Decompositions of partially ordered sets

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Abstract

Generalizing the proof of the theorem describing the closed cone of flag f-vectors of arbitrary graded posets, we give a description of the cone of flag f-vectors of planar graded posets. The labeling used is a special case of a "chain-edge labeling with the first atom property", or FA-labeling, which also generalizes the notion of lexicographic shelling, or CL-labeling. The resulting analogy suggests a planar analogue of the flag h-vector. For planar Cohen-Macaulay posets the two h-vectors turn out to be equal, and the cone of flag h-vectors an orthant, whose dimension is a Fibonacci number. The use of FA-labelings also yields a simple enumeration of the facets in the order complex of an arbitrary graded poset such that the intersection of each cell with the previously attached cells is homotopic to a ball or to a sphere.

Introduction

In a recent paper [2] the authors characterized the cone of linear inequalities holding for the flag f-vectors of all graded posets of a given rank. The proof of this result includes a simple way to label each maximal chain of a graded poset with an antichain of intervals $\mathcal{I}$ in $[1,n] := \{1,2,\ldots,n\}$, such that for every $S \subseteq [1,n]$, the number of partial chains whose set of ranks is S equals the number of all maximal chains whose label $\mathcal{I}$ satisfies the condition that S meets every interval in $\mathcal{I}$. An analysis of the proof of this theorem shows that instead of simply numbering the elements at every rank, we could generate a similar decomposition of the set of maximal chains by using any chain-edge labeling with

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the following property: in every rooted interval of the partially ordered set there is a unique atom, for which the label of the corresponding cover relation is smallest among the labels of cover relations generated by all atoms. We call such labelings chain-edge labelings with the first atom property, or FA-labelings.

A special class of FA-labelings, called lexicographic shellings or CL-labelings, was studied in depth by Björner and Wachs [5, 6]. These labelings linearly order the set of maximal chains in such a way that the size of the classes depends only on the flag f-vector; in fact, they are expressed by the elements of the flag h-vector. Moreover, one may use these labelings to linearly order the facets of a poset in such a way that the intersection of the last attached facet with the previously attached facets is always homeomorphic to a ball or sphere of maximum possible dimension. Such an enumeration is called a shelling of the order complex. Of course, this construction works only for posets which have a CL-labeling.

In section 2 we sketch how any FA-labeling could be used to prove the main theorem in [2], and introduce a generalized flag h-vector giving the class-sizes in the partitioning of the set of maximal chains. This flag h-vector depends on the FA-labeling chosen in general, but specializes to the usual flag h-vector for CL-labelings, as we prove in section 3. In section 4 we show that FA-labelings induced by an embedding of a planar graded partially ordered set into the plane yield an h-vector depending only on the flag f-vector. As with the case of CL-labelings, the definition of planar flag h-vector or p-vector may be extended to the class of all graded partially ordered sets. As in the CL-shellable case, the cone generated by the flag f-vectors of all planar posets turns out to be an orthant, whose extreme rays are exactly the unit vectors in the planar flag h coordinate system.

In section 5 we explore how nicely the two theories of flag h-vectors intersect on CL-shellable planar posets. By a result of Collins [8] we know that all Cohen-Macaulay planar posets belong to this class. For such posets the two flag h-vectors are equal and are supported on sparse sets, *i.e.*, sets not containing two consecutive elements. The cone of flag f-vectors of graded CL-shellable planar posets turns out to be an orthant whose dimension is a Fibonacci number.

Finally, in section 6 we show that we may use any FA-labeling to obtain a weaker version of a shelling: here the intersection of a facet with the already added facets will only be homotopy equivalent to a ball or to a sphere. Yet, since every graded poset has an FA-labeling, this construction works for all posets.

Our work raises several interesting questions. Is there any other way to take a special class of FA-labelings such that we obtain an analogue of the flag h-vector? Are there other classes of posets whose flag f-vectors generate more complicated cones than just orthants, yet still describable by observing that the posets considered allow an FA-labeling of some special kind? Is there an inherent topological property of the order complex of a partially ordered set which – by analogy with the

Cohen-Macaulay property – would guarantee the nonnegativity of its planar h-vector? Drawability in the plane is analogous to shellability in the sense that in both cases a special way of representing the poset guarantees the nonnegativity of the flag-invariants considered.

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1 Preliminaries

1.1 Graded partially ordered sets

Definition 1 *A* graded *poset P is a finite poset with a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, and a* rank function $\mathrm{rank} : P \longrightarrow \mathbb{N}$ *such that we have*

(i) $\mathrm{rank}(\hat{0}) = 0 < \mathrm{rank}(\hat{1})$, *and*

(ii) $\mathrm{rank}(y) - \mathrm{rank}(x) = 1$ *whenever $y \in P$ covers $x \in P$.*

We call $\mathrm{rank}(P) := \mathrm{rank}(\hat{1})$ the rank of the poset P. Given a graded poset P of rank $n+1$, and a subset S of $\{1, 2, \ldots, n\}$ we define the *S–rank-selected subposet of P* to be the poset

$$P_S := \{x \in P \,:\, \mathrm{rank}(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

We denote by $f_S(P)$ the number of maximal chains of P_S. Equivalently, $f_S(P)$ is the number of chains $x_1 \prec \cdots \prec x_{|S|}$ in P such that $\{\mathrm{rank}(x_1), \ldots, \mathrm{rank}(x_{|S|})\} = S$. The function

$$\begin{array}{rccc} f: & 2^{\{1,2,\ldots,n\}} & \longrightarrow & \mathbb{N} \\ & S & \longmapsto & f_S(P) \end{array}$$

is called the *flag f-vector* of P. Whenever it does not cause confusion we will write $f_{s_1 \ldots s_k}$ rather than $f_{\{s_1,\ldots,s_k\}}$; in particular, $f_{\{m\}}$ will always be denoted f_m.

We adopt the following terminology from [3]. The *chain operators* f_S^{n+1} assign $f_S(P)$ to every graded poset P of rank $n+1$, and zero to all other graded posets. These operators are linearly independent and hence they generate a vector space A_{n+1} of dimension 2^n over $\mathbb{R}$. We set $A_0 = \mathbb{R}$.

In [2] we characterized all linear inequalities holding for the flag f-vectors of all graded posets of rank $n+1$. In other words, we described the subset

$$K_{n+1} := \left\{ F = \sum_{S \subseteq \{1,\ldots,n\}} a_S f_S^{n+1} \in A_{n+1} \,:\, \forall P\,(F(P) \geq 0) \right\}$$

of A_{n+1}. K_{n+1} is a polyhedral cone, and its facets can be described in terms of *interval systems*, and their blockers.

Definition 2 *A subset of a partially ordered set P is an* interval*, if it is empty, or of the form*

$$[p,q] := \{x \in P \,:\, p \leq x \leq q\} \quad \text{for some } p,q \in P.$$

An interval system $\mathcal{I}$ on $[1,n]$ is a (possibly empty) family of nonempty subintervals of $[1,n]$.

Definition 3 *Let $\mathcal{S}$ be an arbitrary family of subsets of a finite set X. A subset $T \subseteq X$ is a* blocker *of $\mathcal{S}$, if for every $S \in \mathcal{S}$ we have $S \cap T \neq \emptyset$.*

We denote the set of blockers of $\mathcal{S}$ in X by $\mathbf{B}_X(\mathcal{S})$. In particular, for $\mathcal{S} = \emptyset$ every subset of X is a blocker and so we have $\mathbf{B}_X(\emptyset) = \{T \,:\, T \subseteq X\}$. Note that $\mathbf{B}_X(\mathcal{S}) = \mathbf{B}_X(\mathcal{S}^+)$ where

$$\mathcal{S}^+ = \{T \subseteq X \,:\, \exists S \in \mathcal{S}(S \subseteq T)\}$$

is the *dual ideal* generated by the family $\mathcal{S}$. The following theorem is proved in [2, Theorem 2.1].

Theorem 1 *An expression $\sum_{S\subseteq[1,n]} a_S f_S^{n+1}$ is nonnegative on all graded posets of rank $n+1$ if and only if we have*

$$\sum_{S\in\mathbf{B}_{[1,n]}(\mathcal{I})} a_S \geq 0 \quad \text{for every interval system } \mathcal{I} \text{ on } [1,n]. \tag{1}$$

In the proof of this theorem, the following families of partially ordered sets were useful.

Definition 4 *Given a pair of positive integers n, N and an interval system $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ on $[1,n]$, we define $P(n,\mathcal{I},N)$ to be the following poset. Its elements are all arrays $(i; p_1, \ldots, p_k)$ such that $i \in [0,n+1]$, and for every $j \in [1,k]$ we have*

$$p_j \in \begin{cases} [1,N] & \text{whenever } i \in I_j, \\ \{*\} & \text{otherwise} \end{cases}$$

Here $*$ is a special symbol, different from all integers. We set $(i; p_1, p_2, \ldots, p_k) \leq (i'; p'_1, p'_2, \ldots, p'_k)$ if

(1) $i \leq i'$, and

(2) for every $j \in [1,k]$ we have either $p_j = p'_j$ or $* \in \{p_j, p'_j\}$.

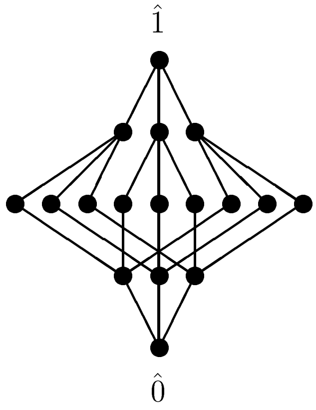

Figure 1: $P(3,\{[1,2][2,3]\},3)$

$P(n,\mathcal{I},N)$ is a graded partially ordered set of rank $n+1$, and for fixed $\mathcal{I}$ and n we have

$$\lim_{N\longrightarrow\infty}\frac{1}{N^{|\mathcal{I}|}}f_S(P(n,\mathcal{I},N))=\begin{cases}1 & \text{when } S\in\mathbf{B}_{[1,n]}(\mathcal{I}),\\ 0 & \text{otherwise.}\end{cases}$$

Figure 1 represents the partially ordered set $P(3,\{[1,2][2,3]\},3)$.

The "virtual posets" $\varepsilon_{\mathcal{I}}^{n+1}$ defined as linear operators on the space A_{n+1} by the formula

$$\varepsilon_{\mathcal{I}}^{n+1}(f_S^{n+1}):=\lim_{N\longrightarrow\infty}\frac{1}{N^{|\mathcal{I}|}}f_S(P(n,\mathcal{I},N))$$

turned out to be the linear functionals defining the facets of K_{n+1} [2].

1.2 Simplicial complexes

A *simplicial complex* Δ is a family of subsets of a finite set V that is closed under taking subsets, *i.e.*, if $S\subset T\in\Delta$ then $S\in\Delta$. We call the elements of Δ *faces*, the maximal elements *facets*. The elements of V are called *vertices*. The *dimension* of a face T is $|T|-1$, the number of k-dimensional faces is denoted by f_k. The *reduced Euler characteristic* of Δ is the invariant

$$\hat{\chi}(\Delta):=-f_{-1}+f_0-f_1+\dots$$

For technical reasons we will consider the empty family of sets, *i.e.* $\Delta=\emptyset$, and the family $\Delta=\{\emptyset\}$ as simplicial complexes. Note that $\hat{\chi}(\emptyset)=0$ while $\hat{\chi}(\{\emptyset\})=-1$.

Given a face $\sigma\in\Delta$ we call the simplicial complex

$$\mathrm{link}_\sigma(\Delta):=\{\tau\subseteq V\setminus\sigma\ :\ \tau\cup\sigma\in\Delta\}$$

on the vertex set $V \setminus \sigma$ the *link of σ in Δ*.

A simplicial complex may be *geometrically represented* in a $|V|$ dimensional Euclidean space, by identifying its vertices with the basis vectors, and adding for every face the convex hull of its vertices. We say that two simplicial complexes are *homeomorphic* or *homotopy equivalent* if the respective statements hold for their geometric representations in the usual sense. In our work we will use the homotopy equivalence of a simplicial complex and its *nerve complex*.

Definition 5 *The nerve complex $\mathcal{N}(\Delta)$ of a simplicial complex Δ is a simplicial complex whose vertices are the facets of Δ and whose faces are $\emptyset$ and those sets $\{F_1, \ldots, F_k\}$ for which $F_1 \cap \cdots \cap F_k \neq \emptyset$ holds.*

The usefulness of the nerve complex is assured by the following result (cf. [4, Theorem 10.7]).

Proposition 1 *Every simplicial complex is homotopy equivalent to its nerve complex.*

1.3 Chain-edge labelings

Given a graded poset P of rank $n+1$, let $\mathcal{M}(P)$ be the set of maximal chains of P, and let

$$\mathcal{E}(P) := \{(x,y) \ : \ x, y \in P, x \prec y\}$$

be the set of cover relations of P. Clearly, $\mathcal{E}(P)$ may be identified with the set of edges in the Hasse diagram of P. Let $\mathcal{ME}(P)$ be the subset of $\mathcal{M}(P) \times \mathcal{E}(P)$ consisting exactly of those pairs $(C, (x,y))$ for which x and y both belong to C, *i.e.*, $x \prec y$ is an edge along the chain C.

Definition 6 *An* edge labeling *of P is a map $\lambda : \mathcal{E}(P) \longrightarrow \Lambda$, where Λ is some poset. A* chain-edge labeling *of P is a map $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$, where Λ is some poset, satisfying the following axiom:*

(CE) If two maximal chains $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \hat{1}$ and $C' : \hat{0} = x'_0 \prec x'_1 \prec \cdots \prec x'_n \prec x'_{n+1} = \hat{1}$ satisfy $x_i = x'_i$ for $i = 1, 2, \ldots, k$, then we have $\lambda(C, (x_{i-1}, x_i)) = \lambda(C', (x'_{i-1}, x'_i)$ for $i = 1, 2, \ldots, k$.

Given a maximal chain $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n \prec x_{n+1} = \hat{1}$, let us denote the set $\{x_0, \ldots, x_i\}$ by $C(i)$. Axiom (CE) may then be rephrased as follows: for a chain-edge labeling $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$ the value of $\lambda(C, (x,y))$ depends only on x, y and the partial chain $C(\mathrm{rank}(x))$. In particular, every edge labeling λ induces a chain-edge labeling λ' defined by $\lambda'(C, (x,y)) := \lambda(x,y)$.

Edge-labelings may be restricted to intervals of P, while chain-edge labelings may be restricted to *rooted intervals*:

Definition 7 *A* rooted interval $[x,y]_r$ *is a pair* $([x,y],r)$ *where* $[x,y]$ *is an interval in* P *and* $r : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{\text{rank}(x)} = x$ *is a maximal chain in the interval* $[\hat{0}, x]$.

Given a rooted interval $[x,y]_r$, and a maximal chain $C : x = x_{\text{rank}(x)} \prec x_{\text{rank}(x)+1} \prec \cdots \prec x_{\text{rank}(y)} = y$ of $[x,y]$, we may be extend C to a maximal chain $C' : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}$ of P by adding the root $r : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{\text{rank}(x)} = x$ and an arbitrary maximal chain $y = x_{\text{rank}(y)} \prec x_{\text{rank}(y)+1} \prec \cdots x_{n+1}$ of the interval $[y, \hat{1}]$. Although C' is not uniquely given, the partial chain $C'(\text{rank}(y))$ consists of the elements of $C \cup r$, and so, by axiom (CE), given a chain-edge labeling $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$, the value of $\lambda(C', (x_{i-1}, x_i))$ for $\text{rank}(x) + 1 \leq i \leq \text{rank}(y)$ does not depend on the particular choice of C' . Hence we may define $\lambda_r : \mathcal{ME}([x,y]) \longrightarrow \Lambda$ by setting $\lambda_r(C, (u,v)) := \lambda(C', (u,v))$ where C' is an arbitrary maximal chain C of P which contains $C \cup r$. It is easy to verify that λ_r is a chain-edge labeling of $[x,y]$.

Given a chain-edge labeling λ, we may associate to each maximal chain $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}$ the ordered string

$$\lambda(C) = (\lambda(C, (x_0, x_1)), \lambda(C, (x_1, x_2)), \ldots, \lambda(C, (x_n, x_{n+1}))))$$

On strings from Λ we introduce the *lexicographic partial order*, defined by

$$(\alpha_1, \ldots, \alpha_p) \leq_L (\beta_1, \ldots, \beta_q)$$

if and only if either $(\alpha_1, \ldots, \alpha_p)$ is an initial segment of $(\beta_1, \ldots, \beta_q)$ or $\alpha_i \neq \beta_i$ for some i and $\alpha_i < \beta_i$ for the least such i.

Definition 8 *A chain edge labeling* $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$ *is a* CR-labeling *(chain rising labeling), if for every rooted interval* $[x,y]_r$ *of* P *there is a unique maximal chain* C *such that* $\lambda_r(C) = (\alpha_1, \ldots, \alpha_p)$ *satisfies* $\alpha_1 < \alpha_2 < \cdots < \alpha_p$. *If for every rooted interval the unique rising chain is lexicographically strictly first, then we call* λ *a CL-labeling (chain lexicographic labeling). A poset that has a CL-labeling is called* CL-shellable.

CL-labelings have the following property (cf. [6, Lemma 5.3]):

Lemma 1 *A CR-labeling is a CL-labeling if and only if the following holds for every rooted interval* $[x,y]_r$*: if* $a_1, a_2, \ldots, a_k$ *are the atoms of* $[x,y]_r$ *and* a_1 *lies on the unique rising chain* C *of* $[x,y]_r$ *then*

$$\lambda_r(C, (x, a_1)) < \lambda_r(C_i, (x, a_i)) \quad \textit{for } i = 2, \ldots k,$$

where C_i *is any maximal chain of* $[x,y]_r$ *containing* a_i.

For detailed information and complete bibliography on this topic we refer the reader to [5] and [6].

2 Chain-edge labelings with the first atom property

The proof of Theorem 2.1 in [2] motivates introducing the following property of chain-edge labelings:

Definition 9 *We say that a chain-edge labeling $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$ has the* first atom property *if in every rooted interval $[x,y]_r$ there is a unique atom a such that*

$$\lambda_r(C,(x,a)) < \lambda_r(C',(x,a'))$$

for every other atom a' of $[x,y]$ and every pair (C,C') of maximal chains of $[x,y]_r$, containing a and a' respectively. We call the atom a the first atom *of the rooted interval $[x,y]_r$ with respect to the chain-edge labeling, and the labeling an* FA-labeling*.*

Note that by axiom (CE), the value of $\lambda_r(C,(x,a))$ depends only on the atom a and the root r, it does *not* depend on the choice of the extending chain C.

Given an FA-labeling, let ϕ be the operation which assigns to every rooted interval $[x,y]_r$ its first atom. This has the following properties:

1. If y covers x then $\phi([x,y]_r) = y$ for any root $r \in \mathcal{M}([\hat{0},x])$.
2. If $a \in [x,y] \subseteq [x,z]$ and $a = \phi([x,z]_r)$ for some root $r \in \mathcal{M}([\hat{0},x])$ then $a = \phi([x,y]_r)$ for the same root r.

It is a consequence of Lemma 1, that every CL-labeling is an FA-labeling. Another example is the labeling implicitly used in the proof of the main theorem of [2]: given an arbitrary graded poset P, we number its elements at any given rank, and consider $\lambda : \mathcal{E}(P) \longrightarrow \mathbb{N}$ which assigns to $(x,y) \in \mathcal{E}(P)$ the number of the element y. It is easy to verify that the chain-edge labeling induced by this edge-labeling has the first atom property: the first atom in every interval will be the one with the smallest number.

Let us outline now how any FA-labeling could be used in the proof of the main theorem of [2]. For every $S \subseteq [1,n]$ the operation M_S assigns to every $i \in [1,n]$ the smallest $j \in S \cup \{n+1\}$ satisfying $j \geq i$. It is easy to show that the set of maximal chains

$$F_S := \left\{C : \hat{0} = p_0 \prec p_1 \prec \cdots \prec p_n \prec p_{n+1} = \hat{1} \ : \ \forall i \in [1,n] \left(p_i = \phi\left([p_{i-1}, p_{M_S(i)}]_{C(i-1)}\right)\right)\right\}$$

has exactly $f_S(P)$ elements. The proof is completely analogous to the one given in [2]. Using the two cited properties of the operation ϕ one may show that for a fixed maximal chain $C : \hat{0} = p_0 \prec p_1 \prec \cdots \prec p_n \prec p_{n+1} = \hat{1}$, and an arbitrary $i \in [1,n]$ there is a largest $j \in [i, n+1]$ such that

$p_i = \phi\left([p_{i-1}, p_j]_{C(i-1)}\right)$. We denote this largest j by $\psi(C,i)$. As in [2] it may be shown that a maximal chain C belongs to F_S if and only if the set S blocks the family of intervals

$$\mathcal{I}_C := \{[i, \psi(C,i)] \ : \ i \in [1,n], \quad \psi(C,i) \neq n+1\}.$$

Of course, it would not be very interesting if we only provided a more complicated proof for an already known result. The interest in allowing a larger class of labelings is that every labeling induces a different partitioning of the set of maximal chains. Whenever for some reason we know that the family of intervals $\mathcal{I}_C$ belongs to some special class, we may transform this information into information on the flag f-vector of the given poset. Moreover our decomposition yields a topologically interesting partitioning of the order complex $\Delta(P)$, as discussed in section 6.

Definition 10 *Given an antichain of intervals $\mathcal{I}$, let $h_{\mathcal{I}}(P)$ denote the quantity*

$$h_{\mathcal{I}}(P) := \left|\left\{C \in \mathcal{M}(P) : \mathcal{I}_C^+ = \mathcal{I}^+\right\}\right|.$$

We call the association $\mathcal{I} \mapsto h_{\mathcal{I}}$ the $\mathcal{I}$-generalized flag h-vector *of the poset P.*

In words, $h_{\mathcal{I}}(P)$ is the number of maximal chains C such that the dual ideal of sets generated by $\mathcal{I}_C$ is the same as the one generated by $\mathcal{I}$. Two dual ideals of sets are equal if and only if they have the same set of minimal elements. In a dual ideal $\mathcal{I}$ the set of minimal elements is an antichain. Conversely, an antichain of intervals $\mathcal{I}$ is the set of minimal elements of $\mathcal{I}^+$. Therefore every maximal chain C contributes to exactly one of the numbers $h_{\mathcal{I}}(P)$. Observe also that exactly one maximal chain satisfies $\mathcal{I}_C = \emptyset$, namely the chain defined recursively by $x_0 = \hat{0}$, $x_i = \phi\left([x_{i-1}, \hat{1}]_{C(i)}\right)$. Hence we always have

$$h_{\emptyset} = 1. \tag{2}$$

Since $C \in F_S$ holds if and only if S blocks $\mathcal{I}_C$, we have the following formula.

$$f_S(P) = \sum_{\substack{\mathcal{I} \\ S \in \mathbf{B}_{[1,n]}(\mathcal{I})}} h_{\mathcal{I}}(P) \tag{3}$$

where the summation on the right hand side is over the antichains of intervals $\mathcal{I}$ with the cited property. In section 3 we will show that for CL-labelings $h_{\mathcal{I}}(P)$ is not zero only if the antichain $\mathcal{I}$ is of the form

$$\mathcal{I}_S := \{\{s\} \ : \ s \in S\}$$

for some $S \subseteq [1,n]$. Introducing h_S as a shorthand notation for $h_{\mathcal{I}_S}$, we may transform (3) into

$$f_S(P) = \sum_{T \subseteq S} h_T(P)$$

which will yield the usual definition of the flag h-vector of a graded poset. It is known that the cone of the flag f-vectors of CL-shellable posets may be characterized by the set of inequalities $h_S^{n+1}(P) \geq 0$ for all $S \subseteq [1,n]$. For the generalized $h_{\mathcal{I}}(P)$ invariants it is necessarily true that they are nonnegative integers, since they count the number of certain maximal chains – we will show in section 3 that for CL-labelings we are counting the same sets of chains as in [5]. It depends on the special class of posets and labelings considered whether these inequalities may be used to describe all inequalities for the flag f-vectors of posets in the class. Two "miracles" are necessary for this to happen:

1. we must be able to express the $h_{\mathcal{I}}(P)$'s in terms of the flag f vector (independently of the FA-labeling used), and
2. we must have a sufficiently large set of posets in the class showing that no more inequalities hold for all flag f-vectors than the ones implied by the inequalities $h_{\mathcal{I}}(P) \geq 0$.

Up to now, lexicographically shellable posets were the only known class with a special labeling satisfying the above two properties. In section 4 we show a new class of graded posets with a similarly fortunate labeling: the class of graded planar posets. As in the case of the "traditional" flag h-vector, the FA-labeling we introduce for planar posets will induce a planar analogue of the flag h-vector that can be defined for any graded poset, although in this generality it will not be always non-negative.

3 CL-shellable posets

In this section we show that for CL-labelings our results on FA-labelings specialize to those of Björner and Wachs in [5, 6]. We also show how the $\mathcal{I}$-generalized flag h-vector specializes to an invariant that does not depend on the labeling, but only encodes the flag f-vector.

Given an arbitrary chain-edge labeling $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$, and a maximal chain $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}$, the set

$$D(C) := \{i \in [1,n] \ : \ \lambda(C(x_{i-1}, x_i)) \not< \lambda(C, (x_i, x_{i+1}))\}$$

is called the *descent set of* C.

Proposition 2 *Let $\lambda : \mathcal{ME}(P) \longrightarrow \Lambda$ be a CL-labeling. Then every maximal chain C satisfies*

$$\mathcal{I}_C^+ = \mathcal{I}_{D(C)}^+$$

where $\mathcal{I}_{D(C)}$ denotes the interval system $\{\{i\} \ : \ i \in D(C)\}$.

Proof: Let us fix a maximal chain $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}$. For every $i \in D(C)$ the element x_i is not the first atom of the rooted interval $[x_{i-1}, x_{i+1}]_{C(i-1)}$, and so we have $\psi(C, i) = i$, implying $\{i\} \in \mathcal{I}_C$. Therefore $\mathcal{I}_{D(C)}$ is a subset of $\mathcal{I}_C$.

We only need to show that every $[i, j] \in \mathcal{I}_C$ contains some $\{k\} \in \mathcal{I}_{D(C)}$, *i.e.*, $[i, j] \cap D(C) \neq \emptyset$. By the definition of $\mathcal{I}_C$, the element x_i is not the first atom in $[x_{i-1}, x_{j+1}]_{C(i-1)}$, and so the saturated chain $x_{i-1} \prec x_i \prec \cdots \prec x_{j+1}$ is not the unique rising chain in $[x_{i-1}, x_{j+1}]_{C(i-1)}$ with respect to the labeling $\lambda_{C(i-1)}$. By the definition of the CL-labeling, this means that there is a $k \in [i, j]$ such that $\lambda_{C(i-1)}(C \cap [x_i, x_{j+1}], (x_{k-1}, x_k)) \not< \lambda_{C(i-1)}(C \cap [x_i, x_{j+1}], (x_k, x_{k+1}))$, and so $\lambda(C, (x_{k-1}, x_k)) \not< \lambda(C, (x_k, x_{k+1}))$ holds. This k belongs to $D(C)$. $\diamond$

As a consequence, [6, Theorem 5.8] follows from Theorem 7.

Corollary 1 *Given a CL-labeling of P, and an antichain of intervals $\mathcal{I}$, we have $h_{\mathcal{I}}(P) = 0$ unless $\mathcal{I}$ is of the form*

$$\mathcal{I}_S = \{\{s\} : s \in S\} \quad \textit{for some } S \subseteq [1, n].$$

Introducing

$$h_S(P) := h_{\mathcal{I}_S}(P)$$

we may write (3) as

$$f_S(P) = \sum_{T \subseteq S} h_T(P). \tag{4}$$

As a consequence of the sieve formula, this system of equalities is equivalent to

$$h_S(P) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T(P). \tag{5}$$

Corollary 2 *For CL-labelings the resulting $\mathcal{I}$-generalized flag h-vector depends only on the flag f-vector, and not on the particular choice of the labeling.*

Using Corollary 2 we may define the operators

$$h_S^{n+1} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_T^{n+1} \tag{6}$$

for every graded partially ordered set; they will form an alternative basis of A_{n+1}. The vector $(h_S(P) : S \subseteq [1, n])$ is what is usually defined in the literature as the *flag h-vector of a graded partially ordered set.*

Corollary 3 *The usual flag h-vector of a CL-shellable poset is nonnegative.*

It is known that the non-negativity of the usual flag h-vector implies all linear inequalities that hold for the flag f-vectors of all graded posets with a CL-labeling (see *e.g.* [17, §III.4] for a discussion of the Cohen-Macaulay case). One way to state this fact is the following.

Theorem 2 *The set K_{n+1}^{CL} of all linear expressions $F := \sum_{T\subseteq[1,n]} a_T f_T^{n+1} \in A_{n+1}$ such that $F(P) \geq 0$ for all CL-shellable posets, is a pointed polyhedral cone. Its extreme rays are the vectors $h_S^{n+1} \in A_{n+1}$ defined by (6).*

Proof: We already know that the vectors h_S^{n+1} belong to A_{n+1}, *i.e.*, they depend only on the operators f_T^{n+1} and not on the particular choice of the CL-labeling, and that they belong to K_{n+1}^{CL}. Since they form a basis of A_{n+1}, every $F \in K_{n+1}^{\mathrm{CL}}$ may be uniquely written in the form

$$F = \sum_{S\subseteq[1,n]} b_S h_S^{n+1}.$$

We only need to show that every coefficient b_S has to be nonnegative.

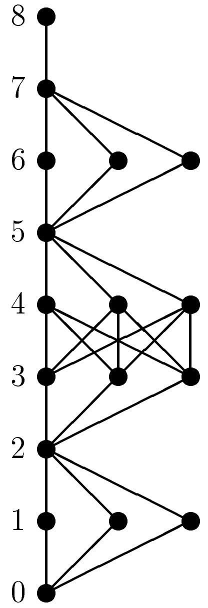

Figure 2: $P(8, \mathcal{I}_{\{1,3,4,6\}}, 3)$

Observe that the posets $P(n, \mathcal{I}, N)$ are CL-shellable if $\mathcal{I}$ is of the form $\mathcal{I} = \mathcal{I}_S$. Hence, every expression $F \in K_{n+1}^{\mathrm{CL}}$ is nonnegative on these graded posets. By letting N go to infinity we obtain

$$\varepsilon_{\mathcal{I}_S}^{n+1}(F) \geq 0 \quad \text{for all } S \subseteq [1,n] \text{ and } F \in K_{n+1}^{\mathrm{CL}}.$$

It is easy to verify that we have

$$\varepsilon_{\mathcal{I}_T}^{n+1}(h_S^{n+1}) = \delta_{S,T},$$

where $\delta_{S,T}$ is the Kronecker delta function. Therefore we have

$$b_T = \varepsilon_{\mathcal{I}_T}^{n+1}\left(\sum_{S \subseteq [1,n]} b_S h_S^{n+1}\right) \geq 0.$$

$\diamond$

Note that the proof of Theorem 2 also implies the following

Theorem 3 *The closure of the cone generated by the flag f-vectors of all graded CL-shellable posets of rank $n+1$ is an orthant. Its extreme rays may be given as*

$$\lim_{N \longrightarrow \infty} \frac{1}{N^{|T|}} \left(f_S(P(n, \mathcal{I}_T, N)) \ : \ S \subseteq [1,n]\right)$$

where T may be any subset of $[1,n]$.

4 Graded planar posets

In this section we discuss the class of graded *planar* posets, for which there is a natural FA-labeling given by the planar embedding. Planar posets have been studied by Collins as to their relationship to shellability [8], and they have arisen in the study of universal algebra [10], motion planning [16], poset dimension and book numbers of graphs [1] [14] and representation theory of the symmetric group [11]. Using FA-labelings, the theory of planar posets parallels that of CL-shellable posets. Besides giving the inequalities for a widely studied class of posets, it raises the question of whether there are other classes of posets for which a naturally defined FA-labeling leads to the complete description of the cone generated by their flag f-vectors.

Definition 11 *A poset P is* planar *if its Hasse diagram can be drawn in the plane with straight, non-crossing edges, such that whenever $y \in P$ covers $x \in P$, the vertex representing y appears above the vertex representing x.*

Remark 1 A graded planar partially ordered set is always a lattice, see [13].

Definition 12 *An antichain of intervals* $\mathcal{I} = \{[a_1, b_1], \ldots, [a_k, b_k]\}$, *where* $a_1 < \cdots < a_k$ *and* $b_1 < \cdots < b_k$, *is* sparse *if for every* $i \in [1, k-1]$ *we have* $a_{i+1} \geq b_i + 2$.

Example 1 The antichain of intervals $\{[1, 2], [4, 5]\}$ is sparse, while $\{[1, 2], [3, 5]\}$ is not.

There is a natural way of defining a bijection between sparse interval systems on $[1, n]$, and subsets of $[1, n]$. Given $S \subseteq [1, n]$ there is a unique sparse interval system $\mathcal{I}[S] = \{[a_1, b_1], \ldots, [a_k, b_k]\}$ satisfying $S = \bigcup_{i=1}^{k} [a_i, b_i]$. In fact, the only way to construct $\mathcal{I}[S]$ is to group the elements of S into the longest possible intervals. Conversely, given a sparse interval system $\mathcal{I} = \{[a_1, b_1], \ldots, [a_k, b_k]\}$, the only $S \subset [1, n]$ satisfying $\mathcal{I} = \mathcal{I}[S]$ is $S = S = \bigcup_{i=1}^{k} [a_i, b_i]$.

Proposition 3 *Let P be a graded planar poset. Then there is a chain-edge labeling with the first atom property such that for every maximal chain C the minimal elements of $\mathcal{I}_C$ form a sparse antichain of intervals.*

Proof: Let us draw the Hasse diagram of P so that it satisfies the conditions set in the definition of planarity. For every rooted interval $[x, y]_r$ let $\phi([x, y]_r)$ be the leftmost atom of the interval $[x, y]$. We want to show that for this labeling $[i, j], [k, l] \in \mathcal{I}_C$ and $i + 1 \leq k \leq j + 1$ imply $[k, l] \subseteq [i, j]$. Assume by way of contradiction the contrary, *i.e.*, the existence of $[i, j], [k, l] \in \mathcal{I}_C$ for some maximal chain C such that $i + 1 \leq k \leq j + 1$ and $l \geq j + 1$ hold. Let us denote the elements of C by $x_1 \prec \cdots \prec x_n$. By definition, $[i, j] \in \mathcal{I}_C$ means that x_i is the first atom in $[x_{i-1}, x_j]$ but not the first atom in $[x_{i-1}, x_{j+1}]$. Hence there is a path $x'_{i-1} = x_{i-1}, x'_i \neq x_i, x'_{i+1} \neq x_{i+1}, \ldots, x'_j \neq x_j, x'_{j+1} = x_{j+1}$ from x_{i-1} to x_{j+1} that is left of the path $x_{i-1}, x_i, \ldots, x_j, x_{j+1}$. Similarly, $[k, l] \in \mathcal{I}_C$ implies the existence of a path $x''_{k-1} = x_{k-1}, x''_k \neq x_k, \ldots, x''_l \neq x_l, x''_{l+1} = x_{l+1}$ on the left hand side of C. (See Figure 3.) Since the Hasse diagram of P contains no crossing edges, $l \geq j + 1$ implies that the two paths intersect in some $x'_m = x''_m$ for some $m \in [k, \min(j + 1, l)]$. But then we obtain $x_k \neq \phi([x_{k-1}, x_{j+1}])$ since x''_k is left from x_k and it belongs to the interval $[x_{k-1}, x_{j+1}])$ because of the existence of the path

$$x_{k-1} \prec x''_k \prec x''_{k+1} \prec \cdots \prec x''_m = x'_m \prec x'_{m+1} \prec \cdots \prec x'_j \prec x_{j+1}.$$

Hence we have reached a contradiction. $\diamond$

Definition 13 *The FA-labeling used in the proof of Proposition 3 is called the* FA-labeling induced by the planar embedding of P.

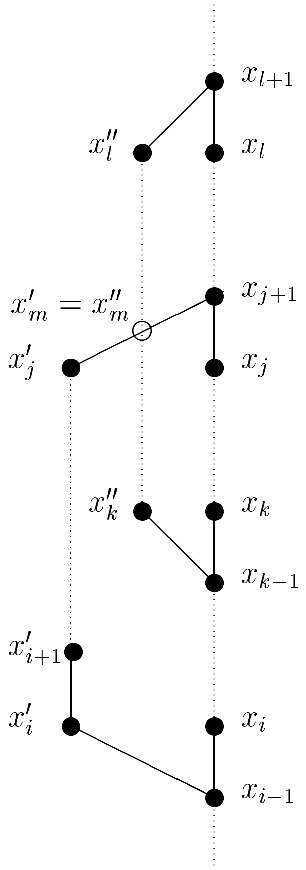

Figure 3: Intersecting paths in the Hasse-diagram

Corollary 4 *Let P be a planar graded poset, and consider an FA-labeling induced by a planar embedding. Then for every antichain of intervals $\mathcal{I}$ we have $h_{\mathcal{I}}(P) = 0$ unless $\mathcal{I}$ is sparse,* i.e., *$\mathcal{I} = \mathcal{I}[S]$ for some $S \subseteq [1, n]$.*

Definition 14 *Given a planar graded poset P and an FA-labeling induced by its drawing into the plane, we introduce the notation*

$$p_S(P) := h_{\mathcal{I}[S]}(P).$$

We call the vector $(p_S(P) \,:\, S \subseteq [1, n])$ the planar flag h-vector *of P.*

Using this notation, (3) may be written as follows.

$$f_S(P) = \sum_{\substack{T \subseteq [1,n] \\ S \in \mathbf{B}_{[1,n]}(\mathcal{I}[T])}} p_T(P). \tag{7}$$

By analogy with Corollary 2, we will use this system of linear equations to show that the numbers $p_T(P)$ depend only on the flag f-vector of P, and not on the particular choice of the drawing of P into the plane. For this purpose, we need a peculiar analogue of the sieve formula.

Definition 15 *Given a set $S \subseteq [1,n]$, we call*

$$\mathrm{N}(S) := \{t \in [1,n] \setminus S \,:\, \{t-1, t+1\} \cap S \neq \emptyset\}$$

the neighborhood *of S.*

In other words, $\mathrm{N}(S)$ consists of those elements of $[1,n]$ which do not belong to S, but at least one of their neighbors does. Using this notion we have the following analogue of the sieve formula.

Lemma 2 ("Planar sieve formula") *Let $u : \mathrm{P}([1,n]) \longrightarrow \mathbb{R}$ and $v : \mathrm{P}([1,n]) \longrightarrow \mathbb{R}$ be functions, assigning a value to every subset of $[1,n]$. Then the system of equalities*

$$u(S) = \sum_{\substack{T \subseteq [1,n] \\ S \in \mathbf{B}_{[1,n]}(\mathcal{I}[T])}} v(T)$$

is equivalent to the system

$$v(S) = (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S}} (-1)^{|T|} u(T).$$

Proof: It is sufficient to prove that the first system of equation implies the second, since that will imply that the $2^n \times 2^n$ matrix connecting the vector $(u(S) \,:\, S \subseteq [1,n])$ with the vector $(v(S) \,:\, S \subseteq [1,n])$ is invertible.

Assume that the first system holds and consider

$$v^*(S) := (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S}} (-1)^{|T|} u(T).$$

We obtain that

$$\begin{aligned} v^*(S) &= (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S}} (-1)^{|T|} \sum_{\substack{R \subseteq [1,n] \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[R])}} v(R) \\ &= (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{R \subseteq [1,n]} v(R) \sum_{\substack{T \subseteq [1,n] \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[R])}} (-1)^{|T|}. \end{aligned}$$

First we show that the coefficient of $v(R)$ in $v^*(S)$ is zero unless $S \subseteq R$. Suppose that there is an element $s \in S \setminus R$. This element is not contained in any of the intervals of $\mathcal{I}[R]$, hence $T \subseteq [1,n] \setminus \{s\}$ blocks $\mathcal{I}[R]$ if and only if $T \cup \{s\}$ blocks $\mathcal{I}[R]$. Similarly, since $\mathrm{N}(S)$ and S are disjoint, $T \subseteq [1,n] \setminus \{s\}$ is a set containing $\mathrm{N}(S)$ and contained in $\mathrm{N}(S) \cup S$ if and only if the same holds for $T \cup \{s\}$. Thus the coefficient of $v(R)$ is

$$(-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \setminus \{s\} \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[R])}} \left((-1)^{|T|} + (-1)^{|T \cup \{s\}|}\right) = 0$$

in this case. Hence we may assume $S \subseteq R$. This also means that every interval of the sparse system $\mathcal{I}[S]$ is contained in some interval of $\mathcal{I}[R]$. We may ignore the case when some $I \in \mathcal{I}[R]$ does not contain any interval of $\mathcal{I}[S]$ since otherwise $T \cap I \neq \emptyset$ implies $T \not\subseteq N(S) \cup S$ so the coefficient of $v(S)$ is an empty sum. Hence in the case $S \neq R$ we may assume that some interval $[a,b]$ of $\mathcal{I}[S]$ is shorter than the containing interval $[a',b']$ of $\mathcal{I}[R]$. Without loss of generality we may assume $[a, b+1] \subseteq R$. Then $b+1 \in \mathrm{N}(S)$ blocks the interval of $[a',b']$ and so a subset $T \subset [1,n]$ containing $\mathrm{N}(S)$ and not containing a blocks $\mathcal{I}[R]$ if and only if $T \cup \{a\}$ does. Similarly T is contained in $\mathrm{N}(S) \cup S$ if and only if $T \cup \{a\}$ is. Thus for a proper superset R of S we obtain that the coefficient of $v(R)$ is

$$(-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \setminus \{a\} \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[R])}} \left((-1)^{|T|} + (-1)^{|T \cup \{a\}|}\right) = 0.$$

Therefore the only potentially nonzero term in the expression for $v^*(S)$ is $v(S)$, and its coefficient is

$$(-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T \subseteq [1,n] \\ \mathrm{N}(S) \subseteq T \subseteq \mathrm{N}(S) \cup S \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[S])}} (-1)^{|T|}.$$

Since no element of $\mathrm{N}(S)$ blocks any interval of $\mathcal{I}[S]$, we may replace T with $T \setminus \mathrm{N}(S)$ and obtain that the same coefficient is

$$(-1)^{|\mathcal{I}[S]|} \sum_{\substack{T \subseteq S \\ T \in \mathbf{B}_{[1,n]}(\mathcal{I}[S])}} (-1)^{|T|} = (-1)^{|\mathcal{I}[S]|} \prod_{I \in \mathcal{I}[S]} \left(\sum_{\substack{T \subseteq I \\ T \neq \emptyset}} (-1)^{|T|} \right) = (-1)^{|\mathcal{I}[S]|} \cdot (-1)^{|\mathcal{I}[S]|} = 1.$$

◇

Corollary 5 *The planar flag h-vector of a planar poset P is independent of the planar embedding of P and is given by the formula*

$$p_S(P) = (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T\subseteq[1,n] \\ \mathrm{N}(S)\subseteq T\subseteq \mathrm{N}(S)\cup S}} (-1)^{|T|} f_T(P). \tag{8}$$

As in with the case of CL-shellable posets we may use Corollary 5 to introduce the chain operators

$$p_S^{n+1} := (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T\subseteq[1,n] \\ \mathrm{N}(S)\subseteq T\subseteq \mathrm{N}(S)\cup S}} (-1)^{|T|} f_T^{n+1} \tag{9}$$

as an alternative basis for A_{n+1}. Following Corollary 3 we may state

Corollary 6 *The planar flag h-vector of a planar poset is nonnegative.*

Finally, we have the following analogue of Theorem 2.

Theorem 4 *The set K_{n+1}^{PL} of all linear expressions $F := \sum_{T\subseteq[1,n]} a_T f_T^{n+1} \in A_{n+1}$ such that $F(P)\geq 0$ for all planar posets, is the pointed polyhedral cone given by the inequalities $\sum_{S\in\mathbf{B}_{[1,n]}(\mathcal{I})} a_S \geq 0$, for all sparse antichains of intervals $\mathcal{I}$ on $[1,n]$. Its extreme rays are the vectors $p_S^{n+1}\in A_{n+1}$ defined by (9).*

Proof: By analogy with the CL-shellable case, we observe that for a sparse antichain of intervals $\mathcal{I}$ on $[1,n]$, the partially ordered sets $P(n,\mathcal{I},N)$ are planar graded posets. Hence their planar h-vector is nonnegative. By letting N go to infinity we obtain

$$\varepsilon_{\mathcal{I}[S]}^{n+1}(F)\geq 0 \quad \text{for all } S\subseteq[1,n] \text{ and } F\in K_{n+1}^{\mathrm{PL}}.$$

Again, it is easy to verify that we have

$$\varepsilon_{\mathcal{I}[T]}^{n+1}(p_S^{n+1}) = \delta_{S,T},$$

where $\delta_{S,T}$ is the Kronecker delta function, and so we may reach the same conclusion. $\diamond$

As in the CL-shellable case the proof of Theorem 2 gives a description of the cone of flag f-vectors.

Theorem 5 *The closure of the cone generated by the flag f-vectors of all graded planar posets of rank $n+1$ is an orthant. Its extreme rays may be given as*

$$\lim_{N\longrightarrow\infty} \frac{1}{N^{|\mathcal{I}[T]|}} \left(f_S(P(n,\mathcal{I}[T],N)) \,:\, S\subseteq[1,n]\right)$$

where T may be any subset of $[1,n]$.

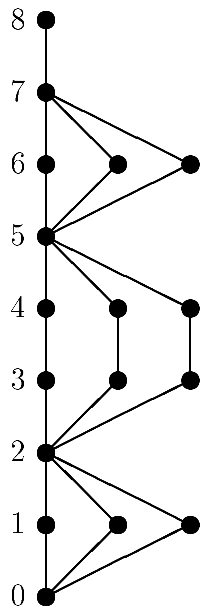

Figure 4: $P(8,\mathcal{I}[\{1,3,4,6\}],3)$

5 Planar and CL-shellable graded posets

In this section we describe the cone of flag f-vectors of those graded posets, which are simultaneously planar and CL-shellable. In [8] it is shown that for planar partially ordered sets the Cohen-Macaulay property is equivalent to the existence of a CL-labeling. Hence the content of this section provides also the description of the cone of flag f-vectors of graded planar Cohen-Macaulay posets.

Using the formulas connecting the flag f-vector with the two flag h-vectors, we first determine the coefficients connecting the planar and the usual flag h-vector.

Proposition 4 *For any $S \subseteq [1,n]$, $p_S^{n+1} = (-1)^{|S|+|\mathcal{I}(S)|} \sum_{T\subseteq \mathrm{N}(S)} h_{S\cup T}^{n+1}$.*

Proof: Using (7) and (4) we obtain

$$\begin{aligned}
p_S^{n+1} &= (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T\subseteq[1,n]\\ \mathrm{N}(S)\subseteq T\subseteq \mathrm{N}(S)\cup S}} (-1)^{|T|} f_T^{n+1} = (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{\substack{T\subseteq[1,n]\\ \mathrm{N}(S)\subseteq T\subseteq \mathrm{N}(S)\cup S}} (-1)^{|T|} \sum_{R\subseteq T} h_R^{n+1} \\
&= (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{R\subseteq \mathrm{N}(S)\cup S} h_R^{n+1} \sum_{\substack{T\subseteq[1,n]\\ \mathrm{N}(S)\cup R\subseteq T\subseteq \mathrm{N}(S)\cup S}} (-1)^{|T|} \\
&= (-1)^{|\mathcal{I}[S]|+|\mathrm{N}(S)|} \sum_{R\subseteq \mathrm{N}(S)\cup S} h_R^{n+1} \delta_{\mathrm{N}(S)\cup R, \mathrm{N}(S)\cup S} (-1)^{|\mathrm{N}(S)\cup S|},
\end{aligned}$$

which, using that fact that $\mathrm{N}(S)$ and S are disjoint, is easily seen to be equivalent to the stated formula. $\diamond$

Definition 16 *We say that a subset S of $[1,n]$ is* sparse *if no two consecutive integers belong to S simultaneously.*

In other words, S is sparse if and only if every interval in $\mathcal{I}[S]$ has length one, or if $\mathcal{I}_S$ is a sparse interval system.

Proposition 5 *If P is a planar CL-shellable graded poset and S is not sparse, then $h_S(P) = 0$.*

Proof: Assume first that $|S| + |\mathcal{I}[S]|$ is odd. Then, by Proposition 4, $p_S(P)$ is the sum of $-h_S(P)$ and of other terms of the form $-h_{S\cup T}(P)$. By Corollaries 3 and 6, $p_S(P)$ is nonnegative, while the terms $-h_S(P)$ and $-h_{S\cup T}(P)$ are nonpositive. The only way that both sides can be equal is all terms involved being zero.

We are left with the case when $|S| + |\mathcal{I}[S]|$ is even, and S is not sparse. Let a be the left end of the first interval in $\mathcal{I}[S]$ which is longer than 1. Then $\mathcal{I}[S \setminus \{a\}]$ has the same number of intervals as $\mathcal{I}[S]$, so $|S \setminus \{a\}| + |\mathcal{I}[S \setminus \{a\}]|$ is odd. Again by Proposition 4, $p_{S\setminus\{a\}}(P)$ is the sum of nonpositive entries. Since a is in the neighborhood of $S \setminus \{a\}$, the term $h_S(P)$ occurs on the right hand side of the expression. Both sides being necessarily equal to zero, we get $h_S(P) = 0$. $\diamond$

Proposition 6 *For planar CL-shellable graded posets the planar flag h-vector is equal to the usual flag h-vector.*

Proof: When we add to $S \subseteq [1,n]$ an element from its neighborhood, the resulting set is surely not sparse. Hence, by Proposition 5, we have $h_{S\cup T}(P) = 0$ for every nonempty subset T of $\mathrm{N}(S)$, and the formula given in Proposition 4 simplifies to $p_S(P) = (-1)^{|S|+|\mathcal{I}[S]|} h_S(P)$. When S is not sparse, both sides equal zero. When S is sparse, then $|\mathcal{I}[S]| = |S|$, and so $p_S(P) = h_S(P)$. $\diamond$

Since, by Propositions 5 and 6, some nontrivial linear equalities hold for the cone of inequalities holding for the flag f-vectors of graded planar CL-shellable posets, the analogue of Theorems 2 and 4 is not true. We can still state, however, the analogue of Theorems 3 and 5:

Theorem 6 *The closure of the cone generated by the flag f-vectors of all graded planar CL-shellable posets of rank $n+1$ is an orthant. Its extreme rays may be given as*

$$\lim_{N\longrightarrow\infty}\frac{1}{N^{|T|}}\left(f_S(P(n,\mathcal{I}_T,N))\ :\ S\subseteq[1,n]\right)$$

where T may be any sparse subset of $[1,n]$.

Proof: The argument can be more easily presented by passing to (the usual) flag h-vectors, and describing the cone generated by all vectors $(p_S(P)\ :\ S\subseteq[1,n])$, where P runs over all planar CL-shellable posets of rank $n+1$. By Corollary 3 and Proposition 5, we know that all vectors of this cone have non-negative coordinates, and zero at every coordinate indexed with a non-sparse subset of $[1,n]$. On the other hand, for a sparse subset T of $[1,n]$ we have $\mathcal{I}_T=\mathcal{I}[T]$, the graded posets $P(n,\mathcal{I}_T,N)=P(n,\mathcal{I}[T],N)$ are simultaneously planar and CL-shellable, and the vector

$$\lim_{N\longrightarrow\infty}\frac{1}{N^{|T|}}(h_S(P(n,\mathcal{I}_T,N))\ :\ S\subseteq[1,n])$$

has coordinate 1 for $S=T$ and zero elsewhere. Hence the cone in question is the cone generated by these vectors. $\diamond$

Corollary 7 *The dimension of the vector space generated by the flag f-vectors of graded planar CL-shellable posets is a Fibonacci number.*

6 "Weak shellings" induced by FA-labelings

The following theorem generalizes [6, Theorem 5.8].

Theorem 7 *Let P be a graded poset of rank $n+1$ and $\lambda:\mathcal{ME}(P)\longrightarrow\Lambda$ an FA-labeling, and let $C_1,C_2,\ldots,C_m$ be an enumeration of the maximal chains of P which extends the lexicographic partial order on their labels. Then for an $S\subseteq[1,n]$ and for any set $\{y_1,\ldots,y_{|S|}\}\subset P$ satisfying $\{\mathrm{rank}(y_1),\ldots,\mathrm{rank}(y_{|S|})\}=S$, the first maximal chain $C:\hat{0}=x_0\prec x_1\prec\cdots\prec x_{n+1}=\hat{1}$ containing $\{y_1,\ldots,y_{|S|}\}$ is given by*

$$x_j=\begin{cases} y_k \text{ where } \mathrm{rank}(y_k)=j & \text{if } j\in S,\\ \phi\left(\left[x_{j-1},x_{M_S(j)}\right]_{C(j-1)}\right) & \text{if } j\not\in S.\end{cases}\tag{10}$$

Proof: Note first that the chain C given by (10) is well defined: $x_0=\hat{0}$ and, assuming we know $x_0,\ldots,x_{j-1}$, the element x_j is uniquely determined by (10). Let $C':\hat{0}=x'_0\prec x'_1\prec\cdots\prec x'_{n+1}=\hat{1}$

be an arbitrary maximal chain satisfying $x'_j \in \{y_1, \ldots, y_{|S|}\}$ whenever $j \in S$. If C' is different from C, then there is a smallest $i \in \{1, 2, \ldots, n\}$ such that $x_i \neq x'_i$. This i cannot be an element of S since C and C' agree at ranks belonging to S. The element x_i is the first atom in the rooted interval $\left[x_{i-1}, x_{M_S(i)}\right]_{C(i-1)}$ and so we have

$$\lambda_{C(i-1)}(C, (x_{i-1}, x_i)) < \lambda_{C'(i-1)}(C', (x'_{i-1}, x'_i))$$

by the first atom property, implying that the string of labels associated to C is less than the one associated to C' in the lexicographic partial order. $\diamond$

As in the penultimate paragraph of the proof of [2, Propostition 2.8] we may conclude:

Corollary 8 *If we enumerate the maximal chains of P as in Theorem 7, at the moment of adding the facet $C : \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_{n+1} = \hat{1}$, exactly those faces $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\Delta(P)$ occur the first time, for which $\{i_1, \ldots, i_k\}$ blocks $\mathcal{I}_C$.*

This corollary motivates our interest in the following class of simplicial complexes.

Definition 17 *Given an interval system $\mathcal{I}$ on $[1, n]$ we call the simplicial complex*

$$\Delta_{\mathcal{I}} := \{S \subseteq [1, n] \ : \ \exists I \in \mathcal{I} \text{ such that } S \cap I = \emptyset\}$$

the interval-generated simplicial complex induced by $\mathcal{I}$.

As a consequence, $\Delta_{\mathcal{I}} = \mathrm{P}([1, n]) \setminus \mathbf{B}_{[1,n]}(\mathcal{I})$.

Remark 2 Here we see the reason while we defined the empty family and the empty face alone to be simplicial complexes. Evidently, $\Delta_{\emptyset} = \emptyset$ and $\Delta_{\{[1,n]\}} = \{\emptyset\}$ hold.

Interval-generated simplicial complexes are closed under taking the nerve complex.

Lemma 3 *If Δ may be represented as an interval-generated simplicial complex induced by an interval system, then so does its nerve complex $\mathcal{N}(\Delta)$.*

Proof: Assume $\Delta = \Delta_{\mathcal{I}}$, where $\mathcal{I}$ is an interval system on $[1, n]$. Observe first that the facets of $\Delta_{\mathcal{I}}$ are those sets $[1, n] \setminus I$ for which $I \in \mathcal{I}$ is a minimal interval of $\mathcal{I}$. Without loss of generality we may assume that $\mathcal{I}$ consists only of these intervals, *i.e.*, $\mathcal{I} = \{[a_1, b_1], \ldots, [a_m, b_m]\}$, where $a_1 < \cdots < a_m$,

$b_1 < \cdots < b_m$, and $a_i \leq b_i$ $(i = 1, \ldots, m)$ hold. This numbering of the minimal intervals induces a linear order on the facets of $\Delta_{\mathcal{I}}$, *i.e.*, on the vertices of $\mathcal{N}(\Delta)$. Let us set $F_i = [1, n] \setminus [a_i, b_i]$.

Assume now $\{F_{i_1}, \ldots, F_{i_k}\}$ is a facet of $\mathcal{N}(\Delta_{\mathcal{I}})$. It is sufficient to show that the set $\{i_1, \ldots, i_k\}$ is the complement of an interval contained in $[1, m]$. For this purpose it is sufficient to show that whenever $j < j'$ are not in $\{i_1, \ldots, i_k\}$ then none of the elements of the interval $[j, j']$ belongs to $\{i_1, \ldots, i_k\}$.

Since $\{F_{i_1}, \ldots, F_{i_k}\}$ is a facet of $\mathcal{N}(\Delta_{\mathcal{I}})$, we have $F_{i_1} \cap \cdots F_{i_k} \neq \emptyset$, and $l \notin \{i_1, \ldots, i_k\}$ holds if and only if

$$F_l \cap F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset.$$

By passing to complements we obtain $[a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}] \neq [1, n]$, and $l \notin \{i_1, \ldots, i_k\}$ holds if and only if

$$[a_l, b_l] \cup [a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}] = [1, n],$$

or equivalently

$$[a_l, b_l] \supseteq [1, n] \setminus \left([a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}]\right).$$

Hence if j and j' are both outside of $\{i_1, \ldots, i_k\}$ then we have

$$[a_j, b_j] \supseteq [1, n] \setminus \left([a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}]\right) \quad \text{and} \quad [a_{j'}, b_{j'}] \supseteq [1, n] \setminus \left([a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}]\right).$$

Given any $j'' \in [j, j']$ the interval $[a_{j''}, b_{j''}]$ necessarily contains $[a_j, b_j] \cap [a_{j'}, b_{j'}]$, and so it also contains $[1, n] \setminus \left([a_{i_1}, b_{i_1}] \cup \cdots \cup [a_{i_k}, b_{i_k}]\right)$. Therefore $j'' \notin \{i_1, \ldots, i_k\}$. $\diamond$

In the proof of Lemma 3 we may also observe the following: the number of vertices of $\mathcal{N}(\Delta)$ is less than the number of vertices of Δ, unless we started with one of the following situations:

(i) $\Delta = \emptyset$,

(ii) $\Delta = \{\emptyset\}$,

(iii) Δ may be represented as $\Delta_{\{\{1\},\{2\},\ldots,\{n\}\}}$ on $[1, n]$ for some $n \geq 2$.

Note that case (i) is "isolated" in the sense that the nerve complex of a nonempty family of faces will always contain at least the empty set as a face, and will thus be not the empty family. In case (ii), Δ is homeomorphic to a ball, in case (iii) it is homeomorphic to a sphere. Repeated use of Lemma 3 and Proposition 1 yields the following.

Theorem 8 *Every interval-generated simplicial complex is contractible or homotopy equivalent to a sphere.*

It was pointed out to the authors by A. Björner that Theorem 8 is equivalent to a result of Kahn [12] on the homotopy type of *interval-generated lattices*, rediscovered independently by Linusson [15]. Actually, the first proof of [15, Theorem 4.1] uses an idea which is equivalent to Lemma 3, modulo the correspondence outlined below.

Definition 18 *Let $\mathcal{I}$ be a family of intervals of $[1,n]$. The interval-generated lattice $L(\mathcal{I})$ is the family of all subsets of $[1,n]$, ordered by inclusion, which may be written as a union of some intervals from the family I.*

Interval-generated lattices were first studied by Greene who proved in [11] that their Möbius functions only take on values 0, 1 or -1. Kahn generalized Greene's result in [12] by showing that the order complex of interval-generated lattices is contractible or homotopy equivalent to a sphere. Björner and Wachs have independently reproved this theorem in a special case in [7], while Linusson gave two new proofs of Kahn's result in [15].

The connection between interval-generated lattices and simplicial complexes is the following. When $\mathcal{I}$ is an antichain of intervals, $\mathcal{N}(\Delta_{\mathcal{I}})$ is isomorphic to the *crosscut complex* generated by the atoms of $L(\mathcal{I})$, *i.e.*, to the simplicial complex whose faces are those subsets of atoms whose join is not the maximum element of $L(\mathcal{I})$. By the homotopy crosscut theorem (see [4], Theorem 10.8), any crosscut complex of $L(\mathcal{I})$ is homotopy equivalent to the order complex of $L(\mathcal{I})$. Thus Kahn's theorem is *equivalent* to our Theorem 8. Since this correspondence establishes only homotopy equivalence between the order complexes of interval-generated lattices and interval-generated simplicial complexes, for stronger (*e.g.*, homeomorphy) results, the direct study of interval-generated simplicial complexes seems to be inevitable. For example, in the case of CL-shellable partially ordered sets we have seen that every interval system occurring was an antichain of singletons, and the corresponding interval generated complexes were homeomorphic to balls or spheres of maximum dimension. For planar posets we obtained a labeling for which the interval systems in question consist of disjoint intervals with gaps in between them, and the corresponding interval-generated simplicial complexes were cones over a nonempty simplex. These more special and more specific results can not be translated into the language of interval-generated lattices via the above correspondence.

Since by (2) only one chain satisfies $\mathcal{I}_C^+ = \emptyset$, we have the following corollary of Theorem 7. For a face F in a simplicial complex, we denote the subcomplex of its faces by $\overline{F}$.

Corollary 9 *The order complex of every graded poset has an enumeration of facets $F_1, F_2, \ldots, F_m$ such that for every $i \in \{2, \ldots m\}$ the complex $\overline{F_i} \cap \bigcap_{j=1}^{i-1} \overline{F_j}$ is homotopy equivalent to a ball or to a sphere.*

Remark 3 The only chain satisfying $\mathcal{I}_C^+ = \emptyset$ will necessarily be the facet F_1 in the enumeration.

We call such a decomposition a *weak shelling* of the order complex. We would get an actual shelling if $\overline{F_i}\cap\bigcap_{j=1}^{i-1}\overline{F_j}$ was not only homotopy equivalent but also homeomorphic to a ball or sphere of dimension $\dim F_i - 1$. For a definition of shelling we refer the reader to [6].

Let us note finally that Corollary 9 implies that *any* partially ordered set with a rank function has a weak shelling. In fact, any such poset may be completed to a graded poset Q by adding a unique minimum element $\hat{0}$, a unique maximum element $\hat{1}$, by adding to each rank k minimal element x of P a unique path of new elements $x_1 \prec \cdots \prec x_{k-1}$ connecting x to $\hat{0}$ and by adding to each rank k maximal element y of P a unique path of new elements $y_{k+1} \prec \cdots \prec \hat{1}$ connecting y to $\hat{1}$. The restriction of a weak shelling of Q to P induces a weak shelling of P.

Appendix

For completeness sake we put on record here the formulas connecting the flag f-vector with the two flag h-vectors, and the equivalent formulations of Theorem 1 in different bases. Some of these formulas were useful in finding our results but not used in the shortest proofs. Being aware of them may develop the reader's intuition about the subject and lead to new results.

A.1 Basis transformations

We have already seen the formulas (4) and (5) connecting the flag f-vector and the flag h-vector. Using these formulas we may show that the connection formulas for

$$F = \sum_{S\subseteq[1,n]} a_S f_S^{n+1} = \sum_{S\subseteq[1,n]} b_S h_S^{n+1}$$

are

$$b_S = \sum_{\substack{T\subseteq[1,n]\\ T\supseteq S}} a_T \quad \text{and} \quad a_S = \sum_{\substack{T\subseteq[1,n]\\ T\supseteq S}} (-1)^{|T\setminus S|} b_T. \tag{11}$$

Similarly, (7) and (8) yield for

$$F = \sum_{S\subseteq[1,n]} a_S f_S^{n+1} = \sum_{S\subseteq[1,n]} c_S p_S^{n+1}$$

the formulas

$$c_S = \sum_{\substack{T\subseteq[1,n]\\ T\in\mathbf{B}_{[1,n]}(\mathcal{I}[S])}} a_T \quad \text{and} \quad a_S = (-1)^{|S|} \sum_{\substack{T\subseteq[1,n]\\ \mathrm{N}(T)\subseteq S\subseteq \mathrm{N}(T)\cup T}} (-1)^{|\mathcal{I}[T]|+|\mathrm{N}(T)|} c_T. \tag{12}$$

Let us see finally the formulas connecting the two flag h-vectors. The expression in Proposition 4 giving the planar flag h-vector in terms of the usual one may be rewritten as

$$p_S^{n+1} = (-1)^{|S|+|\mathcal{I}[S]|} \sum_{\substack{T \\ S \subseteq T \subseteq S \cup \mathrm{N}(S)}} h_T^{n+1}. \tag{13}$$

Proposition 7 *We have $h_\emptyset^{n+1} = p_\emptyset^{n+1} = 1$ and*

$$h_S^{n+1} = (-1)^{|S|} \sum_{\substack{T \\ T \supseteq S}} \hat{\chi}\left(\Delta_{\mathcal{I}[T]} \cap \mathrm{P}(S)\right) p_T^{n+1} \quad \textit{for } S \neq \emptyset.$$

The proof consists of composing (5) with (7). Using Propositions 4 and 7 we may show that for

$$F = \sum_{S \subseteq [1,n]} b_S h_S^{n+1} = \sum_{S \subseteq [1,n]} c_S p_S^{n+1},$$

we have $c_\emptyset = b_\emptyset$ and, for $S \neq \emptyset$,

$$b_S = \sum_{\substack{T \subseteq [1,n] \\ T \subseteq S \subseteq T \cup \mathrm{N}(T)}} (-1)^{|T|+|\mathcal{I}[T]|} c_T \quad \text{and} \quad c_S = \sum_{\substack{T \subseteq S \\ T \neq \emptyset}} (-1)^{|T|} \hat{\chi}\left(\Delta_{\mathcal{I}[S]} \cap \mathrm{P}(T)\right) b_T. \tag{14}$$

A.2 Equivalent forms of Theorem 1

Using (11), for an interval system $\mathcal{I}$ on $[1,n]$ we may transform $\sum_{S \in \mathbf{B}_{[1,n]}(\mathcal{I})} a_S$ as follows:

$$\begin{aligned} \sum_{S \in \mathbf{B}_{[1,n]}(\mathcal{I})} a_S &= \sum_{S \in \mathbf{B}_{[1,n]}(\mathcal{I})} \sum_{\substack{T \subseteq [1,n] \\ T \supseteq S}} (-1)^{|T \setminus S|} b_T = \sum_{T \subseteq [1,n]} (-1)^{|T|} b_T \sum_{\substack{S \subseteq T \\ S \in \mathbf{B}_{[1,n]}(\mathcal{I})}} (-1)^{|S|} \\ &= \sum_{T \subseteq [1,n]} (-1)^{|T|} b_T \left(\sum_{S \subseteq T} (-1)^{|S|} - \sum_{\substack{S \subseteq T \\ \exists I \in \mathcal{I}(S \cap I = \emptyset)}} (-1)^{|S|} \right). \end{aligned}$$

For $T = \emptyset$, the first sum in the brackets is 1, and the second is 0 whenever $\mathcal{I} = \emptyset$ and 1 otherwise. For nonempty T, the first sum is 0, and the second is the reduced Euler characteristic of the restriction of the simplicial complex $\Delta_\mathcal{I}$ to the vertex set T. Therefore we have the following reformulation of Theorem 1.

Theorem 9 *An expression $\sum_{S \subseteq [1,n]} b_S h_S^{n+1}$ is nonnegative on all graded posets of rank $n+1$ if and only if $b_\emptyset \geq 0$, and for every nonempty interval system $\mathcal{I}$ on $[1,n]$ we have*

$$\sum_{\substack{S \subseteq [1,n] \\ S \neq \emptyset}} (-1)^{|S|} \hat{\chi}\left(\Delta_\mathcal{I} \cap \mathrm{P}(S)\right) b_S \geq 0. \tag{15}$$

Similarly, using (12) we may prove

Theorem 10 *An expression* $\sum_{S\subseteq[1,n]} c_S p_S^{n+1}$ *is nonnegative on all graded posets of rank* $n+1$ *if and only if* $c_\emptyset \geq 0$ *and for every nonempty interval system* $\mathcal{I}$ *on* $[1,n]$ *we have*

$$\sum_{\substack{S\subseteq[1,n]\\ \mathrm{N}(S)\in\Delta_{\mathcal{I}}, S\neq\emptyset}} (-1)^{|\mathcal{I}[S]|}\hat{\chi}\left(\mathrm{link}_{\mathrm{N}(S)}(\Delta_{\mathcal{I}})\cap \mathrm{P}(S)\right) c_S \geq 0. \tag{16}$$

Proof: We have

$$\begin{aligned}
\sum_{S\in\mathbf{B}_{[1,n]}(\mathcal{I})} a_S &= \sum_{S\in\mathbf{B}_{[1,n]}(\mathcal{I})} (-1)^{|S|} \sum_{\substack{T\subseteq[1,n]\\ \mathrm{N}(T)\subseteq S\subseteq \mathrm{N}(T)\cup T}} (-1)^{|\mathcal{I}[T]|+|\mathrm{N}(T)|} c_T \\
&= \sum_{T\subseteq[1,n]} (-1)^{|\mathcal{I}[T]|+|\mathrm{N}(T)|} c_T \sum_{\substack{S\in\mathbf{B}_{[1,n]}(\mathcal{I})\\ \mathrm{N}(T)\subseteq S\subseteq \mathrm{N}(T)\cup T}} (-1)^{|S|}.
\end{aligned}$$

For $\mathcal{I}=\emptyset$ the coefficient of c_T is zero unless $T=\emptyset$, which has coefficient 1. For nonempty $\mathcal{I}$ the coefficient of $c_\emptyset$ is zero and we may write

$$\sum_{S\in\mathbf{B}_{[1,n]}(\mathcal{I})} a_S = \sum_{\substack{T\subseteq[1,n]\\ T\neq\emptyset}} (-1)^{|\mathcal{I}[T]|+|\mathrm{N}(T)|} c_T \left(\sum_{\substack{S\subseteq[1,n]\\ \mathrm{N}(T)\subseteq S\subseteq \mathrm{N}(T)\cup T}} (-1)^{|S|} - \sum_{\substack{S\in\Delta_{\mathcal{I}}\\ \mathrm{N}(T)\subseteq S\subseteq \mathrm{N}(T)\cup T}} (-1)^{|S|} \right).$$

The first sum in between the brackets is zero, and the second is not zero only if $\mathrm{N}(T)$ is a face of $\Delta_{\mathcal{I}}$. In that case the second sum is $(-1)^{|\mathrm{N}(T)|}$ times the reduced Euler characteristic of the simplicial complex $\mathrm{link}_{\mathrm{N}(T)}(\Delta_{\mathcal{I}})\cap \mathrm{P}(T)$. ◇

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