# Noncommutative Enumeration in Graded Posets

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**Abstract**

We define a noncommutative algebra of flag-enumeration functionals on graded posets and show it to be isomorphic to the free associative algebra on countably many generators. Restricted to Eulerian posets, this ring has a particularly appealing presentation with kernel generated by Euler relations. A consequence is that even on Eulerian posets, the algebra is free, with generators corresponding to odd jumps in flags. In this context, the coefficients of the **cd**-index provide a graded basis.

## 1 Flag enumeration in graded posets

A natural setting for the study of enumeration of flags of faces in polytopes is the family of ranked posets, that is, posets $P$ such that for any $x \in P$, every maximal chain $a < b < \cdots < x$ has the same number of elements. We assume for convenience that each such $P$ is *graded*, that is, has a unique minimal element $\widehat{0}$ and maximal element $\widehat{1}$. Thus if $x \in P$ has a maximal chain

$$\widehat{0} = x_0 < x_1 < \cdots < x_k = x,$$

we say that $x$ has *rank* $k$, denoted $r(x) = k$ (and so $r(\widehat{0}) = 0$). Further, we define the rank of $P$ to be $r(P) := r(\widehat{1})$. For a graded poset P of rank $d+1$ and a subset $S \subset \{1, \ldots, d\} =: [d]$, we denote by $f_S(P)$ the number of flags (*i.e.*, chains) in $P$ having elements with precisely the ranks in $S$. Note that the ranks 0 and $d+1$ are not included here. When $S = \{i, j, \ldots, k\}$, we will often write $f_{i,j,\ldots,k}$ or $f_{ij\cdots k}$ for $f_{\{i,j,\ldots,k\}}$. The function $S \mapsto f_S(P)$ is often called the *flag f-vector* of $P$.

As a principal example, we can take $P$ to be the lattice of all faces of a $d$-polytope. Here $f_S(P)$ denotes the number of flags of faces with ranks in $S$, where the rank of a face is one more than its dimension, and $r(P) = d+1$.

We can view $f_S$ as a *chain operator* on graded posets. We write $f_S^n$ as the operator applied to posets of rank $n$, with the convention that $f_S^n(P) = 0$ if $r(P) \neq n$. We think of $f_\emptyset^n$, which counts the empty chain, as an operator in its own right, different for each value of $n$, and different from the number 1, even though it takes that value when applied to any rank $n$ poset. Chain operators for a fixed rank were studied in [3], where all the *linear* relations were derived for the operators $f_S^{d+1}$, $S \subset [d]$, restricted to face lattices of $d$-polytopes or, more generally, to all rank $d+1$ Eulerian posets. (See also [2].) We consider this case later.

We first show that there are no linear relations holding for *all* graded posets. By a *form* in chain operators we mean any linear combination $F = \sum_S \alpha_S f_S^n$ with rational coefficients, where $n$ is fixed.

**Proposition 1.1** *As operators on the family of all graded posets of rank $n$, the chain operators $f_S^n$ are linearly independent over* $\mathbf{Q}$.

*Cornell University, Ithaca, NY 14853-7901. Supported in part by NSF Grants DMS-9207700 and 9500581 and by ARO contract DAAL03-91-C-0027.

*Proof:* Suppose the form $F = \sum \alpha_S f_S^n$ vanishes on all graded posets. If $F \neq 0$, choose $T \subset [n-1]$ of maximum cardinality such that $\alpha_T \neq 0$ and consider the rank $n$ poset $P_T^N$, having (in addition to $\widehat{0}$ and $\widehat{1}$) $N$ elements of each rank $i \in T$ and one element of each other rank $k \in [n-1]$, where $x < y$ whenever $r(x) < r(y)$. Then $f_S^n(P_T^N) = N^{|T \cap S|}$, so $F(P_T^N)$ is a polynomial in $N$ of degree $|T|$ having $\alpha_T$ as its leading coefficient. That $F$ vanishes identically on graded posets implies that $\alpha_T = 0$, a contradiction. $\diamond$

There are, however, certain nonlinear relations holding among the operators $f_S^n$. For $k \in S \subset [n-1]$, define $S_{[k]} := S \cap [k-1]$ and $S^{[k]} := \{i - k \mid i \in S, i > k\}$. If $P$ is a poset and $x \in P$, then define the lower and upper intervals defined by $x$ to be $P_x := \{y \in P \mid y \leq x\}$ and $P^x := \{y \in P \mid x \leq y\}$, respectively. We get the following straightforward relations.

**Proposition 1.2** *Let $P$ be a rank $d$ poset and $S \subset [d-1]$. Then for any $i \in S$, we have*

$$f_S^d(P) = \sum_{x \,:\, r(x) = i} f_{S_{[i]}}^i(P_x) f_{S^{[i]}}^{d-i}(P^x). \tag{1.1}$$

Based on (1.1), we define the *convolution* of two chain operators by

$$f_S^d f_T^e = f_{S \cup \{d\} \cup (T+d)}^{d+e} \tag{1.2}$$

where $T + d := \{i + d \mid i \in T\}$. This product was first defined by Kalai [13], as a means of generating new linear inequalities for flag $f$-vectors of convex polytopes, and studied further by Meisinger [15].

We call a family $\mathcal{P}$ of graded posets *hereditary* if it is closed under taking intervals, that is, if $P \in \mathcal{P}$, then for any $x \leq y$ in $P$, we have $P_y$, $P^x$ and therefore $[x, y] := P^x \cap P_y$ are also in $\mathcal{P}$. There are three main hereditary families that will concern us here: all graded posets, all face lattices of polytopes and all Eulerian posets. Recall that a graded poset is said to be *Eulerian* if its Möbius function $\mu$ satisfies $\mu(x, y) = (-1)^{r(y) - r(x)}$ for every pair $x \leq y$. See [19] for general background in this area.

We say a linear form on chain operators $F^d = \sum \alpha_S f_S^d$ is *nonnegative on* $\mathcal{P}$, denoted $F^d \geq 0$, if for any $P \in \mathcal{P}$, $F^d(P) = \sum \alpha_S f_S^d(P) \geq 0$. We can similarly define nonpositivity, equality and so on. The following proposition follows easily from Proposition 1.2. In the case $\mathcal{P}$ is the family of all face posets of polytopes, the first assertion is due to Kalai [13, Lemma 6.1].

**Proposition 1.3** *For any hereditary family $\mathcal{P}$ of graded posets,*

*(i) the convolution of two nonnegative forms is nonnegative, and*

*(ii) the convolution of a zero form with any form is a zero form.*

*If $\mathcal{P}$ is the family of all graded posets, then the convolution of two nonzero forms is nonnegative if and only if both are nonnegative or both are nonpositive.*

Proposition 1.3 gives us a systematic way of generating new linear conditions from others already known to hold. It shows that the zero forms form an "ideal" in some sense, while the nonnegative forms form a "multiplicative cone". For Eulerian posets, we will see that this ideal of zero forms is generated by Euler relations.

While Proposition 1.1 shows there are no nontrivial zero forms on all graded posets, there are nontrivial nonnegative forms on the entire family. For example, if $S \supset T$ then $f_S - f_T \geq 0$ for every graded poset. Less trivial perhaps is the inequality $f_{13} - f_1 - f_3 + f_2 \geq 0$ that can be seen to hold for any graded poset of rank at least 4. In [10], the cone of all nonnegative forms on the family of all graded posets is described by giving the (finite) minimal set of generators for the closure of the cone spanned by flag $f$-vectors of all graded posets. Finding such descriptions for the families of all Eulerian posets and all face lattices of polytopes or hyperplane arrangements remain interesting open problems. (See [6, 8, 9] for recent results along these lines.) From the point of view of the algebras discussed in this paper, it is interesting to note that in the case of all graded posets, the minimal generating set (that is, the set of *extreme rays*) for the cone of nonnegative forms is closed under convolution, except for a well-defined set of excluded factors (see [10, Theorem 3]).

In §2, we study the algebra structure on the set of chain operators on graded posets and see it to be the free associative algebra $A = \mathbf{Q}\langle y_1, y_2, \ldots\rangle$ on countably many generators. In §3, we restrict to the family of all Eulerian posets and see that the algebra $A_{\mathcal{E}}$ of chain operators on these, a quotient of $\mathbf{Q}\langle y_1, y_2, \ldots\rangle$, is itself isomorphic to the free associative algebra $\mathbf{Q}\langle y_1, y_3, y_5, \ldots, y_{2k+1}, \ldots\rangle$. While much of the rest of the paper can be viewed as a survey, in this algebraic setting, of enumeration theory for Eulerian posets, this result is new. Its proof bears on algorithmic issues involving the verification of relations in $A_{\mathcal{E}}$. In §4, we study the components of the flag $h$-vector as elements in $A$ and $A_{\mathcal{E}}$, deriving recursive relations between flag $h$-vectors of different ranks. This leads to a derivation of the Dehn-Sommerville relations for the flag $h$-vectors as identities in $A_{\mathcal{E}}$ and a relatively simple derivation of the coefficients of the **cd**-index of Eulerian posets as elements of $A_{\mathcal{E}}$. The section concludes with a discussion of the toric $h$-vector. Section 5 contains a description of $A$ as the dual algebra to a coalgebra and a discussion of related module structures associated to posets of simplicial, cubical and simple polyhedra.

The development here extends that of [14]. We are grateful to Richard Ehrenborg, Mike Stillman and Moss Sweedler for helpful discussions on various parts of this research, and to Clara Chan, Vesselin Gasharov, Gábor Hetyei and Margaret Readdy for useful comments on earlier versions of this paper.

## 2 The algebra of chain operators

In this section we let $\mathcal{P}$ be the family of all graded posets. Throughout, we let $\mathbf{Q}$ be the field of rational numbers. For each $n > 0$, define the vector space over $\mathbf{Q}$ of forms on chain operators

$$A_n := \left\{ \sum_{S \subset [n-1]} \alpha_S f_S^n \mid \alpha_S \in \mathbf{Q} \right\}.$$

We have by Proposition 1.1 that $\dim A_n = 2^{n-1}$. We set $A_0 = \mathbf{Q}$. The graded vector space $A := \bigoplus_{n \geq 0} A_n$ can be made into a (noncommutative) graded $\mathbf{Q}$-algebra by means of the convolution product (1.2). We see first that this algebra has a particularly simple description.

**Theorem 2.1** *As graded algebras $A \cong \mathbf{Q}\langle y_1, y_2, \ldots\rangle$, the free graded associative algebra on generators $y_i$, where the degree of $y_i$ is taken to be $i$.*

*Proof:* Define an algebra map

$$\varphi : \mathbf{Q}\langle y_1, y_2, \ldots\rangle \longrightarrow A \tag{2.1}$$

by $\varphi(y_j) = f_\emptyset^j$. By repeated use of (1.2), we can easily see that

$$\begin{aligned} f_{i_1 i_2 \ldots i_k}^n &= f_\emptyset^{i_1} f_\emptyset^{i_2 - i_1} \ldots f_\emptyset^{i_k - i_{k-1}} f_\emptyset^{n - i_k} && (2.2)\\ &= \varphi(y_{i_1} y_{i_2 - i_1} \cdots y_{i_k - i_{k-1}} y_{n - i_k}), && (2.3) \end{aligned}$$

so the chain operators $f_\emptyset^j$ generate all chain operators by convolution (*cf.* [15, Theorem 3.2]) and thus $\varphi$ is onto. That it is one-to-one is a direct consequence of Proposition 1.1. Homogeneity is clear. $\diamond$

We henceforth will consider the algebra $A$ to be identical with the free associative algebra $\mathbf{Q}\langle y_1, y_2, \ldots\rangle$, interchangeably using expressions like $y_i y_j$ for $f_i^{i+j}$ and $y_i y_j(P)$ for $f_i^{i+j}(P)$, and identifying the set $\{f_S^n | S \subset$

$[n-1]\}$ with the set $M_n := \{y_{i_1} y_{i_2} \cdots y_{i_k} | k \geq 0, i_1+i_2+\cdots+i_k = n\}$ of all degree $n$ monomials in $y_1, \ldots, y_n$. For completeness, we take $M_0 := \{1\}$. Note that $|M_n| = 2^{n-1}$ when $n \geq 1$.

We consider first the effect of poset duality on the algebra $A$. Recall that for a poset $P$, the dual poset $P^*$ is defined as having the same underlying set with the reverse partial order, so that $x \leq_{P^*} y$ if and only if $y \leq_P x$. Define the involution $F \mapsto \overline{F}$ on $A$ by $\overline{y_{i_1} \cdots y_{i_k}} = y_{i_k} \cdots y_{i_1}$. It is easy to check that for $F \in A$, $F(P^*) = \overline{F}(P)$. Define $A^i = \{F \in A | F = (-1)^i \overline{F}\}$, $i = 0, 1$, and note that $A = A^0 \oplus A^1$. We call elements of $A^0$ and $A^1$ *symmetric* and *antisymmetric*, respectively. Symmetric forms are precisely those taking the same value on $P$ and $P^*$.

**Proposition 2.2** *For $n \geq 1$, the subspace $A^0_n$ of degree $n$ symmetric forms on all graded posets has dimension $2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$.*

*Proof:* For any $F \in A$, we have $F + \overline{F} \in A^0$, $F - \overline{F} \in A^1$ and $2F = (F + \overline{F}) + (F - \overline{F})$. Thus if $M^0_n$ is the set of symmetric elements of $M_n$, then the set $M^0_n \cup \{m + \overline{m} \mid m \in M_n \setminus M^0_n\}$ is a basis for $A^0_n$, and so

$$\dim A^0_n = |M^0_n| + \frac{1}{2}(2^{n-1} - |M^0_n|) = 2^{n-2} + \frac{|M^0_n|}{2}.$$

It suffices to show that $|M^0_n| = 2^{\lfloor n/2 \rfloor}$. To this end, suppose $m \in M_n$ and $m = \overline{m}$. Then $m = u y_i \overline{u}$, where $u \in M_j$, for some $j = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ with $n = 2j + i$. Thus $|M^0_n| = 1 + 1 + 2 + \cdots + 2^{\lfloor n/2 \rfloor - 1} = 2^{\lfloor n/2 \rfloor}$. $\diamond$

**Example 2.3** For $n = 4$, $\dim A^0_4 = 6$ and

$$\begin{aligned} A^0_4 &= \operatorname{span}\{y_4, y_1y_2y_1, y_1^4, y_2^2, y_1y_3 + y_3y_1, y_1y_1y_2 + y_2y_1y_1\} \\ &= \operatorname{span}\{f^4_\emptyset, f^4_{13}, f^4_{123}, f^4_2, f^4_1 + f^4_3, f^4_{12} + f^4_{23}\}. \end{aligned} \tag{2.4}$$

# 3 Enumeration in Eulerian posets

When $\mathcal{P}$ is a proper hereditary family of graded posets then for a form $F \in A$, it may be that $F(P) = 0$ for all posets $P \in \mathcal{P}$. We let $A_\mathcal{P}$ denote the set of operators in $A$ *considered as operators on* $\mathcal{P}$. Thus the map

$$\varphi_\mathcal{P} : \mathbf{Q}\langle y_1, y_2, \ldots \rangle \longrightarrow A_\mathcal{P} \tag{3.1}$$

as in (2.1) may have a nontrivial kernel,

$$I_\mathcal{P} := \{f \in \mathbf{Q}\langle y_1, y_2, \ldots \rangle \mid f(P) = 0 \text{ for all } P \in \mathcal{P}\}.$$

By Proposition 1.3, $I_\mathcal{P}$ is a (two-sided) ideal in $\mathbf{Q}\langle y_1, y_2, \ldots \rangle$, the ideal of *vanishing forms* on $\mathcal{P}$. By the convention that $f^d_S(P) = 0$ if $r(P) \neq d$, $I_\mathcal{P}$ is a homogeneous ideal. Thus the $\mathbf{Q}$-algebra $A_\mathcal{P}$ of all forms on $\mathcal{P}$ is isomorphic to $\mathbf{Q}\langle y_1, y_2, \ldots \rangle / I_\mathcal{P}$ as a graded algebra. When $\mathcal{P}$ is the family of all graded posets, we have $I_\mathcal{P} = 0$.

Henceforth, we restrict our attention to the family $\mathcal{E}$ of Eulerian posets and certain subfamilies. We denote the ideal of vanishing forms on all Eulerian posets by $I_\mathcal{E}$ and the algebra of forms $\mathbf{Q}\langle y_1, y_2, \ldots \rangle / I_\mathcal{E}$ by $A_\mathcal{E}$. Our first task is to describe the ideal $I_\mathcal{E}$. To this end, we recall the so-called *generalized Dehn-Sommerville relations* for Eulerian posets [3, Theorem 2.1].

**Theorem 3.1** *Given a rank $d$ Eulerian poset $P$ and subset $S \subseteq \{1, \ldots, d-1\}$, if $\{i, k\} \subseteq S \cup \{0, d\}$, $i < k$, and $S$ contains no $j$ such that $i < j < k$, then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f^d_{S \cup j}(P) = f^d_S(P)(1 - (-1)^{k-i-1}). \tag{3.2}$$

The relations (3.2) are shown in [3] to generate all linear relations on chain operators over the class of face posets of polytopes, and thus over all Eulerian posets.

When $S = \emptyset$, $i = 0$ and $k = d$, (3.2) reduces to the *Euler relation* for Eulerian posets of rank $d$,

$$f_\emptyset^d - f_1^d + f_2^d - \cdots + (-1)^{d-1} f_{d-1}^d + (-1)^d f_\emptyset^d = 0, \tag{3.3}$$

which, with $f_i^d = f_\emptyset^i f_\emptyset^{d-i} = y_i y_{d-i}$, can be written as

$$y_d - y_1 y_{d-1} + y_2 y_{d-2} - \cdots + (-1)^{d-1} y_{d-1} y_1 + (-1)^d y_d = 0. \tag{3.4}$$

For $d \geq 1$, we define $\chi_d := \sum_{i+j=d} (-1)^i y_i y_j$, where the sum is over all $i, j \geq 0$ and we set $y_0 = 1$ for convenience. This is the element of $A$ corresponding to the rank $d$ Euler relation. We call $\chi_d$ the $d^{th}$ *Euler form*. Note that $\chi_d$ is homogeneous of degree $d$. For example, $\chi_1 = 0$, $\chi_2 = 2y_2 - y_1^2$ and $\chi_3 = y_2 y_1 - y_1 y_2$.

The relations (3.2) were derived in [3] directly from the Euler relations for arbitrary intervals. This has been observed more explicitly in various forms in [13, §6, Remark 7] and [15, Proposition 3.3]. Perhaps the clearest way to state this dependence is the following.

**Proposition 3.2** *The two-sided ideal $I_\mathcal{E}$ of forms in $A$ vanishing on all Eulerian posets is generated by the Euler forms $\chi_d$, $d \geq 1$.*

*Proof:* This follows by noting that the relations (3.2) are all in the ideal generated by the relations (3.3). That is, in terms of convolution of chain operators, (3.2) can be written $f_{S_{[i]}}^i \chi_{k-i} f_{S^{[k]}}^{d-k} = 0$. ◇

We next show that only the even degree Euler forms are needed to generate $I_\mathcal{E}$. This leads to showing that $A_\mathcal{E}$ is essentially the free associative algebra generated by the $y_{2n+1}$, $n \geq 0$, which, in turn, allows an easy computation of its Hilbert series.

**Proposition 3.3** *For $n \geq 1$,*

$$\chi_{2n+1} = -\frac{1}{2} \sum_{i=1}^{2n} \left[\, y_i \; \chi_{2n+1-i} \;+\; (-1)^i \; \chi_{2n+1-i} \; y_i \,\right].$$

*Therefore, $I_\mathcal{E} = \langle \chi_d, d \geq 1 \rangle = \langle \chi_{2n}, n \geq 1 \rangle$.*

*Proof:* For $n \geq 1$, if $r$ is the expression on the right side, then

$$\begin{aligned}
-2r &= \sum_{\substack{i+\ell=2n+1 \\ 0<i<2n+1}} \Big( y_i \chi_\ell + (-1)^i \chi_\ell y_i \Big) \\
&= \sum_{\substack{i+j+k=2n+1 \\ i>0}} \Big( (-1)^j y_i y_j y_k + (-1)^{i+k} y_k y_j y_i \Big) \\
&= \sum_{\ell+k=2n+1} \Big( (-1)^\ell \Big( \chi_\ell - y_\ell \Big) y_k + (-1)^{k+\ell} y_k \Big( \chi_\ell - (-1)^\ell y_\ell \Big) \Big) \\
&= -2 \sum_{\ell+k=2n+1} (-1)^\ell y_\ell y_k \;=\; -2\chi_{2n+1},
\end{aligned}$$

where all indices are constrained to be nonnegative unless otherwise noted. ◇

Proposition 3.3 enables us to show that $A_\mathcal{E}$ is essentially a polynomial algebra in the odd-degree generators.

**Theorem 3.4** *There is a graded isomorphism of* $\mathbf{Q}$*-algebras*

$$A_{\mathcal{E}} \cong \mathbf{Q}\langle y_1, y_3, y_5, \ldots, y_{2k+1}, \ldots\rangle. \tag{3.5}$$

*Proof:* We think of the Euler form $\chi_d = \chi_d(y_1, \ldots, y_d)$ as a polynomial in the $y_i$ and recursively define the following homogeneous elements of $A$ by substitution:

$$q_2(y_1, y_2) = y_2 - \frac{1}{2}\chi_2(y_1, y_2)$$

$$q_4(y_1, y_2, y_3, y_4) = y_4 - \frac{1}{2}\chi_4(y_1, q_2, y_3, y_4)$$

$$\vdots$$

$$q_{2k}(y_1, y_2, \ldots, y_{2k}) = y_{2k} - \frac{1}{2}\chi_{2k}(y_1, q_2, y_3, q_4, \ldots, y_{2k-3}, q_{2k-2}, y_{2k-1}, y_{2k})$$

$$\vdots$$

We claim that $q_{2k}$ is a polynomial in the generators $y_1, y_3, \ldots, y_{2k-1}$ only. This is clear for $k = 1$. Supposing it is true for $k < n$, we can see $q_{2n}$ involves only $y_1, y_3, y_5, \ldots, y_{2n-1}, y_{2n}$, but the coefficient of $y_{2n}$ is zero. It follows from the definition that

$$I_{\mathcal{E}} = \langle \chi_{2k} \mid k \geq 1\rangle = \langle y_{2k} - q_{2k} \mid k \geq 1\rangle. \tag{3.6}$$

We define maps $\theta$ and $\psi$ between $\mathbf{Q}\langle y_1, y_3, y_5, \ldots, y_{2k+1}, \ldots\rangle$ and $A_{\mathcal{E}}$ as follows. The map

$$\theta : \mathbf{Q}\langle y_1, y_3, y_5, \ldots, y_{2k+1}, \ldots\rangle \hookrightarrow A \longrightarrow \mathbf{Q}\langle y_1, y_2, \ldots\rangle / I_{\mathcal{E}} = A_{\mathcal{E}},$$

is the restriction of the canonical projection $A \to A_{\mathcal{E}}$. Note that the map $A \to \mathbf{Q}\langle y_1, y_3, y_5, \ldots\rangle$ defined by $y_{2i+1} \mapsto y_{2i+1}$ and $y_{2i} \mapsto q_{2i}$ has $I_{\mathcal{E}}$ in its kernel by (3.6), and so it extends to a map

$$\psi : A_{\mathcal{E}} = \mathbf{Q}\langle y_1, y_2, \ldots\rangle / I_{\mathcal{E}} \longrightarrow \mathbf{Q}\langle y_1, y_3, y_5, \ldots\rangle.$$

It is clear that $\theta$ and $\psi$ are graded homomorphisms with $\psi \circ \theta = 1$ and $\theta \circ \psi = 1$. ◇

As noted earlier $I_{\mathcal{E}}$ is a homogeneous ideal; in fact, the elements $\chi_d$ are homogeneous generators. Thus $A_{\mathcal{E}} = \mathbf{Q}\langle y_1, y_2, \ldots\rangle / I_{\mathcal{E}}$ is a graded $\mathbf{Q}$-algebra, and as a vector space over $\mathbf{Q}$, $A_{\mathcal{E}} = \mathbf{Q} \oplus A_{\mathcal{E}_1} \oplus A_{\mathcal{E}_2} \oplus \ldots \oplus A_{\mathcal{E}_d} \oplus \ldots$, where $A_{\mathcal{E}_d}$ is the set of all degree $d$ elements of $A_{\mathcal{E}}$. The Hilbert function of $A_{\mathcal{E}}$ is now an easy computation. Recall the Fibonacci numbers $a_d$ defined by $a_1 = a_2 = 1$ and $a_i = a_{i-1} + a_{i-2}$, for $i \geq 3$.

**Corollary 3.5** *For* $d \geq 1$*, the dimension of* $A_{\mathcal{E}_d}$ *is* $a_d$*, the* $d$*-th Fibonacci number.*

*Proof:* We have graded maps

$$A_{\mathcal{E}_d} \hookrightarrow A_{\mathcal{E}} \cong \mathbf{Q}\langle y_1, y_3, \ldots\rangle$$

so by Theorem 3.4, $A_{\mathcal{E}_d}$ is isomorphic to the $d$-th graded part of $\mathbf{Q}\langle y_1, y_3, \ldots\rangle$. Thus a basis for $A_{\mathcal{E}_d}$ consists of all degree $d$ words in the odd-degree $y_i$. Each of these begins with $y_{2i-1}$, where $1 \leq i \leq k = \lfloor \frac{d+1}{2} \rfloor$. It follows that

$$A_{\mathcal{E}_d} = y_1 A_{\mathcal{E}_{d-1}} \oplus y_3 A_{\mathcal{E}_{d-3}} \oplus \cdots \oplus y_{2k-1} A_{\mathcal{E}_{d+1-2k}}.$$

Hence, for $d \geq 2$,

$$\begin{aligned}\dim_{\mathbf{Q}}(A_{\mathcal{E}_d}) &= \dim_{\mathbf{Q}}(A_{\mathcal{E}_{d-1}}) + \left(\dim_{\mathbf{Q}}(A_{\mathcal{E}_{d-3}}) + \cdots + \dim_{\mathbf{Q}}(A_{\mathcal{E}_{d+1-2k}})\right) \\ &= \dim_{\mathbf{Q}}(A_{\mathcal{E}_{d-1}}) + \dim_{\mathbf{Q}}(A_{\mathcal{E}_{d-2}}),\end{aligned}$$

while it is clear that $\dim_{\mathbf{Q}}(A_{\mathcal{E}_1}) = \dim_{\mathbf{Q}}(A_{\mathcal{E}_2}) = 1$. ◇

Using the generating function for the Fibonacci numbers [19], we obtain the Hilbert series for $A_{\mathcal{E}}$. For completeness, we let $a_0 = 1$.

**Corollary 3.6** *The Hilbert series for $A_{\mathcal{E}}$ is*

$$\sum_{d \geq 0} \dim_{\mathbf{Q}}(A_{\mathcal{E}}^d) t^d = \sum_{d \geq 0} a_d t^d = 1 + \frac{t}{1 - t - t^2}.$$

We call a subset $S \subset [d]$ *sparse* if $d \notin S$ and no $\{i, i+1\}$ is a subset of $S$. In [3], it was shown that the set of $f_S^{d+1}$, where $S$ is a sparse subset of $[d]$, is a basis for the flag $f$-vector of Eulerian posets, in the sense that any $f_T^{d+1}$ is uniquely a linear combination of these. A shortcoming of the sparse flag numbers is that they are not closed under the product on $A$; for example $f_1^3 f_1^3 = f_{134}^6$. However, Theorem 3.4 provides a multiplicative basis for the flag $f$-vectors of Eulerian posets. We say $S = \{i_1, \ldots, i_k\} \subset [d]$ has *odd jumps* if for $j = 0, 1, \ldots, k$, $i_{j+1} - i_j$ is odd, where we take $i_0 = 0$ and $i_{k+1} = d + 1$.

**Corollary 3.7** *The flag numbers $f_S^{d+1}$, where $S \subset [d]$ has odd jumps, form a multiplicatively closed basis for the vector space $A_{\mathcal{E}}$.*

As in the case of graded posets, we can consider the set of forms in $A_{\mathcal{E}}$ that are invariant under polarity. As before, we denote by $A_{\mathcal{E}}^0$ the subring of symmetric forms on Eulerian posets, and by $A_{\mathcal{E}_d}^0$ the subspace of those of degree $d$.

**Corollary 3.8** *For $d \geq 0$, the subspace $A_{\mathcal{E}_d}^0$ of symmetric forms of degree $d$ in $A_{\mathcal{E}}$ has dimension*

$$\dim_{\mathbf{Q}}(A_{\mathcal{E}_d}^0) = \begin{cases} \frac{a_d + a_k}{2} & \text{if } d = 2k \\ \frac{a_d + a_{k+1}}{2} & \text{if } d = 2k - 1. \end{cases}$$

*Proof:* Since $F \in I_{\mathcal{E}}$ if and only if $\overline{F} \in I_{\mathcal{E}}$ and, in fact, the Euler forms are themselves symmetric, the isomorphism (3.5) commutes with the involution $F \mapsto \overline{F}$. So it is enough to determine the dimensions of the symmetric graded components of $\mathbf{Q}\langle y_1, y_3, y_5, \ldots, y_{2k+1}, \ldots \rangle$. Denote by $\widetilde{M}_d$ and $\widetilde{M}_d^0$ the set of monomials, respectively, symmetric monomials, of degree $d$ in the odd-degree generators $y_1, y_3, y_5, \ldots$.

As before, we have

$$\dim A_{\mathcal{E}_d}^0 = |\widetilde{M}_d^0| + \frac{1}{2}(a_d - |\widetilde{M}_d^0|) = \frac{a_d + |\widetilde{M}_d^0|}{2},$$

so it suffices to show that $|\widetilde{M}_d^0|$ is equal to $a_k$ or $a_{k+1}$, depending on whether $d = 2k$ or $d = 2k - 1$. Indeed, if $m \in \widetilde{M}_d$ with $m = \overline{m}$ and $d = 2k$, then $m = u\overline{u}$ for $u \in \widetilde{M}_k$, and so $|\widetilde{M}_d^0| = |\widetilde{M}_k| = a_k$. On the other hand, if $d = 2k - 1$, then $m = u y_i \overline{u}$, where $i = 1, 3, 5, \ldots, 2k - 1$, $u \in \widetilde{M}_\ell$, and $2\ell + i = 2k - 1$. Thus $|\widetilde{M}_d^0| = \sum_{i=0}^{k-1} a_i = a_{k+1}$, by properties of the Fibonacci numbers. ◇

**Example 3.9** For $d = 4$, $\dim A^0_{\mathcal{E}_4} = \frac{1}{2}(a_4 + a_2) = 2$ and $\{y_1^4, y_1y_3 + y_3y_1\}$ (equivalently, $\{f^4_{123}, f^4_1 + f^4_3\}$) is a basis for $A^0_{\mathcal{E}_4}$. For $d = 5$, $\dim A^0_{\mathcal{E}_5} = \frac{1}{2}(a_5 + a_4) = 4$ and we have $\{y_1^5, y_1y_3y_1, y_5, y_1y_1y_3 + y_3y_1y_1\}$ (equivalently, $\{f^5_{1234}, f^5_{14}, f^5_{\emptyset}, f^5_{12} + f^5_{34}\}$) is a basis for $A^0_{\mathcal{E}_5}$.

Finally, it is of some interest to be able to determine quickly whether two expressions in $A = \mathbf{Q}\langle y_1, y_2, \ldots\rangle$ represent the same element in the algebra $A_{\mathcal{E}}$. This is equivalent to determining whether an element of $A$ lies in the Euler ideal $I_{\mathcal{E}}$. By Proposition 3.3 and the proof of Theorem 3.4, two elements $F, G \in A$ represent the same element of $A_{\mathcal{E}}$ if and only if they agree after the successive substitutions

$$y_{2i} \mapsto q_{2i} = y_{2i} - \frac{1}{2}\chi_{2i}, \; i = k, \ldots, 1, \tag{3.7}$$

where $y_{2k}$ is the largest even degree variable appearing in either $F$ or $G$. Thus the generators $\chi_{2n}$ constitute a *minimal* (noncommutative) *Gröbner basis* for the ideal $I_{\mathcal{E}}$ [22, §2.4]. The result of the substitutions (3.7) in an element $F \in \mathbf{Q}\langle y_1, y_2, \ldots\rangle$ is the *normal form* of $F$, an element of $\mathbf{Q}\langle y_1, y_3, \ldots\rangle$. Implicit in this is an underlying term order on words, which we take to be first by degree, then lexicographically among words of the same degree.

# 4 Flag $h$-vector, cd-index and toric $h$

In this section we discuss three enumerative invariants of Eulerian posets and derive some of their properties in the context of the algebra $A_{\mathcal{E}}$.

## 4.1 The flag $h$-vector

The flag $h$-vector is another numerical invariant of graded posets and polytopes (and more generally of balanced simplicial complexes [2, 17]). For a poset $P$ of rank $d$, and for $S \subset [d-1]$, define $h_S(P) := \sum_{T\subseteq S}(-1)^{|S|-|T|} f_T(P)$. The function $S \mapsto h_S(P)$, $S \subset [d]$, is called the *flag $h$-vector* of $P$. The relation between $f$ and $h$ can be inverted, and for all $S$, $f_S(P) = \sum_{T\subseteq S} h_T(P)$.

For polytopes, it follows from topological considerations [17] that the flag $h$-vector satisfies the relations $h_S = h_{\overline{S}}$. Since any relations holding for all polytopes must hold for all Eulerian posets [3], these relations must be consequences of the generalized Dehn-Sommerville equations (3.2) and so must be derivable as identities in the algebra $A_{\mathcal{E}}$.

As in the case of the flag $f$-vector, we can consider the flag $h$-vector to give operators on graded posets, and hence define elements of the algebra $A$. If $T = \{i_1, \ldots, i_k\} \subseteq [d-1]$, where $i_1 < \cdots < i_k$, we define $y^d_T := y_{i_1}y_{i_2-i_1}\cdots y_{d-i_k}$. We let $y^d_{\emptyset} := y_d$. Define the flag-$h$ operators by

$$h^d_S = \sum_{T\subseteq S}(-1)^{|S|-|T|} f^d_T = \sum_{T\subseteq S}(-1)^{|S|-|T|} y^d_T, \tag{4.1}$$

for all $S \subseteq [d-1]$.

For $S \subset [d]$, $k \in [d]$, recall $S_{[k]} = S \cap [k-1]$ and $S^{[k]} = \{i - k \mid i \in S, i > k\}$. Note that if $T = \{i_1, i_2, \ldots i_n\}$ and $y^d_T = y_{i_1}y_{i_2-i_1}\ldots y_{d-i_n} = y_{j_1}y_{j_2}\ldots y_{j_n}$, then $y^{i_k}_{T_{[i_k]}} = y_{j_1}\ldots y_{j_k}$ and $y^{d-i_k}_{T^{[i_k]}} = y_{j_{k+1}}\cdots y_{j_n}$.

We begin with the multiplicative formula for the flag-$h$ operators.

**Proposition 4.1** *If $S \subset [k-1]$ and $T \subset [l-1]$, then*

$$h^k_S h^l_T = h^{k+l}_{S\cup(T+k)} + h^{k+l}_{S\cup\{k\}\cup(T+k)}. \tag{4.2}$$

*Proof:* We show an equivalent equality: for $S \subset [d-1]$, $k \notin S$,

$$\begin{aligned} h^d_S + h^d_{S\cup k} &= (-1)^{|S|+1} \sum_{k\in T\subseteq S\cup k} (-1)^{|T|} f^d_T \\ &= \sum_{T_1\subseteq S_{[k]}} \sum_{T_2\subseteq S^{[k]}} (-1)^{|S_{[k]}|-|T_1|}(-1)^{|S^{[k]}|-|T_2|} f^k_{T_1} f^{d-k}_{T_2} \\ &= h^k_{S_{[k]}} h^{d-k}_{S^{[k]}}. \end{aligned}$$

◇

The following relation between $h^d_S$ and $h^k_T$, $k < d$, provides a useful inductive tool for studying the flag $h$-vector. It appears to be new. In the following, we let $h^0_\emptyset = y_0 = 1$ and denote by $\min(T)$ the least element in the nonempty set $T$.

**Theorem 4.2** *Let $S \subseteq [d-1]$. Then*

$$h^d_S = \sum_{i\in S\cup\{d\}} (-1)^{|S_{[i]}|}\ h^i_\emptyset\ h^{d-i}_{S^{[i]}} = \sum_{i\in S\cup\{0\}} (-1)^{|S^{[i]}|}\ h^i_{S_{[i]}}\ h^{d-i}_\emptyset.$$

*Proof:* From (4.1) we get

$$\begin{aligned} h^d_S &= (-1)^{|S|} y_d \ + \sum_{i\in S} y_i \sum_{T\subseteq S, \min(T)=i} (-1)^{|S|-|T|} y^{d-i}_{T^{[i]}} \\ &= (-1)^{|S|} y_d \ + \sum_{i\in S} (-1)^{|S_{[i]}|} y_i h^{d-i}_{S^{[i]}} \\ &= \sum_{i\in S\cup\{d\}} (-1)^{|S_{[i]}|} h^i_\emptyset h^{d-i}_{S^{[i]}} \end{aligned}$$

The second equality follows similarly. ◇

**Corollary 4.3** *For each $S \subseteq [d-1]$, $h^d_S = h^d_{\overline{S}}$, $\overline{S} = [d-1] \setminus S$, holds as an identity in $A_{\mathcal{E}}$.*

*Proof:* We proceed by induction on $d$ and on $|S|$. The statement is clear for $d = 1$. For $|S| = d-1$, it follows from Theorem 4.2 and by induction that

$$h^d_{[d-1]} = \sum_{j=1}^{d} (-1)^{j-1} y_j h^{d-j}_{[d-j-1]} \tag{4.3}$$

$$= \sum_{j=1}^{d} (-1)^{j-1} y_j h^{d-j}_\emptyset \tag{4.4}$$

$$= y_d - \chi_d = h^d_\emptyset. \tag{4.5}$$

For $d > 1$, $|S| < d-1$ and $k \notin S$, we show first that

$$h^d_S - h^d_{\overline{S}} = h^d_{\overline{S}\setminus k} - h^d_{S\cup k}. \tag{4.6}$$

Indeed, letting $T = \overline{S} \setminus k$, we have by Proposition 4.1

$$h_S^d + h_{S\cup k}^d = h_{S_{[k]}}^k h_{S^{[k]}}^{d-k}$$

and

$$h_T^d + h_{T\cup k}^d = h_{T_{[k]}}^k h_{T^{[k]}}^{d-k}.$$

Since $S_{[k]}$ and $T_{[k]}$ are complementary sets in $[k-1]$, we have by induction that $h_{S_{[k]}}^k = h_{T_{[k]}}^k$. Similarly, $h_{S^{[k]}}^{d-k} = h_{T^{[k]}}^{d-k}$, and therefore

$$h_S^d + h_{S\cup k}^d = h_T^d + h_{T\cup k}^d,$$

which gives (4.6). By induction, the right-hand side of (4.6) is zero, completing the proof. $\diamond$

## 4.2 The **cd**-index

The **cd**-index of an Eulerian poset $P$ is a noncommutative polynomial in two variables that provides an efficient encoding of the flag $f$-vector or flag $h$-vector of $P$. From the point of view of the algebra $A_{\mathcal{E}}$, it can be seen as giving an interesting graded basis.

Suppose $P$ is rank $d+1$ and $S \subseteq [d]$. Let $u_i = \mathbf{a}$ if $i \notin S$ and $u_i = \mathbf{b}$ if $i \in S$. Define $u_S = u_S^d = u_1 u_2 \dots u_d$. It was shown in [4] that if one considers the generating function

$$\Phi(P) = \sum_{S\subseteq[d]} h_S(P) u_S, \tag{4.7}$$

then this is a polynomial in $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ if and only if the generalized Dehn-Sommerville equations hold for $P$. Define $\deg(\mathbf{c}) = 1$, $\deg(\mathbf{d}) = 2$. We can write $\Phi(P)$ as

$$\Phi(P) = \sum_w \phi_w(P) w$$

where the summation runs over all degree $d$ monomials in $\mathbf{c}$ and $\mathbf{d}$; there are $a_{d+1}$ such monomials. We call this **cd**-polynomial $\Phi(P)$ the **cd**-*index* of $P$.

We derive the existence of the **cd**-index as a consequence of the Euler relations. One can consider the expression (4.7) as defining a polynomial $\Phi^n := \sum_{S\subset[n-1]} h_S^n u_S^{n-1}$ in $\mathbf{a}$ and $\mathbf{b}$ with coefficients in $\mathbf{Q}\langle y_1, y_2, \dots\rangle$, or in $A_{\mathcal{E}}$. Note that $\Phi^n$ is bihomogeneous, of degree $n$ in the $y_i$ and of degree $n-1$ in $\mathbf{a}$ and $\mathbf{b}$. If we let $A_{\mathcal{E}}\langle \mathbf{a}+\mathbf{b}, \mathbf{ab}+\mathbf{ba}\rangle = A_{\mathcal{E}}\langle \mathbf{c}, \mathbf{d}\rangle$ denote the ring of all (noncommutative) polynomials in $\mathbf{c}$ and $\mathbf{d}$ with coefficients in $A_{\mathcal{E}}$, then the existence of the **cd**-index is equivalent to the following.

**Theorem 4.4** *As a polynomial with coefficients in $A_{\mathcal{E}}$, $\Phi^n \in A_{\mathcal{E}}\langle \mathbf{c}, \mathbf{d}\rangle$.*

Before we prove the theorem, we note the straightforward identities,

$$\sum_{S\subset[k]} (-1)^{|S|} u_S^k = (\mathbf{a} - \mathbf{b})^k, \tag{4.8}$$

for any $k$, and

$$(\mathbf{a}-\mathbf{b})^k(\mathbf{b} + (-1)^k \mathbf{a}) = \begin{cases} (\mathbf{c}^2 - 2\mathbf{d})^{\frac{k}{2}} \mathbf{c}, & \text{if } k \text{ is even, and} \\ -(\mathbf{c}^2 - 2\mathbf{d})^{\frac{k+1}{2}}, & \text{if } k \text{ is odd.} \end{cases} \tag{4.9}$$

The second of these can be found in [20].

*Proof of Theorem 4.4:* The proof is by induction on $n$; note $\Phi^1 = y_1$ and $\Phi^2 = y_2(\mathbf{a}+\mathbf{b})$ are in $A_{\mathcal{E}}\langle \mathbf{c},\mathbf{d}\rangle$. Using Theorem 4.2 and (4.8) we can write

$$\begin{aligned}
\Phi^n &= \sum_{S\subseteq[n-1]} h_S^n u_S^{n-1} = \sum_{S\subseteq[n-1]} \left( \sum_{i\in S\cup\{n\}} (-1)^{|S_{[i]}|} y_i h_{S[i]}^{n-i} \right) u_S^{n-1} \\
&= \sum_{i=1}^{n-1} y_i \sum_{T\subseteq[i-1]} (-1)^{|T|} u_T^{i-1} \ \mathbf{b} \sum_{V\subseteq[n-i-1]} h_V^{n-i} u_V^{n-i-1} + \ y_n \sum_{S\subset[n-1]} (-1)^{|S|} u_S^{n-1} \\
&= \sum_{i=1}^{n} y_i \ (\mathbf{a}-\mathbf{b})^{i-1} \ \mathbf{b} \ \Phi^{n-i},
\end{aligned}$$

where we define $\mathbf{b}\Phi^0 = \mathbf{a}\Phi^0 = 1$ for convenience.

Since $h_S = h_{\overline{S}}$, we have $\Phi^n = \sum_S h_S u_S = \sum_S h_{\overline{S}} u_S = \sum_S h_S u_{\overline{S}}$. This means we can obtain another such formula for $\Phi^n$ by changing all $\mathbf{a}$'s into $\mathbf{b}$'s and vice-versa. Therefore,

$$\Phi^n = \sum_{i=1}^{n} y_i \ (-1)^{i-1} \ (\mathbf{a}-\mathbf{b})^{i-1} \ \mathbf{a} \ \Phi^{n-i}.$$

Adding, we get

$$2\Phi^n = \sum_{i=1}^{n} y_i \ (\mathbf{a}-\mathbf{b})^{i-1} \ (\mathbf{b}+(-1)^{i-1} \ \mathbf{a}) \ \Phi^{n-i}. \tag{4.10}$$

Thus, by (4.9) and the induction hypothesis, $\Phi^n \in A_{\mathcal{E}}\langle \mathbf{c},\mathbf{d}\rangle$. $\diamond$

This proof, while formally different than that of [20, Theorem 1.1] (which operates in the incidence algebra of a poset $P$), appears to cover similar ground. This is likely a consequence of the development in §5.1. We note that (4.10), which is essentially dual to [20, Equation (11)], can be used with (4.9) to compute **cd**-indices recursively as elements of $A_{\mathcal{E}}\langle \mathbf{c},\mathbf{d}\rangle$. As an illustration, we compute the polynomial $\Phi^3$. Recall that the basic relations in $A_{\mathcal{E}}$ through degree 3 are $2y_2 = y_1^2$ and $y_2y_1 = y_1y_2$.

**Example 4.5** For rank 3 posets, we get from the definition

$$\begin{aligned}
\Phi^3 &= y_3\mathbf{a}^2 + (y_1y_2 - y_3)\mathbf{ba} + (y_2y_1 - y_3)\mathbf{ab} + (y_1y_1y_1 - y_2y_1 - y_1y_2 + y_3)\mathbf{b}^2 \\
&= y_3(\mathbf{c}^2 - 2\mathbf{d}) + y_1y_2\mathbf{d} = y_3 \ \mathbf{c}^2 + (y_1y_2 - 2y_3) \ \mathbf{d}
\end{aligned}$$

and so $\phi_{\mathbf{c}^2} = y_3$ and $\phi_{\mathbf{d}} = y_1y_2 - 2y_3$. Alternatively, from (4.10) and (4.9) we get

$$\begin{aligned}
2\Phi^3 &= y_1 \ \mathbf{c} \ \Phi^2 - y_2 \ (\mathbf{c}^2 - 2\mathbf{d}) \ \Phi^1 + y_3 \ (\mathbf{c}^2 - 2\mathbf{d})\,\mathbf{c} \ \Phi^0 \\
&= y_1y_2\mathbf{c}^2 - y_2y_1(\mathbf{c}^2 - 2\mathbf{d}) + 2 \ y_3 \ (\mathbf{c}^2 - 2\mathbf{d}).
\end{aligned}$$

## 4.3 Toric $h$-vector

Finally, we can easily describe the so-called toric $h$-vector as an element of the polynomial ring $A[x]$ in one indeterminate over $A$. We define polynomials $h^n = h^n(x) = \sum_{i=0}^{n-1} k_i^n x^i$ and $g^n = g^n(x) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (k_i^n - k_{i-1}^n) x^i$ in $A[x]$ by $h^0 = g^0 = 1$, and

$$h^n = \sum_{i=1}^{n} g^{n-i} \ y_i \ (x-1)^{i-1}, \tag{4.11}$$

for $n > 0$. It is easy to see that this definition is equivalent to that given in [18] or [19, §3.14]. Again, $h^n(x)$ is bihomogeneous, of degree $n$ in the $y_i$ and degree $n-1$ in $x$.

For example, from (4.11) we get $h^1 = g^0 y_1 = y_1$ and

$$h^2 = g^1 y_1 + g^0 y_2(x-1) = y_2 x + (y_1^2 - y_2).$$

and

$$\begin{aligned} h^3 &= g^2 y_1 + g^1 y_2(x-1) + g^0 y_3 (x-1)^2 \\ &= y_3 x^2 + (y_1 y_2 - 2y_3)x + (y_1^3 - y_2 y_1 - y_1 y_2 + y_3). \end{aligned}$$

Note that the lead term of $h^n$ is always $y_n$. Over $A_{\mathcal{E}}$, $h^2$ and $h^3$ reduce to $y_2 x + y_2$ and $y_3 x^2 + (y_1 y_2 - 2y_3)x + y_3$, respectively. To see that the constant term of $h^n$ always reduces to $y_n$ over $A_{\mathcal{E}}$, observe that by (4.11) and induction, this is equal to $\sum_{i=1}^{n} (-1)^{i-1} g_0^{n-i} y_i = \sum_{i=1}^{n-1} (-1)^{i-1} y_{n-i} y_i + (-1)^{n-1} y_n$, which equals $y_n$ in $A_{\mathcal{E}}$ by (3.4).

To distinguish them from terms of the flag $h$-vector, we denote the components of the *toric* h-*vector* by $(\widehat{h}_0^n, \ldots, \widehat{h}_{n-1}^n)$, where $\widehat{h}_i^n := k_{n-1-i}^n$. We have seen that $\widehat{h}_i^n = \widehat{h}_{n-1-i}^n$ holds in $A_{\mathcal{E}}$ when $i = 0$. In fact, this identity holds in $A_{\mathcal{E}}$ for all $i$ since it holds for all Eulerian posets [19, 3.14.9]. Similarly, it follows from [4, Theorem 7] that $g_k^{2k+1} \in A_{\mathcal{E}}^0$, that is, $g_k^{2k+1} = \overline{g_k^{2k+1}}$ (see Corollary 3.8). One advantage of this formulation is that it is clear from (4.11) that the $\widehat{h}_i^n$ are integer linear combinations of the $f_S^n$ [4, theorem 6]. In fact, one gets a recursive formula $\widehat{h}_k^n = \sum_{j=0}^{n-1-k} \sum_{i=j+1}^{n} (-1)^{i-j-1} \binom{i-1}{j} g_{n-1-k-j}^{n-i}\, y_i$ (*cf.* [5]).

# 5 Related algebraic structures

There are two algebraic structures related to those discussed so far, one involving coalgebras studied in a similar context in [11], the other involving modules associated to classes of Eulerian posets having restrictions on their lower or upper intervals.

## 5.1 Duality and the associated coalgebras

We discuss first the connection between the algebras studied here to the coalgebras of [11]. We will see that the algebra $A$ can be viewed as the graded dual to the coalgebra $\mathbf{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. For the basic definitions of coalgebras we refer to [16] or [21]. That $A$ appeared to be the dual of some coalgebra was first pointed out to us by Moss Sweedler. The discussion in this subsection was suggested to us by Richard Ehrenborg and is included here for completeness.

Let $C$ be a coalgebra with coassociative coproduct

$$\Delta : C \longrightarrow C \otimes C.$$

We do not assume that $C$ has a counit. Then the vector space dual $C^* = \mathrm{Hom}_{\mathbf{Q}}(C, \mathbf{Q})$ is an algebra (possibly without a unit), with multiplication defined, for $f, g \in C^*$, by

$$(f * g)(x) = \sum_x f(x_{(1)}) \cdot g(x_{(2)}). \tag{5.1}$$

(See [21, Proposition 1.1.1] or [16, Lemma 1.2.2].) Here we use the Sweedler notation $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$ for the coproduct.

Suppose, in addition, $C = \bigoplus_{n \geq 1} C_n$ is a graded vector space, and $C$ is graded as a coalgebra, that is, $\Delta(C_n) \subset \bigoplus_{i+j=n} C_i \otimes C_j$. Then if $C_n^*$ is the vector space dual of $C_n$, the *graded* dual of $C$, $C^g := \bigoplus_{n \geq 1} C_n^*$, is a subalgebra of $C^*$. Here we have an orthogonal direct sum, in the sense that if $f \in C_k^*$, $c \in C_m$ and $k \neq m$ then $f(c) = 0$.

We are interested in the underlying vector space of the free associative algebra $C = \mathbf{Q}\langle \mathbf{a}, \mathbf{b} \rangle$. As in [11], we make $C$ into a coalgebra (without counit) by defining a coproduct $\Delta : C \longrightarrow C \otimes C$ by

$$\Delta(v_1 \cdot v_2 \cdots v_n) = \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_{i+1} \cdots v_n,$$

for an **ab**-word $v_1 \cdot v_2 \cdots v_n$. For example, $\Delta(\mathbf{aba}) = 1 \otimes \mathbf{ba} + \mathbf{a} \otimes \mathbf{a} + \mathbf{ab} \otimes 1$. As a coalgebra, $C = \bigoplus C_n$ is graded, where $C_n$ is the span of all **ab**-words of degree $n-1$.

We define elements $v_S^n = v_1 \cdots v_{n-1} \in C_n$, $S \subset [n-1]$, by

$$v_i = \begin{cases} \mathbf{b} & \text{if } i \in S, \text{ and} \\ \mathbf{a} - \mathbf{b} & \text{if } i \notin S. \end{cases}$$

For fixed $n$, the elements $v_S^n$ form a **Q**-basis for the vector space $C_n$. The effect of the coproduct on these elements is straightforward.

**Lemma 5.1** *For each $n$ and each $S \subset [n-1]$,*

$$\Delta(v_S^n) = \sum_{i \in S} v_{S_{[i]}}^i \otimes v_{S^{[i]}}^{n-i}.$$

The connection with the algebra $A$ is given by the following.

**Proposition 5.2** *The graded dual of the coalgebra $C$ is isomorphic to the subalgebra $A^+ = \bigoplus_{n \geq 1} A_n$ as a graded **Q**-algebra.*

*Proof:* Let $e_S^n$ be a dual basis to the basis $v_S^n$ of $C_n$, *i.e.*, $\langle e_S^n, v_T^k \rangle = \delta_{n,k} \cdot \delta_{S,T}$. Then by (5.1) and Lemma 5.1,

$$\begin{aligned} \langle e_S^n \cdot e_T^k, v_R^m \rangle &= \sum_{i \in R} \left\langle e_S^n, v_{R_{[i]}}^i \right\rangle \cdot \langle e_T^k, v_{R^{[i]}}^{n-i} \rangle \\ &= \delta_{n+k,m} \cdot [n \in R] \cdot \left\langle e_S^n, v_{R_{[n]}}^n \right\rangle \cdot \langle e_T^k, v_{R^{[n]}}^k \rangle \\ &= \delta_{n+k,m} \cdot [n \in R] \cdot \delta_{S,R_{[n]}} \cdot \delta_{T,R^{[n]}} \\ &= \delta_{n+k,m} \cdot \delta_{R,S\cup\{n\}\cup(T+n)} = \left\langle e_{S\cup\{n\}\cup(T+n)}^{n+k}, v_R^m \right\rangle, \end{aligned}$$

where $[n \in R]$ is 1 if $n \in R$ and 0 otherwise, showing the map $C \to A^+$ given by $e_S^n \mapsto f_S^n$ to be an isomorphism of **Q**-algebras. ⋄

### 5.2 Simplicial and cubical quotients

Interesting classes of Eulerian posets are provided by the face lattices of simplicial and cubical polytopes, that is, polytopes such that every face is, respectively, a simplex or a cube. From the poset point of view, this condition becomes one on lower intervals and leads to consideration of certain quotients of the algebra $A_{\mathcal{E}}$ by one-sided ideals. To this end, we consider the following.

A subfamily $\mathcal{P}$ of graded posets is said to be *lower hereditary* if it is closed under taking lower intervals, *i.e.*, if $P \in \mathcal{P}$ and $x \in P$, then $P_x \in \mathcal{P}$. Similarly $\mathcal{P}$ is *upper hereditary* if $P^x \in \mathcal{P}$ for every $x \in P \in \mathcal{P}$. The following is analogous to Proposition 1.3.

**Proposition 5.3** *Suppose $F \in A$ is a zero form on a class $\mathcal{P}$ of graded posets and $G \in A$ is any form. If $\mathcal{P}$ is lower (upper) hereditary, then the form $F \cdot G$ (respectively, $G \cdot F$) is also a zero form. Thus the subspace $I_{\mathcal{P}} \subset A$ of zero forms on $\mathcal{P}$ is a right (left) ideal in the algebra $A$.*

A lower hereditary family of posets $\mathcal{P}$ is said to be *lower uniform* if there are $Q_i \in \mathcal{P}$, $i \geq 1$, with $r(Q_i) = i$ such that for any $P \in \mathcal{P}$ and any $x \in P$, $x \neq \hat{1}$, $P_x \simeq Q_{r(x)}$. Similarly, $\mathcal{P}$ is *upper uniform* if its dual is lower uniform. Thus, for a poset $P$ in a lower uniform family $\mathcal{P}$, every rank $i$ proper lower interval of $P$ is isomorphic to $Q_i$. Our primary examples of lower uniform families of posets are the face posets of simplicial or cubical polytopes. Here $Q_i$ is, respectively, the face poset of an $(i-1)$-dimensional simplex or cube. Examples of upper uniform families can be obtained from these by duality. Of particular interest is the family of face posets of simple polytopes, that is, those whose duals are simplicial.

Note that for $\mathcal{P}$ lower uniform and $P \in \mathcal{P}$ of rank $d$, if $S = \{s_1, \ldots, s_k\} \subset [d-1]$, $s_1 < \cdots < s_k$, then $f^d_S(P) = f^{s_k}_{S \setminus s_k}(Q_{s_k}) \cdot f^d_{s_k}(P)$. Thus for a uniform family of posets, the study of chain operators reduces to that of ordinary $f$-vectors. This leads us to the following.

Given constants $c_{ij} \in \mathbf{Q}$ $(i, j \geq 1)$, a *lower uniform ideal* of $A$ is a right ideal $I_U$ generated by the elements $y_i y_j y_k - c_{ij} y_{i+j} y_k$, $i, j, k \geq 1$. Similarly an *upper uniform ideal* is a left ideal generated by the elements $y_k y_i y_j - c_{ij} y_k y_{i+j}$. If $I_U$ is a uniform ideal, then the quotient $A_U := A/(I_{\mathcal{E}} + I_U)$ is a right or left module over $A$, which we will refer to as a *uniform module.* Note that a uniform ideal is always homogeneous and so uniform modules are always graded.

**Proposition 5.4** *If $A_{U_d}$ is the subspace of degree $d$ elements in some uniform module $A_U$, then*

$$\dim_{\mathbf{Q}}(A_{U_d}) \leq \left\lfloor \frac{d+1}{2} \right\rfloor.$$

*Proof:* We assume $U$ is lower uniform and so $A_U$ is a right $A$-module. Successive applications of the relation $y_i y_j y_k = c_{ij} y_{i+j} y_k$ to (the last three terms of) a degree $d$ monomial ending with $y_k$ for some $k$ reduce the monomial to a multiple of $y_{d-k} y_k$. So any two monomials ending with $y_k$ are linearly dependent in $A^d_U$.

By Theorem 3.4, we know that any degree $d$ element in $A_U$ can be written as linear combination of monomials ending with $y_{2i-1}$, $i \leq k = \lfloor \frac{d+1}{2} \rfloor$. Then by the above observation, any degree $d$ element in $A_U$ is a linear combination of $y_{d-1}y_1, y_{d-3}y_3, \ldots, y_{d+1-2k}y_{2k-1}$. So $\dim_{\mathbf{Q}}(A_{U_d}) \leq \lfloor \frac{d+1}{2} \rfloor$. $\diamond$

Note this bound is not tight in general. If $c_{ij} = 0$ for all $i, j$, then $y_i y_j y_k = 0$ for all $i, j, k > 0$. Thus when $d$ is odd, $y_d$ spans $A^d_U$; hence, $\dim_{\mathbf{Q}}(A^d_U) = 1$ in this case.

For face posets of simplicial $d$-polytopes, the flag $f$-vectors are determined by the $f$-vectors as noted above; for example, if $1 \leq i < j \leq d$, then $f_{ij} = \binom{j}{i} f_j$. For cubical polytopes, we have that $f_{ij} = 2^{j-i}\binom{j-1}{i-1} f_j$. From this, the following is straightforward.

**Proposition 5.5** *The uniform families of posets consisting of face posets of simplicial, respectively, cubical polytopes correspond to uniform ideals in $A_{\mathcal{E}}$ generated by elements $y_i y_j y_k - c_{ij} y_{i+j} y_k$, $i, j, k \geq 1$, where*

$$c_{ij} = \begin{cases} \binom{i+j}{i} & \textit{for simplicial polytopes, and} \\ 2^j \binom{i+j-1}{j} & \textit{for cubical polytopes.} \end{cases}$$

Proposition 5.4 allows us to conclude that the linear span of the flag $f$-vectors of all simplicial $d$-polytopes (or all cubical $d$-polytopes) has dimension at most $\lfloor \frac{d+1}{2} \rfloor$. Actually equality holds in both cases [12], and thus all linear relations on simplicial or cubical polytopes are spanned by convolutions of linear forms with Euler relations or with the forms given in Proposition 5.5.

Note that the relation between the simplicial and the cubical constants is suggestive of the relation between the usual simplicial $h$-vector and the cubical $h$-vector introduced by Adin [1]. It would be of interest to find liftings to the flag $f$-vector of the linear inequalities given by the nonnegativity of the simplicial $g$-vector for simplicial convex polytopes (the generalized lower bound theorem) and the conjectured nonnegativity of the cubical $g$-vector for cubical polytopes (the cubical lower bound conjecture). See [7] for a discussion of these and other face number inequalities.

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