***Kethuboth* rule** for an estate h at most half the total claim is:

$$(s_A, s_B, s_C) = (\min\{\lambda, 50\}, \min\{\lambda, 100\}, \min\{\lambda, 150\})$$

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The ***Kethuboth* rule** for an estate h at least half the total claim,

calculate the losses for $d_1 + d_2 + \cdots + d_n - h$,
subtract them from the claims.

So, if $h = 500$ then $(s_A, s_B, s_C) = (66\frac{2}{3}, 166\frac{2}{3}, 266\frac{2}{3})$; and if $h = 400$ then $(s_A, s_B, s_C) = (50, 125, 225)$.

The moral of this tale

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There are infinite numbers of coherent rules . . . but not all seemingly reasonable rules are coherent (as we will see)!

Contents

1 First Presidential Veto

2 Why Proportionality?

3 Fair Representation

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Webster's method

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New York	7,264,183	37.484	37
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Total	9,496,036	49.000	49

But each state must have at least one representative:

State	Population (of 1900)	Quota	Rule
New York	7,264,183	37.522	37
Wyoming	92,531	0.478	1
Total	7,356,714	38.000	38

Webster's method

The rule, where $[x]$ means round x to the nearest integer:

$$(s_{NY}, s_{IA}) = \big(\max\{1, [p_{NY}\lambda]\}, \quad \max\{1, [p_{IA}\lambda]\}\big)$$

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So in general, when h seats are to apportioned and the populations of n states are $(p_1, p_2, \ldots, p_n)$, the method is Daniel Webster's:

$$(s_1, s_2, \ldots, s_n) = \big(\max\{1, [p_1\lambda]\}, \max\{1, [p_2\lambda]\}, \ldots, \max\{1, [p_n\lambda]\}\big)$$

where λ is chosen so that $s_1 + s_2 + \cdots s_n = h$.

Webster's method in his words, April 5, 1832

"To apportion is to distribute by right measure, to set off in just parts, to assign in due and proper proportion. . . [The] apportionment of representative power can never be precise and perfect. . . That which cannot be done perfectly must be done in a manner as near perfections as can be. . . Let the rule be that the population of each State be divided by a common divisor, and, in addition to the number of members resulting from such a division, a member shall be allowed to each State whose fraction exceeds a moiety of the divisor."

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	Quota	Hamilton
NY	37.606	38
PA	32.625	33
IA	11.554	11
VA	9.599	10
NE	5.520	5
ME	3.595	3
OR	2.141	2
VT	1.779	2
⋮	⋮	⋮
Sum	386	386

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	Quota	Hamilton
NY	37.432	37
ME	3.568	4
Sum	41	41

The “Alabama paradox”

House size	350-82	383-385	386	387-88	389-90	391-400
Maine’s seats	3	4	3	4	3	4

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Coherent rules guarantee that when the size of the House h increases, no state can lose seats.

Coherent methods are "divisor methods"

All coherent apportionment methods must be one of these:

$$(s_1, s_2, \ldots, s_n) = \big(\max\{1, \langle p_1\lambda\rangle\}, \max\{1, \langle p_2\lambda\rangle\}, \ldots, \max\{1, \langle p_n\lambda\rangle\}\big)$$

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where λ is chosen so that $s_1 + s_2 + \cdots s_n = h$, and

$\langle x \rangle$ is defined by a **threshold** fixed in each interval $[0, 1]$, $[1, 2]$, $[2, 3]$, ..., $[n, n + 1], \ldots$: above it round-up, below it round-down, at the threshold do either.

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$\langle x\rangle$ is defined by a **threshold** fixed in each interval $[0,1]$, $[1,2]$, $[2,3]$, ..., $[n, n+1], \ldots$: above it round-up, below it round-down, at the threshold do either.

Hill's method, cleverly baptized the "method of equal proportions," is the law of the land in the United States since 1940: its thresholds are the geometric means $\sqrt{n(n+1)}$ of the end points of the intervals $[n, n+1]$.

The five traditional divisor methods

Five divisor methods were considered in the U.S. debate over which method should be used (1920's and 1930's):

- John Quincy Adams's method: *round up* (*"I was all night meditating in search of some device, if it were possible to avert the heavy blow from the State of Massachusetts and from New England"*);

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- Joseph Hill's: *round at the geometric mean*;
- Daniel Webster's: *round at the arithmetic mean*;
- Thomas Jefferson's: *round down* (used, rejected, accused of "*committing a classic rape on a cloud of statistics right in the face of the House*").

1900 apportionments

They give very different results! Going from left to right, bigger states more favored and smaller states less favored.

	Quota	Adams	Dean	Hill	Webstr	Jeffrsn	Hamiltn
NY	37.606	36	37	37	37	39	38
PA	32.625	31	32	32	32	34	33
IA	11.554	11	11	11	12	12	11
NE	5.520	6 $\nearrow$	5	6 $\nearrow$	5	5	5
ME	3.595	4	4	4	4 $\nearrow$	3	3
OR	2.141	3 $\nearrow$	2	2	2	2	2
VT	1.779	2	2	2	2 $\nearrow$	1	2
UT	1.425	2	2 $\nearrow$	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Total	386.000	386	386	386	386	386	386

Willcox *vs.* Huntington and the mathematicians

In a direct violation of the Constitution, there was no reapportionment in 1920. Following the war, the cities had made enormous gains in population. Emmanuel Celler explained:

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> *The issue and the struggle underlying reapportionment is between the large States with large cities on one side and the rural and agricultural States on the other side. That thread of controversy runs through all the political struggles evidenced in this House. . . The issue grows more and more menacing.*

Willcox *vs.* Huntington and the mathematicians

Walter F. Willcox (1861-1964) of Cornell was president at different times of the American Economic, Statistical and Sociological Associations, and a great walker who remarked at the end of his life, "Unfortunately there is some danger that I will be remembered more for my feet than for my head." He championed Webster from 1900 to 1952. His main reason was summarized in 1915, in his presidential address to the AEA:

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> *The use of [Hill's method] has recently been advocated. To use it . . . would . . . result in defeating the main object of the Constitution, which is to hold the scales even between the small and the large states. For the use of [it] inevitably favors the small state.*

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Edward V. Huntington (1874-1952) of Harvard was at different times president of the Mathematical Association of America, vice-president of the American Mathematical Society and the American Association for the Advancement of Science, a charming and witty person, and an adept expert witness. He led the mathematicians in support of Hill's method:

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> *[S]tatistical experts . . . who have examined [Hill's method] have pronounced it the only scientific method. . . The method of [Webster] has a distinct bias in favor of the larger states, while the method of [Dean] has a similar bias in favor of the smaller states. Between these two methods stands the method of [Hill] which has been shown to have no bias in favor of the either the larger or the smaller states.*

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The two gave identical results in 1930: no choice was necessary. Hill gave one more seat to Democratic Arkansas, one less to Michigan in 1940. Hill was chosen.

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What would they have said had there been an even number of methods?

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Theorem

Webster's is the unique unbiased divisor method.

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Walter Willcox's view was *true*: **Webster's** is the method to use!

Senator Arthur Vandenberg

April 19, 1929 address to the Senate:

> *"To identify any one method in this permanent act . . . would be to assume that science itself has traversed the subject with finality. Science is not thus static . . . The last word by no means has been spoken . . . A permanent ministerial apportionment act should be susceptible of accommodation to the progressive state of knowledge."*

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UNOS priorities determined by the total points of a patient: his "score" (all points that are independent of time of waiting) plus his "bonus."

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Postulate four patients, A, B, C, and D, listed in order from the one who has waited the most to the one who has waited the least, and suppose their scores are as given.

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Incoherent, so inequitable: **Yet, it is a simple matter to define a coherent rule!**