The Geometry of Products of Minors [1]

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Abstract

We consider the Newton polytope $\Sigma(m,n)$ of the product of all minors of an $m \times n$ matrix of indeterminates. Using the fact that this polytope is the secondary polytope of the product $\Delta_{m-1} \times \Delta_{n-1}$ of simplices, and thus has faces corresponding to coherent polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$, we study facets of $\Sigma(m,n)$, which correspond to the coarsest, nontrivial such subdivisions. We make use of the relation between secondary and fiber polytopes, which in this case gives a representation of $\Sigma(m,n)$ as the Minkowski average of all $m \times n$ transportation polytopes.

1. Newton, secondary and fiber polytopes.

Given an $m \times n$ matrix $A = (\,a_{ij}\,)$ of indeterminates, we consider the polytope $\Sigma(m,n) \subset \mathbb{R}^{mn}$ defined as the *Newton polytope* of the product of *all* minors of A, that is, the convex hull of all exponent vectors obtained when one expands this product as a polynomial in the a_{ij}. For example, when $m = n = 2$, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, the product of minors is

$$|a_{11}| \cdot |a_{12}| \cdot |a_{21}| \cdot |a_{22}| \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}^2 a_{12} a_{21} a_{22}^2 - a_{11} a_{12}^2 a_{21}^2 a_{22}$$

and so $\Sigma(2,2)$ is the line segment joining $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ in $\mathbb{R}^4$.

We address here the problem of determining the polytope $\Sigma(m,n)$ for general m and n. Determining a polytope can be accomplished in several different ways, for example, by giving a description of its vertices, or by determining a minimal set of linear inequalities which define it. The latter amounts to describing all its faces of codimension 1, that is, its *facets*. This is the point of view we adopt in this paper. We begin a description of $\Sigma(m,n)$ by describing some classes of facets and giving a means to determine the linear inequalities to which they correspond. Note that the approach by description of vertices was taken

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in [1], [12] and [14], where the closely related polytope $\Pi_{m,n}$, the Newton polytope of the product of *maximal* minors, was considered.

Our approach to $\Sigma(m,n)$ depends on the fact that it is equal to the *secondary polytope* $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$ of the product $\Delta_{m-1} \times \Delta_{n-1}$ [8;§3E.3], [7; §7.3.D]. (Here Δ_k denotes the standard k-simplex, defined as the convex hull of the $k+1$ unit vectors in $\mathbb{R}^{k+1}$.) Thus, the lattice of faces of $\Sigma(m,n)$ is isomorphic to the poset of all coherent polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$ that add no new vertices.

A subdivision is *coherent* if it supports a strictly convex piecewise linear function, where *strictly* convex means there is a different linear function on each maximal cell of the subdivision. See [3], [5], [7] or [8] for a discussion of secondary polytopes and coherent (or *regular*) subdivisions. Vertices of $\Sigma(m,n)$ correspond to the coherent triangulations of $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, while facets correspond to the coarsest possible coherent subdivisions (see, e.g., [7; §7.2.B]). It is known that $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$ has only coherent triangulations when (m,n) satisfies $(m-2)(n-2) < 4$. In all other cases, there is at least one triangulation that is not coherent [6], [13].

Thus to study faces of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, it is sufficient to study the structure of the associated coherent subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. To do this, we make use of the fact that secondary polytopes are (up to scalar multiple) equal to fiber polytopes of projections of simplices, in this case, the projection of the $(mn-1)$-simplex onto $\Delta_{m-1} \times \Delta_{n-1}$. For definitions and basic results about fiber polytopes, see [5]. A discussion of fiber polytopes can also be found in [7; §7.1.E] and [16; Lecture 9].

In §2, we consider the general case of fiber polytopes and give a concrete description of the isomorphism between faces and coherent polyhedral subdivisions. This is specialized in §3 to the case of faces of $\Sigma(m,n)$ and coherent polyhedral subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. In particular, for each $\Theta \in \mathbb{R}^{mn}$, we give a description of the subdivision Π_Θ corresponding to the face of $\Sigma(m,n)$ having outward normal Θ. If Θ is a **0-1** matrix, the maximal cells of Π_Θ can be read directly from the minimal line covers of the $1's$ of Θ (sets of rows and columns including all the $1's$). In some cases the resulting subdivision can be shown to be coarsest possible, and so the corresponding Θ will be normal to a facet of $\Sigma(m,n)$. In other cases we can give an upper bound on the dimension the corresponding faces of $\Sigma(m,n)$.

We study edges of $\Sigma(m,n)$ in §4, and use this information to give lower bounds on the dimensions of certain faces. In some cases we can combine these bounds to fix the dimension of a face. We do this, in particular, for almost all faces of $\Sigma(m,n)$ admitting a **0-1** normal. In §5, we use homological methods to construct facets that need arbitrarily large integers for any integral normal; in particular, these admit no **0-1** normal.

In §6, formulas for the support function are derived for fiber and secondary polytopes. In the case of $\Sigma(m,n)$, the value of the support function h_{mn} at a **0-1** matrix Θ can be expressed in terms of the minimal line covers of Θ. This calculation is carried out for some special Θ and for small values of m and n.

Finally, we give some notational conventions. For P a convex polyhedron in $\mathbb{R}^n$ and $\theta \in \mathbb{R}^n$, we denote by P^θ and P_θ the faces of P where the linear form $\langle \theta, \cdot \rangle$ achieves its maximum and minimum, respectively. Note that $P^{-\theta} = P_\theta$. For a set $X \subset \mathbb{R}^n$, we denote by *cone* X (respectively, *pos* X) the set of all nonnegative (respectively, positive) linear combinations of the points in X. The set $\{1, \ldots, n\}$ will be denoted $[n]$.

2. Coherent subdivisions and faces of $\Sigma(P, Q)$.

We consider first the general fiber polytope $\Sigma(P, Q)$, where $P = conv\{p_1, \ldots, p_m\} \subset \mathbb{R}^n$, $Q = conv\{q_1, \ldots, q_m\} \subset \mathbb{R}^d$ and $\pi : P \to Q$ a linear map such that $\pi(p_i) = q_i$. By Theorem 2.1 of [5], each face $\Sigma(P, Q)^\theta$ of $\Sigma(P, Q)$ corresponds to a certain (coherent) polyhedral subdivision Π_θ. We describe this correspondence in this section and specialize it to obtain the correspondence between faces of the secondary polytope $\Sigma(Q)$ and coherent subdivisions of Q. As in [3], [5], [7] and [8], by a *polyhedral subdivision of* Q we mean a collection Π of subsets of $\{q_1, \ldots, q_m\}$ whose convex hulls form a polyhedral complex that covers Q (cf. [7; Definition 7.2.1]). Note this definition requires that if $A, B \in \Pi$ and $conv\,A$ is a face of $conv\,B$, then $A = B \cap conv\,A$.

The fiber $\pi^{-1}(q)$ over a point $q \in Q$ can be written as

$$\pi^{-1}(q) = \{ \sum_{i=1}^{m} \lambda_i p_i \mid \sum \lambda_i q_i = q,\ \sum \lambda_i = 1,\ \lambda_i \geq 0 \} \subset \mathbb{R}^n. \tag{2.1}$$

Let $(y_0, \bar{y}) = (y_0, y_1, \ldots, y_d)$ denote a point in $\mathbb{R}^{d+1}$. For $\theta \in \mathbb{R}^n$ define the polyhedron

$$S(\theta) = \{ (y_0, \bar{y}) \in \mathbb{R}^{d+1} \mid y_0 + \langle \bar{y}, q_i \rangle \geq \langle \theta, p_i \rangle,\ i = 1, \ldots, m \}. \tag{2.2}$$

For $y = (y_0, \bar{y}) \in S(\theta)$, define

$$\sigma_y = \{ q_i \mid y_0 + \langle \bar{y}, q_i \rangle = \langle \theta, p_i \rangle \}. \tag{2.3}$$

The following gives a concrete description of the lattice isomorphism given by Theorem 2.1 of [5].

Theorem 2.1. *In the correspondence between faces of the fiber polytope $\Sigma(P, Q)$ and polyhedral subdivisions of Q, the face $\Sigma(P, Q)^\theta$ corresponds to the coherent subdivision*

$$\Pi_\theta = \{ \sigma_y \mid y \in S(\theta) \}.$$

Proof: By general linear programming duality we can write

$$max\{ \langle \theta, p \rangle \mid p \in \pi^{-1}(q) \} = min\{ y_0 + \langle \bar{y}, q \rangle \mid (y_0, \bar{y}) \in S(\theta) \}. \tag{2.4}$$

Since $max\{ \langle \theta, p \rangle \mid p \in \pi^{-1}(q) \}$ is finite precisely when $q \in Q$, we have that $(1, q)$ is an element of the relatively open inner normal cone $N(S(\theta), y)$ to the polyhedron $S(\theta)$ at some point y if and only if $q \in Q$. In fact, for $y \in relint\, S(\theta)_{(1,q)}$, we have

$$(1, q) \in N(S(\theta), y) = pos\{ (1, z) \mid z \in \sigma_y \}.$$

Thus $\Pi := \{ \sigma_y \mid y \in S(\theta) \}$ is a polyhedral subdivision of Q; it is given by the intersection of the (sets of generators of the cones in the) normal fan of $S(\theta)$ with the hyperplane $y_0 = 1$ in $\mathbb{R}^{d+1}$.

By Theorem 2.1 of [5], $\sigma \in \Pi_\theta$ if and only if for $q \in conv\,\sigma$, the maximum value of $\langle \theta, p\rangle$ over the fiber $\pi^{-1}(q)$ is attained on the face

$$\pi^{-1}(\sigma) := conv\{\ p_i \mid q_i = \pi(p_i) \in \sigma\ \} \tag{2.5}$$

of P. That is,

$$\pi^{-1}(q)^\theta = \pi^{-1}(\sigma) \cap \pi^{-1}(q). \tag{2.6}$$

For $q \in relint\,conv\,\sigma$, the face $\pi^{-1}(q)^\theta$ contains a point $p \in relint\,\pi^{-1}(\sigma)$, and so

$$max\{\ \langle \theta, p\rangle \mid p \in \pi^{-1}(q)\ \}$$

is attained at a point $p = \sum \lambda_i p_i$ for which $\lambda_i > 0$ whenever $q_i \in \sigma$. Thus for any $(y_0, \bar{y})$ solving

$$min\{\ y_0 + \langle \bar{y}, q\rangle \mid (y_0, \bar{y}) \in S(\theta)\ \},$$

$q_i \in \sigma$ implies $y_0 + \langle \bar{y}, q_i\rangle = \langle \theta, p_i\rangle$, and so $\sigma \subset \sigma_y$. Thus Π_θ refines Π.

On the other hand, any $p = \sum \lambda_i p_i$ solving $max\{\ \langle \theta, p\rangle \mid p \in \pi^{-1}(q)\ \}$ must satisfy $\lambda_i = 0$ when $q_i \notin \sigma$ and so, by strict complementarity, there is a $(y_0, \bar{y}) \in S(\theta)_{(1,q)}$ with $y_0 + \langle \bar{y}, q_i\rangle > \langle \theta, p_i\rangle$ whenever $q_i \notin \sigma$. Thus $\sigma_y \subset \sigma$, and we conclude $\Pi_\theta = \Pi$. $\triangleleft$

One way to understand this association $\theta \mapsto \Pi_\theta$ is to observe that, for fixed θ, the maximum in (2.4) varies piecewise-linearly in q. The subdivision Π_θ gives the associated regions of linearity. A consequence of the proof of Theorem 2.1 is that the lattice of faces of the subdivision Π_θ is the lattice of faces of the normal fan to the (necessarily unbounded) polyhedron $S(\theta)$.

Corollary 2.2. *The lattice of faces of the subdivision Π_θ is anti-isomorphic to the lattice of faces of the polyhedron $S(\theta)$. In particular, the maximal cells of the subdivision Π_θ correspond to the minimal faces of $S(\theta)$.* $\triangleleft$

In the case of the secondary polytope of Q, we have $m = n$ and $P = \Delta_{n-1}$, and so (2.1), (2.2) and (2.3) simplify, respectively, to $\pi^{-1}(q) = \{\ \lambda \in \mathbb{R}^n \mid \sum \lambda_i q_i = q, \sum \lambda_i = 1, \lambda \geq 0\ \}$, the set of all convex representations of q, $S(\theta) = \{\ (y_0, \bar{y}) \in \mathbb{R}^{d+1} \mid y_0 + \langle \bar{y}, q_i\rangle \geq \theta_i,\ i = 1, \ldots, m\ \}$, the set of all downward pointing halfspaces containing the convex hull of the points (θ_i, q_i), and $\sigma_y = \{\ q_i \mid y_0 + \langle \bar{y}, q_i\rangle = \theta_i\ \}$, the set of points on some top face of this convex hull.

3. Coherent subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$ and faces of $\Sigma(m, n)$.

We consider here the case of the secondary polytope $\Sigma(\Delta_{m-1} \times \Delta_{n-1})$, which corresponds to the special case of the situation described in §2 in which P is the $(mn-1)$-simplex Δ_{mn-1} and $Q = \Delta_{m-1} \times \Delta_{n-1}$. As in the general case, we make more explicit the association between faces and subdivisions of $\Delta_{m-1} \times \Delta_{n-1}$. For faces defined by **0-1** normals, this association leads to consideration of the classical combinatorial notion of a line cover of a **0-1** matrix.

We define the map $\pi : \mathbb{R}^{mn} \to \mathbb{R}^m \times \mathbb{R}^n$ in this case as follows. We consider elements of $\mathbb{R}^{mn}$ to be $m \times n$ matrices; unit vectors are then the **0-1** matrices E_{ij} having a single 1 in the i^{th} row and j^{th} column. Then

$$\begin{aligned}\Delta_{mn-1} &= \{X \in \mathbb{R}^{mn} \mid X = (x_{ij}) \geq 0,\ \sum_{ij} x_{ij} = 1 \ \} \\ &= conv\{\ E_{ij} \mid i \in [m],\ j \in [n]\ \}.\end{aligned}$$

For $X \in \mathbb{R}^{mn}$, let $r(X) \in \mathbb{R}^m$ and $c(X) \in \mathbb{R}^n$ denote the vectors of row sums and column sums of X. The map π is then defined by $\pi(X) = (r(X), c(X))$. We define $v_{ij} := \pi(E_{ij}) = (e_i, f_j)$, where e_i and f_j are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Thus

$$\Delta_{m-1} \times \Delta_{n-1} = conv\{\ v_{ij} \mid i \in [m],\ j \in [n]\ \}.$$

For $(a, b) \in \Delta_{m-1} \times \Delta_{n-1}$, the fiber

$$\pi^{-1}(a, b) = \{\ X \in \mathbb{R}^{mn} \mid X \geq 0,\ r(X) = a,\ c(X) = b\ \}$$

has been studied in the optimization literature under the name *transportation polytope*; see [10] and [15]. Note that in this case, the restriction $\sum_{ij} x_{ij} = 1$ in (2.1) is implied by $r(X) = a$. As in §2, we observe that for $\Theta = (\theta_{ij}) \in \mathbb{R}^{mn}$,

$$\begin{aligned}max\{\ \langle\Theta, X\rangle \mid X \in \pi^{-1}(a, b)\ \} &= max\{\ \langle\Theta, X\rangle \mid X \geq 0,\ r(X) = a,\ c(X) = b\ \} \\ &= min\{\ \langle u, a\rangle + \langle v, b\rangle \mid u_i + v_j \geq \theta_{ij}\ \}.\end{aligned} \tag{3.1}$$

If we let

$$S(\Theta) = \{\ (u, v) \in \mathbb{R}^m \times \mathbb{R}^n \mid u_i + v_j \geq \theta_{ij}\ \} \tag{3.2}$$

and define, for $(u, v) \in S(\Theta)$,

$$\sigma_{uv} = \{\ v_{ij} \mid u_i + v_j = \theta_{ij}\ \}, \tag{3.3}$$

interpreting Theorem 2.1 and Corollary 2.2 in this case leads to the following

Theorem 3.1. *For any $\Theta \in \mathbb{R}^{mn}$, the face $\Sigma(m, n)^\Theta$ of the secondary polytope $\Sigma(m, n) = \Sigma(\Delta_{m-1} \times \Delta_{n-1})$ corresponds to the coherent subdivision $\Pi_\Theta = \{\ \sigma_{uv} \mid (u, v) \in S(\Theta)\ \}$. The lattice of faces of Π_Θ is anti-isomorphic to the lattice of faces of the polyhedron $S(\Theta)$, so Π_Θ is a triangulation precisely when $S(\Theta)$ is simple.* $\triangleleft$

Note that in this case, $S(\Theta)$ is simple if every 1-face is on precisely $m + n - 1$ facets. $S(\Theta)$ has no vertices so a maximal cell of Π_Θ is of the form σ_{uv}, where (u, v) lies on a 1-face.

We consider next subdivisions Π_Θ where $\Theta = (\theta_{ij})$ is a **0-1** matrix. In this case, we define a *line cover* of Θ to be a pair (I, J), with $I \subset [m], J \subset [n]$ such that for all (i, j) with $\theta_{ij} = 1$, either $i \in I$ or $j \in J$. A line cover (I, J) is said to be *minimal* if both I

and J are minimal. Note that line covers of Θ correspond directly to **0-1** vectors in the polyhedron $S(\Theta)$.

Given **0-1** matrix Θ and line cover (I, J), we say θ_{ij} is *exactly covered* by (I, J) if $\theta_{ij} = 1$ and $(i, j) \notin I \times J$, or $\theta_{ij} = 0$ and $(i, j) \in ([m] \setminus I) \times ([n] \setminus J)$. Define

$$\sigma_{IJ} = \{\ v_{ij} \mid \theta_{ij} \text{ exactly covered by } (I, J)\ \}. \tag{3.4}$$

Note that we always have $\sigma_{\emptyset\,[n]} = \sigma_{[m]\,\emptyset}$.

To every **0-1** matrix Θ, there corresponds a bipartite graph G_Θ having bipartition $[m]$ and $[n]$, with (i, j) an edge of G_Θ if and only if $\theta_{ij} = 1$. Note that when G_Θ is connected, both $([m], \emptyset)$ and $(\emptyset, [n])$ are minimal covers. In this case $\sigma_{\emptyset\,[n]} = \sigma_{[m]\,\emptyset}$ has dimension $m + n - 2$. On the other hand, if G_Θ is not connected, then this cell has dimension less than $m + n - 2$.

Corollary 3.2. *For* **0-1** *matrices* Θ*, the maximal cells of the subdivision* Π_Θ *are precisely the cells* σ_{IJ}*, where* (I, J) *is a minimal line cover of* Θ *with* $I \neq [m]$ *and, if* G_Θ *is not connected,* $J \neq [n]$.

Proof: We first show for every $(u, v) \in S(\Theta)$ there is a **0-1** vector $(\bar{u}, \bar{v}) \in S(\Theta)$ with $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$, showing the σ_{IJ} to include all maximal cells of Π_Θ. We can assume that both $u \geq 0$ and $v \geq 0$ since, for example, if $u_1 = min\{u_i, v_j\} < 0$, then adding $|u_1|$ to every u_i and subtracting $|u_1|$ from every v_j yields $(u', v') \geq 0$ with $\sigma_{u'v'} = \sigma_{uv}$. In this case, set $\bar{u}_i = [u_i \geq \frac{1}{2}]$ and $\bar{v}_j = [v_j > \frac{1}{2}]$ for all i and j, where, for any relation R, $[R]$ is 1 or 0 depending on whether R is true or false. Then $u_i + v_j = 0$ implies $\bar{u}_i + \bar{v}_j = 0$, and $u_i + v_j = 1$ implies $\bar{u}_i + \bar{v}_j = 1$, and so $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$.

Next we show that if (I, J) and (I', J') are distinct minimal line covers, with $I, I' \neq [m]$ and $J \neq [n]$, then $\sigma_{IJ} \not\subset \sigma_{I'J'}$, showing σ_{IJ} to be a maximal cell of Π_Θ. Since both covers are minimal, one of $I' \setminus I$ or $J' \setminus J$ is not empty. Suppose $I' \setminus I \neq \emptyset$. Then if $i \in I' \setminus I$ and $j \in [n] \setminus J$, we have $v_{ij} \in \sigma_{IJ} \setminus \sigma_{I'J'}$.

Finally, if $\sigma_{\emptyset\,[n]} \subset \sigma_{IJ}$ for some $I \neq [m]$ and $J \neq [n]$, then all the edges of G_Θ are in $I \times ([n] \setminus J)$ or $([n] \setminus I) \times J$ and so G_Θ is not connected. Thus $\sigma_{\emptyset\,[n]}$ is also maximal when G_Θ is connected. ◁

Example 3.3. $\Theta = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $m = n = 2$. *Here* Π_Θ *is a triangulation of a square into triangles* $\sigma_1 = \{v_{11}, v_{12}, v_{21}\}$ *and* $\sigma_2 = \{v_{12}, v_{21}, v_{22}\}$*. The cell* σ_1 *corresponds to the minimal line cover* $(\{1, 2\}, \emptyset)$ *(as well as to* $(\emptyset, \{1, 2\})$*), while* σ_2 *corresponds to* $(\{1\}, \{1\})$*. Note that the cell* $\sigma_1 \cap \sigma_2$ *does not correspond to a line cover; however* $\sigma_1 \cap \sigma_2 = \sigma_{uv}$ *for* $u = (1, \frac{1}{2})$ *and* $v = (\frac{1}{2}, 0)$*. Finally, note that if* $\Theta' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$*, then* $\Pi_\Theta = \Pi_{\Theta'}$*.* ◁

More generally, we consider

Example 3.4. *(cf. [7; Example 7.3.14]) Let m=n and let* Θ *be an* $n \times n$ **0-1** *matrix with* $\theta_{ij} = 1$ *if and only if* $i \leq j$*. The minimal line covers of* Θ *are* $([i], [n] \backslash [i])$*,* $i = 0, \ldots, n$*, where we take* $[0] := \emptyset$*. If we define* $A_k := \sigma_{[k],[n]\backslash[k]}$*, then, by Corollary 3.2,* $\Pi_\Theta = \{A_1, \ldots, A_n\}$*. (Note that* $A_n = A_0$*.) It is straightforward to check that the sets* A_k*,* $k = 1, \ldots, n$ *are* *precisely those specified in the coarse subdivision of [7; Example 7.3.14], so we can conclude that the corresponding facet has normal* Θ *(or, equivalently, that* $\Sigma(n,n)^{\Theta}$ *is a facet).* $\triangleleft$

Note that since $\Sigma(P,Q) \subset \mathbb{R}^n$ and generally $dim\, \Sigma(P,Q) = dim\, P - dim\, Q < n$, facet normals of fiber polytopes are only determined up to the addition of an element of $\mathcal{K}(\pi, P) := (\ker \pi)^{\perp} + (\text{aff } P)^{\perp}$. In the case of $\Sigma(m,n) = \Sigma(\Delta_{m-1} \times \Delta_{n-1})$, $\mathcal{K}(\pi, P)$ is the linear span of the $m+n$ **0-1** matrices consisting of a row or column of $1's$. This is the same as the space of all *additive* matrices, that is, matrices of the form $X = (x_{ij})$ with $x_{ij} = \alpha_i + \beta_j$ for $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$. We call matrices Θ and Θ' *equivalent*, denoted $\Theta \sim \Theta'$, if $\Theta - \Theta'$ is additive. In this case $\Sigma(m,n)^{\Theta} = \Sigma(m,n)^{\Theta'}$; that is, equivalent matrices yield the same face (see Example 3.3.).

We can use Theorem 3.1 to give a lower bound on the codimension of certain faces of $\Sigma(m,n)$. Suppose $\Theta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where both A and B are nonnegative nonzero matrices, and let $\bar{\Theta} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. A matrix, all of whose entries are the same, will be called *constant*.

Lemma 3.5. *The subdivision* Π_{Θ} *is a refinement of the subdivision* $\Pi_{\bar{\Theta}}$*, and it is a strict refinement if the matrix* B *is not constant.*

Proof: Given any $(u,v) \in S(\Theta)$, write $(u,v) = ((u^1,u^2),(v^1,v^2))$, where (u^1,v^1) correspond to the rows and columns of A. As in the proof of Corollary 3.2, we can translate (u,v) by a multiple of $(1,\ldots,1,-1,\ldots,-1)$, if necessary, to assure that both $u \geq 0$ and $v \geq 0$ without changing σ_{uv}. If $(\bar{u},\bar{v}) = ((u^1,0),(v^1,0))$, then $(\bar{u},\bar{v}) \in S(\bar{\Theta})$ and $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$ (see (3.2) and (3.3)). Thus the subdivision Π_{Θ} is a refinement of $\Pi_{\bar{\Theta}}$.

To see that the refinement is strict when B is not constant, it is enough to show some maximal cell of Π_{Θ} to be a strict subset of some cell in $\Pi_{\bar{\Theta}}$. For this purpose, let $(u,v) = ((u^1,0),(0,v^2))$ be defined by setting the entries of u^1 and v^2 to be the successive row maxima of A and column maxima of B, respectively. Then $(u,v) \in S(\Theta)$ and so $\sigma_{uv} \in \Pi_{\Theta}$. Since the bipartite graph consisting of all edges (i,j) with $v_{ij} \in \sigma_{uv}$ is connected, the cell σ_{uv} has dimension $m+m-2$ and so is maximal. Letting $(\bar{u},\bar{v}) = ((u^1,0),(0,0))$, as above, then $\sigma_{uv} \subset \sigma_{\bar{u}\bar{v}}$ with equality only if the matrix B has constant columns. In this case, since B cannot also have constant rows, the same argument using a similarly defined (u,v) of the form $((0,u^2),(v^1,0))$ will yield a strict inclusion of σ_{uv} in $\sigma_{\bar{u}\bar{v}}$. $\triangleleft$

We can now prove the following

Theorem 3.6. *If* Θ *is a block-diagonal matrix*

$$\Theta = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix},$$

where $k > 1$ *and* $A_1, A_2, \ldots, A_k$ *are nonnegative and nonzero, then* $\operatorname{codim} \Sigma(m,n)^{\Theta} \geq k-1$. *If at least one of the* A_j *is not constant, then* $\operatorname{codim} \Sigma(m,n)^{\Theta} \geq k$.

Proof: The proof is by induction on k. For $k = 2$, it follows since $\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ defines a face of codimension at least 1 and, by Lemma 3.5, $\Sigma(m,n)^{\Theta}$ is a subface of this, which is proper when A_2 is not constant. The inductive step is completed by defining

$$\bar{\Theta} := \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_{k-1} & \\ & & & & 0 \end{pmatrix},$$

observing that $\operatorname{codim} \Sigma(m,n)^{\bar{\Theta}} \geq k-1$ by induction ($\begin{pmatrix} A_{k-1} & 0 \\ 0 & 0 \end{pmatrix}$ is not constant) and applying Lemma 3.5. $\triangleleft$

4. Edges of $\Sigma(m,n)$.

To discuss edges of $\Sigma(m,n)$, it is helpful to decompose it as a Minkowski sum in a canonical way. Define $\Sigma_k = \Sigma_k(m,n)$ to be the Newton polytope of the product of all $k \times k$ minors of an $m \times n$ matrix $A = (a_{ij})$. (We will usually suppress the dependence on m and n.) Then

$$\Sigma(m,n) = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_{m \wedge n}, \tag{4.1}$$

where $m \wedge n = min\{m,n\}$. The polytope Σ_1 is just a point (the matrix of all 1's) and so has no effect on the face structure of $\Sigma(m,n)$. The polytope $\Sigma_{m\wedge n}$ was studied in [1] and [14] (where it was called $\Pi_{m,n}$).

Each of the polytopes $\Sigma_k(m,n)$ has a further decomposition as a Minkowski sum of Birkhoff polytopes as follows. For $I \subseteq [m]$, $J \subseteq [n]$ with $|I| = |J|$, let $B_{I,J}$ denote the Newton polytope of the I,J-minor of A. The polytope $B_k := B_{[k],[k]}$ is the Birkhoff polytope of all doubly stochastic $k \times k$ matrices (see [4], [15]), and when $|I| = |J| = k$, $B_{I,J}$ is linearly isomorphic to B_k. Further

$$\Sigma_k(m,n) = \sum_{|I|=|J|=k} B_{I,J} \tag{4.2}$$

is a Minkowski sum of all of these.

We note that the Birkhoff polytope B_k has dimension $(k-1)^2$ and so B_2 is a line segment. More directly, $B_2 = \operatorname{conv}\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$. Thus Σ_2 is a Minkowski sum of line segments, that is, a *zonotope*. We will see that a great deal of the combinatorial structure of $\Sigma(m,n)$ is carried by the zonotope Σ_2.

Let $K_{m,n}$ denote the complete bipartite graph with bipartition $[m]$ and $[n]$, with every edge directed from $[m]$ to $[n]$. For each cycle in $K_{m,n}$ with $2k$ edges (which thus involves k vertices in $[m]$ and k vertices in $[n]$) we associate an $m \times n$ oriented incidence matrix $C = (c_{ij})$ obtained by orienting the cycle and setting $c_{i,j} = \pm 1$ for (i,j) in the cycle (depending on whether or not it agrees with the orientation) and $c_{i,j} = 0$ otherwise. The resulting *k-cycle matrix* C is unique up to change of sign and has the property that it is 0 except for some $k \times k$ minor that can be put into the form

$$\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \ddots & \\ -1 & & & & 1 \end{pmatrix} \tag{4.3}$$

by permuting rows and columns.

The following shows that all the edge directions in $\Sigma(m,n)$ are given by the k-cycle matrices for $k \leq m,n$. Its proof is essentially that of Proposition 1.9 of [14].

Proposition 4.1. *If X_1 and X_2 are vertices of $\Sigma(m,n)$ such that $[X_1, X_2]$ is an edge of $\Sigma(m,n)$, then $X_1 - X_2 = \ell(X_1, X_2) \cdot C$, where C is a k-cycle matrix for some $k \leq m,n$ and $\ell(X_1, X_2)$ is a positive integer.*

Proof: By (4.1) and (4.2), $\Sigma(m,n)$ is a Minkowski sum of polytopes linearly isomorphic to Birkhoff polytopes B_k for $k \leq m,n$. For B_k, edge directions are precisely the k'-cycle matrices for $k' \leq k$ (see [15; Theorem 5.1.3] or [4]). Any face F of $\Sigma(m,n)$ has a Minkowski decomposition $F = \sum_{|I|=|J|\leq m,n} F_{I,J}$, where $F_{I,J}$ is a face of $B_{I,J}$. When $F = [X_1, X_2]$ is an edge, all the $F_{I,J}$ are either vertices or are edges parallel to F; $\ell(X_1, X_2) > 0$ is the number of edges. ◁

Remark 4.2. We note that the number $\ell(X_1, X_2)$ has an interpretation in terms of the coherent triangulations Π_1 and Π_2 representing the vertices X_1 and X_2. Edges of secondary polytopes correspond to embedded bistellar operations on triangulations. These amount to modifications (perestroikas) along minimal dependent sets of, say, $p+1$ vertices in which half the boundary of a p-simplex is replaced by the other half. (See [7; Definition 2.9].) If $p-1$ is less than the dimension of the polytope being triangulated (in our case, $m+n-2$), then this dependent set will have the same link in the triangulations Π_1 and Π_2. This link will have $\ell(X_1, X_2)$ maximal simplices.

We note further that parallel edges appear in the sum $F = \sum_{|I|=|J|\leq m,n} F_{I,J}$ when the same k-cycle matrix appears as an edge for different $B_{I,J}$ with $|I| = |J| > k$. For the zonotope $\Sigma_2(m,n)$, edges correspond only to 2-cycle matrices, its *zones*, the $m \times n$ matrices with maximal nonzero minor $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

We can use knowledge of edges of $\Sigma(m,n)$ to determine which normals support facets. For example, Θ will support a face of dimension at least k if Θ is orthogonal to at least k independent edge directions of Σ_2. We use this to compute a lower bound on the dimension of faces supported by **0-1** Θ containing a line of zeros.

Assume Θ is a nonzero $m \times n$ **0-1** matrix having a zero row and no row of $1's$. We assume all zero rows are on the bottom. We define a graph Γ_Θ on the nonzero rows of Θ as follows. A pair of rows (i,j) will be an edge of Γ_Θ if and only if there are columns s and t with $\theta_{is} = \theta_{js} = 1$ and $\theta_{it} = \theta_{jt} = 0$.

Theorem 4.3. *If the graph Γ_Θ has k components, then*

$$\operatorname{codim} \Sigma(m,n)^\Theta \le k.$$

Proof: It will suffice to exhibit at least $(m-1)(n-1)-k$ independent edge directions in Σ_2 orthogonal to Θ. To insure independence, we list these edge directions in an "upper triangular" order defined using the graph Γ_Θ.

We start by ordering the rows of Θ according the components of Γ_Θ. Within each component we choose a spanning tree and extend the order given by distance from a root in this spanning tree. Each edge direction in Σ_2 corresponds to four entries of Θ. Taking each row in order (except the last), we identify, again in order, $n-1$ edge directions orthogonal to Θ ($n-2$ in the case of the first row of any component), each labeled by one of its four entries that has not appeared as an entry of a previously labeled direction. The resulting $(m-1)(n-1)-k$ edge directions are clearly independent.

The choice of edge directions and labelings proceeds as follows. When we first encounter row i, choose an element $\theta_{ij}=0$. Then for every $\theta_{it}=0$, $t \ne j$, choose direction $B_{in,jt}$ and label it by (i,t). If row i has no $1's$, we go to the next row.

If row i has a 1 and if it is not the first row in its component, let (i,i') be the unique edge of its spanning tree with $i'<i$, and choose columns k and k' with $\theta_{ik}=\theta_{i'k}=1$ and $\theta_{ik'}=\theta_{i'k'}=0$ guaranteed by the definition of the graph Γ_Θ. Take direction $B_{ii',kk'}$ and label it by (i,k). If row i has a 1 and is the first row of its component, arbitrarily choose an entry $\theta_{ik}=1$.

In either case, having an entry $\theta_{ik}=1$, we proceed as follows. For every $\theta_{it}=1$, $t\ne k$, choose direction $B_{in,kt}$ and label it by (i,t). $\triangleleft$

Recall, for a **0-1** matrix Θ, the bipartite graph G_Θ having vertex set $[m]\cup[n]$ and edges (i,j) whenever $\Theta_{ij}=1$. Continuing to assume that Θ has a zero row and no row of $1's$, note that if G_Θ has $k\ge 2$ nontrivial components (that is, components having more than one vertex), then Γ_Θ also has k components, so we get the following corollary of Theorem 3.6 and Theorem 4.3.

Corollary 4.4. *If Θ is a* **0-1** *matrix having a zero row and no row of $1's$ such that the bipartite graph G_Θ has $k\ge 2$ nontrivial components, then*

$$\operatorname{codim} \Sigma(m,n)^\Theta = k.$$

We note that this conclusion does not necessarily follow when $k=1$: one can check that if $\Theta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then G_Θ has one nontrivial component, but $\operatorname{codim}\Sigma(4,3)^\Theta = 3$, which is the number of components in Γ_Θ.

We consider a case in which the $1's$ in Θ form the union of two rectangles. We call a **0-1** matrix Θ a *generalized hook* if there are proper nonempty subsets $I'\subseteq I\subset[m]$ and $J'\subseteq J\subset[n]$ such that $\theta_{ij}=1$ if and only if $(i,j)\in(I'\times J)\cup(I\times J')$. Included in this class are all **0-1** matrices having all $1's$ only in one row or one column.

Corollary 4.5. *Suppose Θ is a* **0-1** *matrix which is a generalized hook. Then the face $\Sigma(m,n)^\Theta$ is a facet.*

We note that if we let $I=[m]$ or $J=[n]$ in the definition of generalized hook (but keep both I' and J' to be proper), then the resulting matrix is equivalent to a **0-1** matrix in which the $1's$ form a proper rectangle, and thus defines a facet by Corollary 4.5. These include all of the facets corresponding to coarse subdivisions of $\Delta_{m-1}\times\Delta_{n-1}$ into two cells (cf. [7; §7.2.B(2)]).

Example 4.6. *Suppose f is a linear form on $\mathbb{R}^{m+n}$ such that*

$$\{(x,y)\in\Delta_{m-1}\times\Delta_{n-1}\mid f(x,y)\geq 0\} \text{ and } \{(x,y)\in\Delta_{m-1}\times\Delta_{n-1}\mid f(x,y)\leq 0\}$$

are both $(m+n-2)$-dimensional polytopes all of whose vertices are vertices of $\Delta_{m-1}\times\Delta_{n-1}$. Then if $\sigma_+:=\{v_{ij}=(e_i,f_j)\mid f(e_i,f_j)\geq 0\}$ and $\sigma_-:=\{v_{ij}=(e_i,f_j)\mid f(e_i,f_j)\leq 0\}$, we have $\Pi:=\{\sigma_+,\sigma_-\}$ is a coarsest proper polyhedral subdivision of $\Delta_{m-1}\times\Delta_{n-1}$ and so corresponds to a facet of $\Sigma(m,n)$.

To determine a normal vector for this facet, note that since

$$\{(x,y)\in\Delta_{m-1}\times\Delta_{n-1}\mid f(x,y)=0\}=\operatorname{conv}(\sigma_+\cap\sigma_-)$$

is a codimension 1 cell meeting the interior of $\Delta_{m-1}\times\Delta_{n-1}$ and spanned by vertices (i.e., a wall of its chamber complex), the linear form f must be (up to scalar multiple) of the form $f(x,y)=x(I)-y(J):=\sum_I x_i-\sum_J x_j$, with I and J proper nonempty subsets of $[m]$ and $[n]$, respectively. Let Θ be the **0-1** matrix defined by $\theta_{ij}=1$ precisely for $(i,j)\in I\times J$. The matrix Θ has two minimal line covers, $(I,\emptyset)$ and $(\emptyset,J)$, with $\sigma_{I,\emptyset}=\sigma_-$ and $\sigma_{\emptyset,J}=\sigma_+$, so $\Pi=\Pi_\Theta$ corresponds to the facet $\Sigma(m,n)^\Theta$. Alternatively, defining Θ' by $\Theta'_{ij}=(f(e_i,f_j))^+$ (as in [7; §7.2.B(2), (2.3)]), it is straightforward to check that $\Sigma(m,n)_{\Theta'}=\Sigma(m,n)^\Theta$.

5. The cohomology of faces of Σ_2.

Using a cubical 2-complex associated to each face of Σ_2, and dual norms on its first homology and cohomology, we construct examples of facets of $\Sigma(m,n)$ requiring arbitrarily large numbers for any integral normal.

Recall from (4.1) that $\Sigma(m,n)=\Sigma_1+\Sigma_2+\cdots+\Sigma_{m\wedge n}$, and further that Σ_2 is a zonotope in $C_1:=\mathbb{R}^n\otimes\mathbb{R}^m$, the space of n by m matrices with standard basis $\{e_i\otimes f_k=E_{ik}|1\leq i\leq m,1\leq k\leq n\}$. The zones (edge directions) in Σ_2 are $\{E_{ik}-E_{il}+E_{jl}-E_{jk}|1\leq i<j\leq m,1\leq k<l\leq n\}$, corresponding to the 2 by 2 minors. Thus Σ_2 is also the image of the unit cube in $C_2:=\bigwedge^2\mathbb{R}^n\otimes\bigwedge^2\mathbb{R}^m$ with standard basis $\{e_i\wedge e_j\otimes f_k\wedge f_l|1\leq i<j\leq m,1\leq k<l\leq n\}$, under the linear map $\partial_2:C_2\to C_1$ with

$$\partial_2(e_i\wedge e_j\otimes f_k\wedge f_l)=(e_i-e_j)\otimes(f_k-f_l)=E_{ik}-E_{il}+E_{jl}-E_{jk}.$$

Note that the range of ∂_2 is the linear span of Σ_2 as well as that of $\Sigma(m,n)$. Thus the facet normals of Σ_2 are also facet normals for Σ. Accordingly we will restrict attention to the zonotope Σ_2 for the rest of this section.

Summarizing, if we take K to be the cubical 2-complex with 1-skeleton the complete bipartite graph $G_{n,m}$ and a square for each length 4 circuit in G, then we have

Lemma 5.1. *If $C_2 \to C_1 \to C_0$ is the cellular chain complex for K then the image of the unit cube in C_2 is Σ_2.*

Thus a normal to a face of Σ_2 will be in C_1^* and two such cochains must support the same face if they differ by a coboundary $\partial_1^* : C_0^* \to C_1^*$ (see the discussion following Example 3.4). Since K is simply connected this sequence is exact at C_1, so two normals support the same facet if and only if they differ by a coboundary.

Given a matrix $\Theta \in C_1^*$ define K_Θ to be the maximal subcomplex of K for which $\partial_2^*\Theta = 0$. Thus K and K_Θ have the same 1-skeleton, and a square of K appears in K_Θ if and only if the determinant of the associated 2 by 2 minor in Θ is zero. Since the dimension of the face Σ_2^Θ is the number of independent vanishing 2×2 minors of Θ, we have

Lemma 5.2.

$$\operatorname{codim} \Sigma_2^\Theta = dim\, H_1(K_\Theta) = \beta_1(K_\Theta).$$

Thus opposite pairs of facets of Σ_2 are in bijection with maximal subcomplexes K' of K with full 1-skeleton and with $H_1(K') = \mathbb{R}$.

Take K' to be a subcomplex of K containing the 1-skeleton of K. To identify facets without **0-1** normals we introduce norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on $H^1(K')$ and $H_1(K')$, respectively. With the standard dual basis, C_1^* has a natural infinity norm which is extended to the subquotient $H^1(K')$ by taking the norm of a class to be the infimum of the infinity norms of all representatives of that class. Similarly, one can extend the usual 1-norm on C_1 to $H_1(K')$. We note that the norm $\|\cdot\|_1$ on $H_1(K')$ is the dual norm to $\|\cdot\|_\infty$ on $H^1(K')$, so for $\tau \in H_1(K')$ and $\Theta \in H^1(K')$, we have $\|\Theta\|_\infty \|\tau\|_1 \geq |\langle \Theta, \tau \rangle|$.

Suppose $\beta_1(K') = \beta^1(K') = 1$. We define maps $H_1(K', \mathrm{Z}) \to H_1(K', \mathbb{R}) = H_1(K')$ and $H^1(K', \mathrm{Z}) \to H^1(K', \mathbb{R}) = H^1(K')$ by $\tau \mapsto \tau \otimes 1$ and $\Theta \mapsto \Theta \otimes 1$, respectively. That the images $H_1(K', \mathrm{Z}) \otimes 1$ and $H^1(K', \mathrm{Z}) \otimes 1$ are nonzero dual lattices follows from the universal coefficient theorem, which yields isomorphisms $H_1(K', \mathrm{Z}) \otimes \mathbb{R} \to H_1(K')$ and $H^1(K', \mathrm{Z}) \to \mathrm{Hom}(H_1(K', \mathrm{Z}), \mathrm{Z})$. Since these lattices are dual, there is a $\Theta \in H^1(K', \mathrm{Z}) \otimes 1$ with $\|\Theta\|_\infty \leq \frac{1}{2}$ if and only if for each nonzero $\tau \in H_1(K', \mathrm{Z}) \otimes 1$, $\|\tau\|_1 \geq 2$.

Theorem 5.3. *If $\Theta \in C_1^*$ is a normal to a facet F and τ and τ' are circuits such that the length of τ is 4 and $|\langle \theta, \tau' \rangle| \neq 0$, then every integral normal to F must have some entry with absolute value at least $\frac{|\langle \Theta, \tau \rangle|}{4|\langle \Theta, \tau' \rangle|}$.*

Proof: If Θ' is any integral normal for Σ_2^Θ then since $\beta_1(K_\Theta) = 1$ and Θ' and τ' are nonzero integral classes, $|\langle \Theta', \tau' \rangle| \geq 1$, and $\frac{|\langle \Theta', \tau \rangle|}{4|\langle \Theta', \tau' \rangle|} = \frac{|\langle \Theta, \tau \rangle|}{4|\langle \Theta, \tau' \rangle|}$. Since τ is represented by a circuit of length 4, $\|\tau\|_1 \leq 4$. By the duality of the norms $\|\Theta'\|_\infty \|\tau\|_1 \geq |\langle \Theta', \tau \rangle|$. Thus

$$\|\Theta'\|_\infty \geq \frac{|\langle \Theta', \tau \rangle|}{4} \geq \frac{|\langle \Theta', \tau \rangle|}{4|\langle \Theta', \tau' \rangle|} = \frac{|\langle \Theta, \tau \rangle|}{4|\langle \Theta, \tau' \rangle|}. \quad \triangleleft$$

Note that with the hypotheses of the theorem above, any nonnegative integral normal must have some entry with absolute value at least $\frac{|\langle \Theta, \tau \rangle|}{2|\langle \Theta, \tau' \rangle|}$.

When $min\{m, n\} \leq 2$ or $m = n = 3$, all facets are known to have **0-1** normals. When m or n is 2, $\Sigma(m, n)$ is a permutohedron. The case $m = n = 3$ can be verified by exact computation. (The case $(m,n) = (3,5)$ seems to have only **0-1** normals as well.) The minimal known examples of such non **0-1** supported facets correspond to the minimal examples of complexes having cycles of norm less than 2 and can be given by the following normals Θ. In each case the verification is immediate from the above theorem.

Example 5.4. *Consider the matrices*

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & \dots & k-1 & k-1 & k & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first two are normals to facets of $\Sigma(\Delta_3 \times \Delta_3)$ and $\Sigma(\Delta_2 \times \Delta_5)$, respectively, that do not have **0-1** normals. The third is a normal to a facet of $\Sigma(\Delta_2 \times \Delta_{2k-1})$ not representable by an integer normal having all entries less than $\frac{k}{4}$ in absolute value. $\triangleleft$

Theorem 5.5. *A facet F of Σ_2 with normal Θ has a* **0-1** *normal if and only if $\|\Theta\|_\infty \leq \frac{1}{2}$.*

Proof: It is clear that if Θ is **0-1** then $\|\Theta\|_\infty \leq \frac{1}{2}$. It will suffice to show that a class $\Theta \in H^1(K', \mathrm{Z}) \otimes 1$ with $\|\Theta\|_\infty \leq \frac{1}{2}$ is equivalent to a **0-1** matrix. To do this, choose a representative of Θ in C_1^* with $\|\Theta\|_\infty \leq \frac{1}{2}$. We can assume the maximum entry in Θ is $\frac{1}{2}$; this can be arranged by adding an appropriate multiple of the matrix of all $1's$. For simplicity we assume that $\theta_{11} = \frac{1}{2}$. Now define $E_{i*} := \sum_j E_{ij}$, $E_{*j} := \sum_i E_{ij}$ and $J = \sum_{ij} E_{ij}$, and take

$$\begin{aligned} \Theta' = \Theta &+ \sum_{i>1} \left(\frac{1}{2} - \theta_{i1} + \lfloor \theta_{i1} \rfloor \right) E_{i*} \\ &+ \sum_{j>1} \left(-\frac{1}{2} - \theta_{1j} + \lceil \theta_{1j} \rceil \right) E_{*j} + \frac{1}{2} J. \end{aligned} \tag{5.1}$$

The matrix $\Theta' \sim \Theta$ is **0-1** in the first row and first column. Since pairing Θ' with any length 4 circuit τ yields only integer values, all other entries of Θ' must be integral. Now the coefficients in the first summation of (5.1) lie in $(-\frac{1}{2}, \frac{1}{2}]$, while those in the second lie in $[-\frac{1}{2}, \frac{1}{2})$, so entries in Θ' differ from those in Θ by amounts in $(-\frac{1}{2}, \frac{3}{2})$, showing Θ' to be **0-1**. $\triangleleft$

Corollary 5.6. *A facet F of Σ_2 with normal Θ has a* **0-1** *normal if and only if every nonzero 1-cycle $\tau \in H_1(K_\Theta; Z)$ has $\|\tau\|_1 \geq 2$.*

6. The support function.

The *support function* $h_P : \mathbb{R}^n \to \mathbb{R}$ of a polytope $P \subset \mathbb{R}^n$ is defined by

$$h_P(\theta) := max\{ \langle x, \theta \rangle \mid x \in P \}$$

for $\theta \in \mathbb{R}^n$; thus

$$P^\theta = \{ x \in P \mid \langle x, \theta \rangle = h_P(\theta) \}. \tag{6.1}$$

Support functions are always positively homogeneous ($h_P(t\theta) = th_P(\theta)$, for $t \geq 0$) and subadditive ($h_P(\theta + \theta') \leq h_P(\theta) + h_P(\theta')$), and we have

$$P = \{ \; x \in \mathbb{R}^n \mid \langle x, \theta \rangle \leq h_P(\theta), \quad \text{for all} \quad \theta \in \mathbb{R}^n \; \}.$$

See, for example, [11] for a general discussion of support functions. We note that if $P = \{ \; x \mid Ax \leq b \; \}$, then $h_P(\theta) = min\{ \; \langle A_1, \theta \rangle, \ldots, \langle A_m, \theta \rangle \; \}$.

We first consider the support function of a general $\Sigma(P, Q)$. Let $\theta \in \mathbb{R}^n$ and for each $\sigma \in \Pi_\theta$, let $\Delta(\sigma)$ be any fixed triangulation of the polytope σ. For $\tau \in \Delta(\sigma)$, denote by x_τ the centroid of the subset $\pi^{-1}(\tau)$ of P (see (2.5)).

Proposition 6.1. *For $\theta \in \mathbb{R}^n$,*

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\operatorname{vol} \tau) \; \langle \theta, x_\tau \rangle. \tag{6.2}$$

Proof: By definition of fiber polytopes and properties of the Minkowski integral, we have for each $\theta \in \mathbb{R}^n$

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \int_Q h_{\pi^{-1}(q)}(\theta) \; dq = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \int_\sigma h_{\pi^{-1}(q)}(\theta) \; dq. \tag{6.3}$$

The function $h_{\pi^{-1}(q)}(\theta)$ is given by (2.4) and, for fixed θ, is a linear function of $q \in \sigma \in \Pi_\theta$. By (2.6) and (6.1), for a simplex $\tau \in \Delta(\sigma)$ and $q \in \tau$ we have $h_{\pi^{-1}(q)}(\theta) = \langle \theta, p \rangle$, where p is the unique point of $\pi^{-1}(\tau)$ such that $\pi(p) = q$. Thus by (6.3) we get

$$h_{\Sigma(P,Q)}(\theta) = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} \int_\tau h_{\pi^{-1}(q)}(\theta) \; dq = \frac{1}{vol\, Q} \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\operatorname{vol} \tau) \; \langle \theta, x_\tau \rangle. \triangleleft$$

Note that $vol\, \tau = 0$ if τ is not a maximal simplex of $\Delta(\sigma)$. For a triangulation Δ refining Π_θ, Proposition 6.1 follows directly from [5; Cor. 2.6]. There is a slight advantage to the more general formulation here in that it allows one to use arbitrary triangulations of each cell of Π_θ.

To give the support function for secondary polytopes, we must be a bit more careful about the scaling involved in passing from fiber polytopes. Recall that if Q is a polytope with n vertices then the secondary polytope $\Sigma(Q)$ is homothetic to the fiber polytope $\Sigma(\Delta_{n-1}, Q)$, i.e., $\Sigma(Q) = (dim\, Q + 1) \; vol(Q) \; \Sigma(\Delta_{n-1}, Q)$ [5; Thm. 2.5]. Again, letting Π_θ be any coherent subdivision of Q and for each $\sigma \in \Pi_\theta$, $\Delta(\sigma)$ a triangulation of σ, define, for each $\tau \in \Delta(\sigma)$, $e_\tau := \sum_{q_i \in \tau} e_i = (dim\, Q + 1) x_\tau$.

Corollary 6.2. *For $\theta \in \mathbb{R}^n$,*

$$h_{\Sigma(Q)}(\theta) = \sum_{\sigma \in \Pi_\theta} \sum_{\tau \in \Delta(\sigma)} (\operatorname{vol} \tau) \; \langle \theta, e_\tau \rangle. \triangleleft \tag{6.4}$$

Denote by h_{mn} the support function of the Newton polytope $\Sigma(m,n)$. This is a subadditive function on the space of $m \times n$ matrices Θ. Determining $h_{mn}(\Theta)$ for all integral matrices Θ is equivalent to determining a complete set of inequalities determining $\Sigma(m,n)$. Restricted to **0-1** matrices, h_{mn} can be viewed as giving a monotone, subadditive function on subsets of $[m]\times[n]$ (or *shapes* fitting in an $m\times n$ array); i.e., for $S, T \subset [m]\times[n]$, if $S \subseteq T$, then $h_{mn}(S) \leq h_{mn}(T)$, and if $S \cap T = \emptyset$, then $h_{mn}(S \cup T) \leq h_{mn}(S) + h_{mn}(T)$. It would be of interest to determine $h_{mn}(S)$ for all $S \subset [m]\times[n]$. We give below a formula for $h_{mn}(S)$ in terms of line covers of S and describe how to evaluate it for certain simple shapes.

By (6.4), we can relate the evaluation of $h_{mn}(\Theta)$ to the subdivision Π_Θ. Since each simplex in $\Delta_{m-1}\times\Delta_{n-1}$ has the same volume by the unimodularity of its coordinates (see, e.g., [9; Lemma 2]), the support function h_{mn} is obtained from (6.4) by setting $vol\,\tau = 1$ for each maximal simplex τ. In particular, when Θ is a **0-1** matrix, the indicator function of some shape S, we can view this shape as a subset of the vertex set of $\Delta_{m-1}\times\Delta_{n-1}$. In this case, the inner product in (6.4) counts the number of vertices in the set $S \cap \tau$.

Corollary 6.3. *For any shape $S \subset [m]\times[n]$,*

$$h_{mn}(S) = \sum_{(I,J)} \sum_{\tau\in\Delta(I,J)} |S \cap \tau|, \tag{6.5}$$

where the first sum is over minimal line covers (I,J) of S (with $J \neq [n]$) and the second sum is over maximal simplices of any triangulation $\Delta(I,J)$ of the cell σ_{IJ} in (3.4). ◁

The primary difficulty in evaluating (6.5) for any particular **0-1** matrix Θ is determining triangulations of the cells σ_{IJ} for minimal line covers of S. In certain cases this presents no problem. Recall the *standard triangulation* of $\Delta_{m-1}\times\Delta_{n-1}$ whose maximal simplices have vertex sets determined by all *monotone paths* in an $m \times n$ array, i.e., paths from the upper left entry to the lower right entry that only move down or to the right. In [2], it is shown that lexicographically ordering these paths gives a shelling of this triangulation. From this it follows that if one takes an initial segment of this triangulation, in the lexicographic order favoring moves down (so the initial path in this order moves down the first column and across the last row), stopping just prior to the addition of a new vertex, then the simplices so defined form a triangulation of the convex hull of the vertices involved.

Thus if, after reordering rows and columns, the vertices of a cell σ_{IJ} are arranged in a block triangular form, then a triangulation of σ_{IJ} is formed by taking all monotone paths among the $1's$ in this array. For example, if S is a generalized hook, then it is easy to check that the two or three cells σ_{IJ} in Π_S can be put into this form. This also applies to all the cells in the coarse subdivision given in Example 3.4.

It is a considerably more difficult problem to count the number of meetings of these paths with a prescribed fixed set of cells, which is necessary for a complete evaluation of (6.5). In certain cases, this can be done, as is illustrated by the examples below.

Proposition 6.4. *If $\Theta_{(k)}$ is a* **0-1** *$m \times n$ matrix with k $1's$, all contained in a single line* *(row or column), then*

$$h_{mn}(\Theta_{(k)}) = \sum_{j=1}^{k} \binom{m+n-j-1}{m-1}. \tag{6.6}$$

Proof: By Corollary 4.5, such matrices correspond to facets of $\Sigma(m,n)$. We can assume that $\Theta_{(k)}$ has $1's$ as the first k entries in the first row. In this case the subdivision $\Pi_{\Theta_{(k)}}$ consists of two cells σ_1 and σ_2, corresponding, respectively, to the minimal line covers $(\{1\},\emptyset)$ and $(\emptyset,\{1,\ldots,k\})$. By (3.4), these cells have vertices indicated by the $1's$ in the $m\times n$ arrays

$$\begin{pmatrix} 1\ldots10\ldots0 & \\ & \\ E_1 & \end{pmatrix} \text{ and } \begin{pmatrix} 1\ldots1 & \\ & E_2 \\ O & \end{pmatrix},$$

where E_1 is an $(m-1)\times n$ array of $1's$ and E_2 is an $m\times(n-k)$ array of $1's$. By the above discussion, one sees that $vol\,\sigma_1 = \binom{m+n-2}{m-1} - \binom{m+n-k-2}{m-1}$ (which is the number of paths in an $m\times n$ array minus the number in an $m\times(n-k)$ array) and $vol\,\sigma_2 = \binom{m+n-k-2}{m-1}$. Counting the incidence of these paths with the first k entries in row 1, we get

$$h_{mn}(\Theta_{(k)}) = \sum_{j=1}^{k}\left[\binom{m+n-j-1}{m-1} - \binom{m+n-k-2}{m-1}\right] + k\binom{m+n-k-2}{m-1},$$

which is (6.6). $\triangleleft$

Remark 6.5. $\Sigma(2,n)$ is congruent to the permutohedron P_n. (See [14; Prop. 1.12] where this is proved for the Newton polytope of the product of *maximal* minors of a $2\times n$ matrix, from which one obtains $\Sigma(2,n)$ by translation by the $2\times n$ matrix of $1's$.) In this case, the 2^n-2 facets of $\Sigma(2,n)$ have normals given by all matrices $\Theta_{(k)}$ consisting of k $1's$ in the first row, $1<k<n$, and (6.6) reduces to $h_{2,n}(\Theta_{(k)}) = \binom{n+1}{2} - \binom{k+1}{2}$ (c.f., [15; §5.3.1]).

Example 6.6. *The case $m=n=3$. From (6.6) we get $h_{3,3}(\Theta_{(1)})=6$ and $h_{3,3}(\Theta_{(2)})=9$. The only other* **0-1** *matrix corresponding to a connected subgraph of $K_{2,2}$, and thus to a facet of $\Sigma(3,3)$, which is not equivalent to one of these two, is (up to row and column permutation)* $\Theta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Here $h_{3,3}(\Theta)=14$: corresponding to line covers $(\{1,2\},\emptyset)$, $(\emptyset,\{1,2\})$ and $(\{1\},\{1\})$, we have cells σ_1, σ_2 and σ_3, contributing 5, 5 and 4 to the sum (6.5).*

One can check by direct calculation that all row and column permutations of $\Theta_{(1)}$, $\Theta_{(2)}$ and Θ give all the facet normals of $\Sigma(3,3)$. Thus $\Sigma(3,3)$ consists of all 3×3 matrices having row and column sums 10, such that each entry is at most 6, any two entries in the same row or column sum to at most 9, and any 3 entries, two in the same row and the third in one of their columns, sum to at most 14. This polytope has 108 vertices and so $\Delta_2\times\Delta_2$ has 108 (coherent) triangulations.

Θ	$h_{4,4}(\Theta)$	Θ	$h_{4,4}(\Theta)$	Θ	$h_{4,4}(\Theta)$
$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	20	$\begin{pmatrix}1&1&1&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	53	$\begin{pmatrix}1&1&1&0\\1&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	76
$\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	30	$\begin{pmatrix}0&1&1&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	56	$\begin{pmatrix}1&1&0&0\\1&0&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}$	84
$\begin{pmatrix}1&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	34	$\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	62	$\begin{pmatrix}1&1&1&0\\1&1&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$	86
$\begin{pmatrix}1&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	46	$\begin{pmatrix}1&1&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$	71	$\begin{pmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\1&0&0&1\end{pmatrix}$	116
$\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	52	$\begin{pmatrix}0&1&1&0\\1&1&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$	74	$\begin{pmatrix}2&0&1&0\\0&1&1&0\\1&1&0&0\\0&0&0&0\end{pmatrix}$	95

Table 1

Example 6.7. *The case $m = n = 4$. We tabulate in Table 1 the values of $h_{4,4}(\Theta)$ for those* **0-1** *Θ corresponding to facets by Theorem 4.3, as well as one example of a facet normal with no zero row or column and one of the non-***0-1** *normals in Example 5.4.*

We have omitted row and column permutations of $\Theta's$ in the table (which have the same values of $h_{4,4}$). We also omit $\Theta's$ which are equivalent to those on the table, and which thus yield the same facet, although usually different, but easily derivable, values of $h_{4,4}$ (using the fact that elements of $\Sigma(4,4)$ have row and column sums equal to 35). Note the monotonicity and subadditivity of $h_{4,4}$, as illustrated, for example, by the first, second and fourth values in the left table.

A complete list of the normals of facets of $\Sigma(4,4)$ has not been conjectured. The table accounts for all the **0-1** facet normals having a zero row. There appear to be many others. For example, the next-to-last entry in the table can be shown to be normal to a facet by analyzing its edges as in §4. (However, it does not give a facet of Σ_2.) Looking at the Table 1, one is led to ask whether, as a function on **0-1** matrices, h_{mn} can be viewed as giving the energy, in some unspecified sense, of the corresponding configuration of $1's$.

The values in the table were all obtained by application of (6.5). In some cases, the resulting cells σ_{IJ} could only be partially triangulated by initial segments of the standard triangulation. In these cases, a full triangulation was obtained by "placing" the remaining vertices outside the partial triangulation, forming further simplices by joining to exposed facets on the boundary. In this regard, the following observation is useful.

Full-dimensional simplices on the vertices of $\Delta_{m-1} \times \Delta_{n-1}$ correspond to edge sets of spanning trees in the graph $K_{m,n}$. A simplex τ of codimension 1 corresponds to an acyclic subgraph G_τ having 2 components. To form a full-dimensional simplex containing τ, one must add an edge which joins the components of G_τ. Two such simplices lie on opposite sides of the hyperplane spanned by τ if and only if the added edges meet opposite sides of the first component of G_τ.

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