EXERCISES FOR 1920 SECTION, FALL 2008

* Starred exercises are intended as "challenges". Some are more theoretical applications of the course material; others are related to other parts of the same problem but are not necessarily directly related to course material.

$\mathbb{R}^2$ denotes the plane, usually with x and y as coordinates.

$\mathbb{R}^3$ denotes three-dimensional space, usually with x, y, and z as coordinates.

A few of the exercises in this handout use the hyperbolic functions, which are not part of the course material, but technically belong to the prerequisites (although I know it is rare to encounter them in high school). The hyperbolic cosine and hyperbolic sine are defined by

$$\cosh t = \frac{e^t + e^{-t}}{2} \qquad \text{and} \qquad \sinh t = \frac{e^t - e^{-t}}{2}.$$

(The remaining hyperbolic functions are defined by analogy with the trigonometric functions.) You should check that these satisfy the properties

$$\frac{d}{dt}\sinh t = \cosh t \qquad \text{and} \qquad \frac{d}{dt}\cosh t = \sinh t$$

(note: no change of sign in the latter) as well as the "hyperbolic Pythagorean identity"

$$\cosh^2 t - \sinh^2 t = 1.$$

It is useful to note that $\cosh t$ is even, $\sinh t$ is odd, and $\cosh t \geq 1$ for all t.

0. Introduction and review

Exercise 1. Suppose you only had the formula for the circumference of a circle, $C = 2\pi r$. How could you find a formula for the area?

Exercise 2. Suppose an infinitesimally thin circular sheet of plastic with a uniform charge density σ is placed in the (x, y)-plane. The *electric potential* of a point P_0 in space due to this disk is

$$\int_{\text{points } P \text{ in the disk}} \frac{\text{charge at } P}{\text{distance from } P_0 \text{ to } P}.$$

Suppose that the disk is centered at $(0, 0, 0)$ and P_0 is a point on the z-axis.

a. What is the distance from P_0 to a point P of the disk?

b. Note that the points of the disk some fixed distance from P_0 form a circle. The "infinitesimal potential" due to this circle is $\dfrac{2\pi\sigma s\, ds}{\text{distance from this circle to } P_0}$, where s is the radius of the circle. Find an integral that computes the total potential at P_0, and evaluate it.

1. Vectors and the geometry of $\mathbb{R}^3$

Exercise 3 (Thomas, p. 861 #24)**.** Consider three nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^2$ such that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}|$ and any two have an angle of 120° between them. Sketch the following:

$$\mathbf{u}+\mathbf{v}, \qquad \mathbf{u}-\mathbf{v}, \qquad 2\mathbf{u}-\mathbf{v}, \qquad \mathbf{u}-\mathbf{v}+\mathbf{w}, \qquad \mathbf{u}+\mathbf{v}+\mathbf{w}.$$

If you observe any special configurations, try to prove that your guess is right.

Exercise 4. Recall that $|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$.

- **a.** Prove the *parallelogram law*:

$$|\mathbf{u}+\mathbf{v}|^2 + |\mathbf{u}-\mathbf{v}|^2 = 2\left(|\mathbf{u}|^2 + |\mathbf{v}|^2\right).$$

  Use it to prove the Pythagorean theorem.
- **b.** Prove the *polarization identity*:

$$\mathbf{u}\cdot\mathbf{v} = \frac{1}{2}\left(|\mathbf{u}+\mathbf{v}|^2 - |\mathbf{u}|^2 - |\mathbf{v}|^2\right).$$

Exercise 5. How would you find an equation for the plane through $(0,0,0)$, $(1,1,1)$, and $(-2,0,3)$? How would you describe the line perpendicular to this plane and passing through the origin?

Exercise 6.

- **a.** What kinds of data can be used to determine a point in $\mathbb{R}^3$? a line? a plane?
- **b.** How can we tell whether a pair of lines in $\mathbb{R}^3$ is parallel, intersecting, or skew?
- **c.** How can we tell whether a pair of planes in $\mathbb{R}^3$ intersects or is parallel?
- **d.** What are the possible relative positions of three planes in $\mathbb{R}^3$?

2. Parametrized curves

Exercise 7. We parametrize a *helix* by $(\cos at, \sin at, t)$ for some fixed $a > 0$.

- **a.** Show that this parameterization has constant speed.
- **b.** Show that the helix makes a constant angle with the vertical direction.

Exercise 8. The *twisted cubic* is parametrized in $\mathbb{R}^3$ by (t, t^2, t^3) for $-\infty < t < \infty$.

- **a.** Sketch projections of the twisted cubic onto the (x,y)-, (x,z)-, and (y,z)-planes.
- ***b.** Show that any four distinct points on the twisted cubic do not lie in a single plane. (*Hint:* If any four points do lie in a plane, then the vectors from one of the points to the remaining three will span a parallelepiped having zero volume.)

Exercise 9. Why does the curve $y = |x|$ not have a smooth parameterization? Does it have a differentiable parameterization?

Exercise 10. Using the fact that a "cusp" (or "corner") of a differentiable path can only occur when the velocity vector vanishes, how many cusps would you guess lie on the curve parametrized by $(\cos^3 t, \sin^3 t)$? (The resulting curve is called an *astroid*.)

Exercise 11. Starting from the parameterization of a helix, find a parameterization of a *loxodrome* (which spirals from south to north along a fixed compass direction on a sphere) by scaling the points of the helix to lie on the unit sphere.

3. Line integrals, work, and flux

Exercise 12.

- **a.** What is the mass of the circular wire with equation $x^2+y^2=1$ and density $3+x+y$?
- **b.** What density in the plane or in space yields a line integral that computes arclength?

Exercise 13 (Thomas, p. 1223 #4)**.** The path $(\cos t+t\sin t, \sin t-t\cos t)$ traces out a curve called an *involute of the circle*. Find the line integral of $\sqrt{x^2+y^2}$ over the piece of this curve given by $0 \leq t \leq \sqrt{3}$.

Exercise 14. Consider three wires in the plane, all having their endpoints at $(-1,1)$ and $(1,1)$. One is a straight segment, another is an arc of a circle centered at $(0,0)$, and the third is the upper half of the circle with radius 1 centered at $(0,1)$. The density of all three wires is given by $\delta(x,y)=1/y$. Which has the most mass? the least?

Exercise 15. Let the force of gravity on a 1 kg object near the surface of the Earth be approximated by the field $\mathbf{F}(x,y,z)=-10\,\mathbf{k}$.

- **a.** Find the work done by gravity on a 1 kg box as it is carried up one turn of the helix $(\cos 10t, \sin 10t, t)$.
- **b.** Find the work done by gravity on a 1 kg ball dropped from 20 m and following the parabolic path $z=-5y^2+20$ until it hits the ground.

Exercise 16.

- **a.** Sketch the field on $\mathbb{R}^3$ minus the origin such that the vector at each point (x,y,z) points to the origin and whose length is the inverse of the square of the distance from (x,y,z) to the origin.
- **b.** Find a formula that describes this field (*cf.* Thomas p. 1158, #5).
- **c.** What is the total work done by this field over one complete orbit of an object in a circular orbit around the origin? What if the orbit is an ellipse? (*Hint:* You should not need to calculate in either case; use geometric reasoning.)

Exercise 17. Consider the vector field in $\mathbb{R}^2$ given by $\mathbf{F}(x,y)=(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}$.

- **a.** Sketch this field.
- **b.** Find the work done by this field along the semicircular path $x^2+y^2=1$, $y \geq 0$, going right to left. What is the work along the entire unit circle?
- **c.** Find the work done by this field along the circular path centered at $(0,1)$ with radius 1, moving counterclockwise.
- **d.** Can you predict the work done along the circle centered at $(1,0)$ with radius 1, again moving counterclockwise? Give a reason for your answer.
- **e.** Find the flux of $\mathbf{F}$ across each of the curves in **(c)** and **(d)**. Can you predict either of them ahead of time?

Exercise 18. Consider the field $\mathbf{F}(x,y)=-\dfrac{x}{(x^2+y^2)^{3/2}}\,\mathbf{i}-\dfrac{y}{(x^2+y^2)^{3/2}}\,\mathbf{j}$ in $\mathbb{R}^2$ minus the origin. If this represents the gravitational field of a massive body at the origin, then a much smaller object caught in this field might follow a parabolic trajectory described by $y=\frac{1}{4}x^2-1$ (i.e., the focus of this parabola is at the origin). Find the work done by $\mathbf{F}$ on such an object from the time it makes its closest approach to the origin (its *perihelion*) to a point further along its parabolic path. As the distance the object has traveled tends to infinity, does the work done by the field also go to infinity?

Exercise 19. Let $\mathbf{F}(x, y) = x\,\mathbf{j}$ and $\mathbf{G}(x, y) = y\,\mathbf{i}$.
(Each of the paths below should be parametrized to have counterclockwise motion.)

- **a.** Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ along the boundary of a unit square, with corners at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.
- **b.** Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ along the unit circle.
- **c.** Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ along the boundary of a rectangle centered at the origin with width $2a$ and height $2b$.
- **d.** Compute the work done by $\mathbf{F}$ and $\mathbf{G}$ along the boundary of an ellipse centered at the origin with major axis $2a$ and minor axis $2b$.
- **e.** Can you guess what $\int \mathbf{F} \cdot d\mathbf{r}$ and $\int \mathbf{G} \cdot d\mathbf{r}$ are for any closed path $\mathbf{r}$ in the plane?
- **f.** How do your answers to any of the above change if you replace work with flux?

Note: If $\mathbf{r}$ parametrizes a curve C, then $\int \mathbf{F} \cdot d\mathbf{r}$ and $\int \mathbf{G} \cdot d\mathbf{r}$ are more commonly written $\int_C x\,dy$ and $\int_C y\,dx$. In this notation, what does the form $\frac{1}{2}(x\,dy - y\,dx)$ measure when integrated along a closed curve C?

4. Functions of several variables

Exercise 20.

- **a.** What are the level sets of the function $f(x, y, z) = x^2 + y^2 + z^2$?
- **b.** If a, b, and c are not all zero, what are the level sets of the function $g(x, y, z) = ax + by + cz$? What if a, b, and c *are* all zero?
- **c.** Give an example of a function whose level sets are spheres centered at $(1, -1, 2)$.
- **d.** Give an example of a function whose level sets are ellipsoids.
- **e.** Give an example of a function whose level sets are cylinders.

Exercise 21 (*cf.* examples in §14.2 of Thomas)**.** In this exercise, all functions are extended to be defined on all of $\mathbb{R}^2$ by setting their value to 0 at $(0, 0)$.

- **a.** Show that the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$ is not continuous at $(0, 0)$.
- ***b.** Show that the function $g(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ is continuous at $(0, 0)$.
- **c.** Let h be the function $h(x, y) = \dfrac{x^2 y}{x^4 + y^2}$. Show that all lines approaching $(0, 0)$ give the same limiting value for h. Show that approaching along a curve of the form $y = mx^2$ gives a limiting value that depends on m. What can you conclude about the continuity of h at $(0, 0)$?
- ***d.** Define F on $\mathbb{R}^2$ by $F(x, y) = \dfrac{x^d y}{x^{2d} + y^2}$. Show that the limit of F at $(0, 0)$ along any path $y = p(x)$, where p is a polynomial of degree less than d, is zero. Show that F is not continuous at $(0, 0)$.
- ***e.** Suppose $p(x, y)$ is any polynomial such that every term has degree at least d. Show that $G(x, y) = \dfrac{p(x, y)}{(x^2 + y^2)^{d/2}}$ is continuous at $(0, 0)$. (Recall: the degree of a term is the sum of the powers of x and y in that term.)

Note: Parts **(d)** and **(e)** are direct generalizations of **(c)** and **(b)**. It's worth thinking about why they're important as illustrations of what can happen, even if you don't work them out fully. Hint for these: in both cases, you can assume p is a monomial. Why?

Exercise 22. Define $f(x, y) = xy$.

a. Sketch the graph of f. Sketch the level curves of f.
b. Find the partial derivatives of f.
c. At what points (x, y) is f increasing in the x-direction? in the y-direction? When is it increasing faster in the x-direction than in the y-direction?

Exercise 23. Define $F(x, y, z) = xyz$.

a. Sketch or describe the level sets of F.
b. In which of the three cardinal directions (positive x-, positive y-, and positive z-) is F increasing fastest at the point $(1, -2, 3)$?
c. Can F ever be increasing in two of these directions and decreasing in the third?
d. Verify the chain rule for the functions from $\mathbb{R}$ to $\mathbb{R}$ given by evaluating F at each point of the twisted cubic $\mathbf{c}(t) = t\,\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and the helix $\mathbf{h}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$.

5. Tangent planes, differentials, and extreme values

Exercise 24.

a. Find an equation of the tangent line to the curve $y^2 = x^3 + 1$ at $(1, \sqrt{2})$.
b. Find an equation of the tangent plane to the unit sphere in $\mathbb{R}^3$ at a point (x_0, y_0, z_0).
c. Find an equation of the tangent plane to the hyperboloid $z^2 = 1 + x^2 + y^2$ at $(2, 2, 3)$.

Exercise 25. Suppose α, β, and γ are the three angles in a Euclidean triangle T.

***a.** Show that the point $P = (\cot\alpha, \cot\beta, \cot\gamma)$ lies on the hyperboloid H defined by $xy + yz + zx = 1$.
b. Find an equation of the tangent plane to H at P.
***c.** Can you find a way to relate the side lengths of the triangle T to the coordinates of a normal vector to H at P?

Exercise 26 (*cf.* Thomas pp. 1019–1020)**.** Consider the function $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$.

a. Compute the total differential df.
b. What range of values does the linearization of f take for $|x - 3| \leq 0.1$, $|y - 2| \leq 0.1$?
c. What is an upper bound for the error introduced by replacing f with its linearization over these values of x and y?

Exercise 27. Compute the Hessians of $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$ at the origin. How do the shapes of the graphs of these functions relate to each other? Can you glean anything from the observation that $f(x, y) = (x + y)(x - y)$? How does the shape of the graph of f change, locally and globally, by adding a y^3 term?

Exercise 28 (Thomas, p. 1034 #41)**.** Consider a flat circular plate having the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and coldest points on the plate.

Exercise 29. Define f on $\mathbb{R}^2$ by
$$f(x,y) = x^3 - 3xy^2 - 3x.$$
Find all critical points of f. Are any of them local maxima or minima? Which ones? Does f have global extrema? Give reasons.

Exercise 30. Find the maximum and minimum values of the function $f(x,y,z) = z$ on the intersection of the surface defined by $xy + yz + zx = 1$ and the plane $x + y + z = 3$.

6. Double integrals and polar coordinates

Exercise 31. Integrate e^{-y^2} over the triangle bounded by $x = 0$, $y = 1$, and $x = y$.

Exercise 32 (*cf.* Wikipedia, "Fubini's Theorem")**.** Show that the improper integrals
$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dx\,dy \qquad \text{and} \qquad \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,dy\,dx$$
both converge (conditionally), but not to the same value.

Exercise 33.

- **a.** Find the integral of the function $1/y^2$ over the region described by $-1 \le x \le 1$ and $x^2 + y^2 \ge 1$, with $y > 0$. Note that this is an improper integral (the region is not closed and bounded), so our version of Fubini's theorem does not apply. Show that the conclusion of Fubini's theorem still holds by computing in the order "y, then x" and "x, then y".
- ***b.** Find the center of mass of the shape described in part **(a)** with $1/y^2$ used as a density function.
- ***c.** Find the volume contained in the (unbounded) region of $\mathbb{R}^3$ described by
$$x \ge 0, \qquad y \ge 0, \qquad z \ge 0, \qquad xy + yz + zx \le 1.$$

Exercise 34.

- **a.** Consider a circular disk with radius R and constant thickness h. Find the moment of inertia of this disk about its axis of symmetry.
- **b.** Find the moment of inertia of a sphere about any axis through its center. This can be done two ways: think of the sphere as being composed of
  - either stacked infinitesimal disks perpendicular to the axis
  - or nested infinitesimal cylindrical shells sharing the same axis.

Exercise 35.

- **a.** Find an equation in rectangular coordinates for the curve given by $r = \cos\theta$ in polar coordinates.
- **b.** Graph each of the following limaçons (*cf.* Thomas p. 724, #21–24):
$$r = \frac{1}{2} + \cos\theta, \qquad\qquad r = 1 + \cos\theta,$$
$$r = \frac{3}{2} + \cos\theta, \qquad\qquad r = 2 + \cos\theta.$$
- **c.** Find the area contained between the two loops of the first curve in part **(b)**.
- **d.** How do your answers to parts **(a)**–**(c)** change when $\cos\theta$ is replaced with $\sin\theta$?

Exercise 36. Let $0 < a < 1$. Integrate $1/(1-(x^2+y^2))^2$ over the disk of radius a.

Exercise 37.

a. Find the integral of $1/x^2$ over the region in $\mathbb{R}^2$ described by

$$|x| \geq |y|, \qquad 1 \leq x^2 + y^2 \leq 4, \qquad x > 0.$$

***b.** Show that the integral of $1/x^2$ over the region described by

$$m|x| \geq |y|, \qquad a^2 \leq x^2 + y^2 \leq b^2, \qquad x > 0$$

only depends on $m > 0$ and the ratio b/a.

Exercise 38 (Thomas p. 1097, #27)**.** Find the area and centroid of the cardioid $r = 1 + \cos\theta$.

Exercise 39. Use polar coordinates to show that the area of an ellipse with axes of length $2a$ and $2b$ is πab.

Exercise 40 (A proof that $\sum 1/n^2 = \pi^2/6$)**.** This is an extremely challenging problem, both computationally and conceptually, but one that is still accessible using the methods we have developed (plus a bit of the theory of infinite series, which you should have seen before).

***a.** Recall that $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} x^n$. Use this to show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(You will need to switch the order of an integral and an infinite series; this just requires applying another version of Fubini's theorem and is allowed in this case, essentially because everything involved is positive and the integral and series both converge.)

***b.** The integrand in part **(a)** is symmetric in x and y. Show that the integral of this function over the triangle T defined by $0 \leq y \leq x \leq 1$ is

$$\iint_T \frac{1}{1-xy}\,dx\,dy = \frac{\pi^2}{12}$$

by changing to polar coordinates. Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(The usual proof of this fact uses Fourier series, which you will see in later courses.)

7. Triple integrals

Exercise 41.

a. Show that a right circular cone of height h whose base has radius r has volume $\frac{1}{3}\pi r^2 h$.

***b.** Consider a cone whose base is the unit square (defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$) and whose summit is (x_0, y_0, z_0), with $z_0 > 0$. Show that the volume of this cone is $|z_0|/3$. (Note that x_0 and y_0 can be any values.)

***c.** Let R be any shape in the (x, y)-plane of $\mathbb{R}^3$ that has finite area A, and let $P(x_0, y_0, h)$ be any point in $\mathbb{R}^3$ with $h > 0$. Argue (using part **(b)** and Riemann sums, for example) that the volume of the cone with base R and summit P has volume $\frac{1}{3}Ah$. (This includes pyramids.)

d. Find the volume of the *octahedron*, defined in $\mathbb{R}^3$ by $|x| + |y| + |z| \leq 1$.

***e.** Find the area of one of the faces of the octahedron. Use the result from part **(d)** and the formula from part **(c)**, with the distance from the origin to the center of the face as the height of a pyramid.

f. Find the volume of the *tetrahedron*, with four equilateral triangles as faces.

Exercise 42. This exercise uses triple integrals to study *four-dimensional volumes*. Just as a single integral can be interpreted as the area in $\mathbb{R}^2$ between the graph of a function $f(x)$ and the x-axis, and a double integral can be interpreted as the volume in $\mathbb{R}^3$ between the graph of a function $f(x, y)$ and the (x, y)-plane, a triple integral can be interpreted as the 4d volume between the graph of a function $f(x, y, z)$ and the (x, y, z)-space in $\mathbb{R}^4$. (Often we use w as a fourth coordinate, but this is not immediately relevant.)

a. We call a sphere in $\mathbb{R}^4$ a *hypersphere*. The 4d volume of a hemi-hypersphere can be computed as the integral of $\sqrt{R^2 - x^2 - y^2 - z^2}$ over the interior of the sphere of radius R in $\mathbb{R}^3$. Convince yourself that this is true, then find the 4d volume of a hypersphere of radius R. (Spherical coordinates will probably be easiest.) Differentiate with respect to R to find the surface area (really, "surface 3d volume") of a hypersphere.

b. Choose some $h > 0$. Integrate:
- $f(x, y, z) = h(1 - |x| - |y| - |z|)$ over the octahedron $|x| + |y| + |z| \leq 1$. (*Hint*: just compute in the first octant, then multiply by 8. Why does this work?)
- $g(x, y, z) = h(1 - \sqrt{x^2 + y^2 + z^2})$ over the interior of the unit sphere. (This function also makes an appearance in the homework problem #85 on p. 1128.)

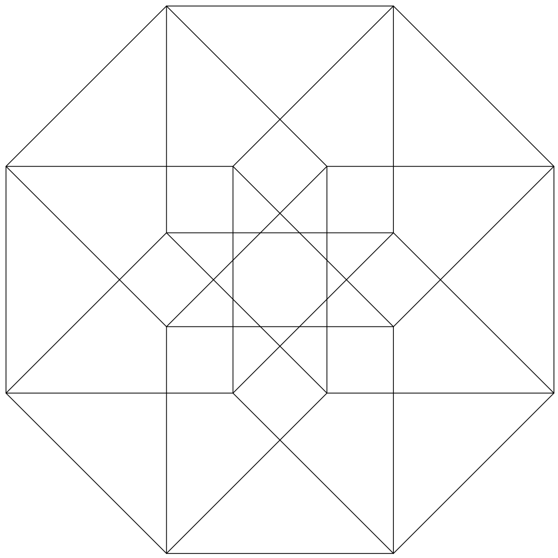
***c.** Make a guess at the formula for the 4d volume of a "cone" whose "base" is a solid in (x, y, z)-space and whose "summit" has h as its fourth coordinate.

***d.** In part **(f)** of the previous exercise you found the volume of a tetrahedron. Here's another way that uses less trigonometry and is analogous to finding the area of an equilateral triangle as in part **(e)** of the previous exercise.

Assuming that the formula you guessed in part **(c)** for the 4d volume of a cone is correct, find the 4d volume of a "pyramid" whose base is one-eighth of the octahedron:

$$x \geq 0, \qquad y \geq 0, \qquad z \geq 0, \qquad x + y + z \leq 1,$$

and whose height is 1. The tetrahedron is the face of this pyramid with vertices at $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Find the distance from the origin in $\mathbb{R}^4$ to the center of this face and again apply your formula from **(c)**. By what factor should you multiply to find the volume of a tetrahedron with unit edge lengths?



A *hypercube* with edges of length 1 provides the measure of unit volume in $\mathbb{R}^4$.

This picture shows a highly symmetric projection of a hypercube into $\mathbb{R}^2$ so that all the edge lengths are equal. Each edge has one of only four directions. By choosing one of these directions and removing all edges in that direction, you can find a pair of 3d cubes that are opposite "faces" of the hypercube.

Check out these references to hypercubes (also called *tesseracts*) in art and literature:

- Salvador Dalí's painting *Corpus Hypercubicus*
- Madeleine L'Engle's novel *A Wrinkle in Time*
- Robert Heinlein's short story "—And He Built a Crooked House"
- the film version (2007) of *Flatland*

Exercise 43. This exercise introduces *hyperbolic coordinates*, which are a twist on spherical coordinates. We start with spherical coordinates, but we replace the radial distance ρ and the zenith angle φ with parameters τ and σ, so that

$$x = \tau \sinh \sigma \cos \theta, \quad y = \tau \sinh \sigma \sin \theta, \quad z = \tau \cosh \sigma.$$

These coordinates are valid on the region defined by $z > \sqrt{x^2 + y^2}$. The volume form is

$$dV = \tau^2 \sinh \sigma \, d\tau \, d\sigma \, d\theta.$$

The parameters ρ and τ really are different: while $\rho^2 = x^2 + y^2 + z^2$, for τ we have instead $\tau^2 = z^2 - x^2 - y^2$, i.e., τ is constant on hyperboloids rather than spheres.

Use hyperbolic coordinates to find the volumes of these regions in the first octant:

- bounded by the hyperboloid $z^2 = 1 + x^2 + y^2$ and the cone $z^2 = 4(x^2 + y^2)$;
- bounded by the hyperboloid $z^2 = 1 + x^2 + y^2$ and the plane $z = x + y$.

8. Divergence, curl, surface integrals, Green's Theorems, Stokes' Theorem, and the Divergence Theorem

Exercise 44. For each of the following vector fields in $\mathbb{R}^2$, sketch the field and determine whether or not it is conservative. Find a potential function for each conservative field, and sketch its graph.

$$\mathbf{F}_1(x, y) = x\,\mathbf{i} + y\,\mathbf{j} \qquad \mathbf{F}_3(x, y) = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$$

$$\mathbf{F}_2(x, y) = -y\,\mathbf{i} + x\,\mathbf{j} \qquad \mathbf{F}_4(x, y) = (x^2 - y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$$

Compute the divergence and the circulation density of each of these fields.

Exercise 45. Let $\mathbf{F}(x,y) = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$. Let C be a simple closed curve in $\mathbb{R}^2$, oriented counterclockwise.

- **a.** Using Green's Theorem, show that the flow and the flux of $\mathbf{F}$ along (or across, respectively) a simple closed curve C in $\mathbb{R}^2$ only depend on M_x and M_y, the moments about the x-axis and y-axis of the region R enclosed by C.
- **b.** Using your computations from part **(a)**, find the flux and the flow when C is each of the following curves (without doing any more calculus!):
  - the rectangle with vertices $(1,2)$, $(1,-2)$, $(3,-2)$, and $(3,2)$;
  - the circle with center $(-3,3)$ and radius 2.

Exercise 46. This exercise illustrates why the hypothesis of simple connectivity is necessary, while also showing a nifty field whose flux across a curve computes an interesting geometric property. Let $\mathbf{\Theta}$ (that's a capital θ) be the vector field defined by

$$\mathbf{\Theta}(x,y) = \frac{x}{x^2+y^2}\,\mathbf{i} + \frac{y}{x^2+y^2}\,\mathbf{j} \qquad \text{on } \mathbb{R}^2 \text{ minus the origin.}$$

- **a.** Find the flux of $\mathbf{\Theta}$ across the circle $x^2 + y^2 = R^2$ using an ordinary line integral.
- **b.** Now try to apply Green's theorem: find the divergence of $\mathbf{\Theta}$ and integrate it over the interior of the circle in part **(a)**.
- **c.** Why do the results of parts **(a)** and **(b)** not contradict Green's theorem?
- **d.** Show that $\mathbf{\Theta}$ satisfies the "component test" $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$. Does this mean that $\mathbf{\Theta}$ is conservative, i.e., the gradient of a function? If so, what is the function?
- ***e.** Let C be any simple closed curve in $\mathbb{R}^2$ that does not pass through the origin. Show that the flux of $\mathbf{\Theta}$ across C is 2π if the origin lies inside C and 0 otherwise. (*Hint*: make clever use of Green's Theorem by adding to the curve you are given when it surrounds the origin. Alternatively, you could parametrize C in polar coordinates.)

Exercise 47. Prove the following theorem, due to Archimedes: let S be the unit sphere in $\mathbb{R}^3$, and let C be the infinite cylinder $x^2 + y^2 = 1$. Choose real numbers a and b such that $-1 \le a < b \le 1$. Then the area of S contained between the two planes $z = a$ and $z = b$ equals the area of C contained between the same two planes.

Exercise 48. The shape of the graph of $\cosh x$ is called a *catenary*. When it is rotated around a certain axis, the surface of revolution thus obtained is called a *catenoid*; this is the shape that soap film makes when it is stretched between two parallel circular wires.

- **a.** Sketch the surface $x^2 + y^2 = \cosh^2 z$, which is a catenoid.
- **b.** Find the surface area of the portion of the catenoid in part **(a)** that is contained between the planes $z = \pm a$.

Exercise 49.

- **a.** Find the flux of the constant field $\mathbf{i}$ across the surface which is the intersection of the plane $z = -x$ and the filled-in cylinder $y^2 + z^2 \le 4$. How does the result change if $z = -x$ is replaced with $mz = x$ for some constant m? Give a physical interpretation (think of water moving through a pipe).
- **b.** Find the flux of the axial field $x\,\mathbf{i} + y\,\mathbf{j}$ across the portion of the paraboloid $z = x^2 + y^2$ beneath the plane $z = 2y$.

Exercise 50.

a. Show that $(\sinh t \cos\theta, \sinh t \sin\theta, \cosh t)$ is a parameterization of one component of the hyperboloid $z^2 - x^2 - y^2 = 1$. Which component is it?

b. Find parameterizations of the cylinder $x^2+y^2 = 1$ and the hyperboloid $x^2+y^2-z^2 = 1$. (*Hint:* use hyperbolic functions for the latter.)

***c.** Find an equation and a parameterization of the circular cylinder in $\mathbb{R}^3$ whose axis is the line (t, t, t) and whose radius is 1.

Exercise 51 (cf. Thomas, p. 1200 #53)**.** Fix values $0 < a < b$, and consider the surface with equation

$$(b - \sqrt{x^2 + y^2})^2 + z^2 = a^2.$$

This surface is called a *torus*. The number a is called the *cross-sectional radius*; b does not have a special name, but it measures the distance from the axis of the torus to the center of the (circular) cross-section.

a. Show that

$$\mathbf{T}(u, v) = \big((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u\big)$$

is a parameterization of the torus.

b. Compute the area of the torus with the above equation.

c. Compute the flux of the field

$$\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

across the surface of the torus, from inside to outside.

d. Integrate the surface areas of the nested tori

$$(b - \sqrt{x^2 + y^2})^2 + z^2 = t^2,$$

where t ranges from 0 to a, to compute the volume of the solid torus. Compare the result with the result of part **(c)**.

Exercise 52. The *pseudosphere* is the surface in $\mathbb{R}^3$ parametrized by

$$\left(\frac{\cos\theta}{\cosh u}, \ \frac{\sin\theta}{\cosh u}, \ u - \frac{\sinh u}{\cosh u}\right)$$

for $0 \leq u < \infty$, $0 \leq \theta < 2\pi$. Show that the pseudosphere has surface area equal to half of the unit sphere.

Exercise 53. Consider the field $\mathbf{F}$ on $\mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + (z^2 - x^2 - y^2)\,\mathbf{k}.$$

a. Show that $\mathbf{F}$ is not conservative. What does $\mathbf{F}$ look like? (*Hint*: try using cylindrical coordinates.)

b. Find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$. Explain the results qualitatively using the geometry of $\mathbf{F}$.

c. Using Stokes' Theorem, compute:

- the circulation of $\mathbf{F}$ along the boundary of square with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, and $(0, 0, 1)$, taken in that order.
- the circulation of $\mathbf{F}$ along any simple closed curve in the plane $z = 2$;

Exercise 54. Let S be the closed cylindrical surface in $\mathbb{R}^3$ bounded by $x^2 + y^2 = 9$, $z = 0$, and $z = 4$. Compute the outward flux of the field $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ across S, both directly and using the divergence theorem. Feel free to use geometric reasoning to simplify calculations.

Exercise 55. Describe the "boundary" of each of the following geometric objects (some may have no boundary).

- the curve $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $-1 \le t \le 1$
- the region $x^2 + 3y^2 \le 9$ in $\mathbb{R}^2$
- the curve $\mathbf{r}(\theta) = 2\cos\theta\,\mathbf{j} + \sin\theta\,\mathbf{k}$, $-\pi \le \theta \le \pi$
- the sphere $x^2 + y^2 + z^2 = 9$
- the portion of the region $1 \le x^2 + y^2 \le 4$ in $\mathbb{R}^3$ contained between $z = 0$ and $z = 5$
- the intersection of the cone $z^2 = x^2 + y^2$ and the plane $x + 2z = 5$

Note: the boundary of a curve is its endpoints, the boundary of a surface is a curve, and the boundary of a solid region in space is a surface. That is, the boundary of an object, when it exists, is something "one dimension lower".

Exercise 56. Find the flux of $\mathbf{F}(x, y, z) = xz\sin yz\,\mathbf{i} + \cos yz\,\mathbf{j} + e^{x^2+y^2}\,\mathbf{k}$ across the portion of the paraboloid $x^2 + y^2 + z = 4$ satisfying $z \ge 0$, oriented away from the origin. (*Hint:* find the divergence of $\mathbf{F}$. Use the divergence theorem to find a simpler way to compute the flux.)

Exercise 57. Take a look at the proofs in the book that $\nabla \times (\nabla f) = \mathbf{0}$ and $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ (or re-derive these equalities yourself). How do the operators grad, curl, and div relate to each other? What conditions are necessary in order to "go backwards"—i.e., under what conditions is it true that if $\nabla \times \mathbf{F} = \mathbf{0}$ then you can always find f so that $\nabla f = \mathbf{F}$, or if $\nabla \cdot \mathbf{F} = 0$, then you can always find $\mathbf{G}$ so that $\nabla \times \mathbf{G} = \mathbf{F}$?

Exercise 58. Consider the field $\mathbf{F}$ on $\mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + (x^2 + y^2 - z^2)\,\mathbf{k}.$$

a. Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$. Is $\mathbf{F}$ conservative?

b. Show that the field

$$\mathbf{G}(x, y, z) = y\left(\frac{z^2}{2} - x^2\right)\mathbf{i} + x\left(y^2 - \frac{z^2}{2}\right)\mathbf{j}$$

satisfies $\nabla \times \mathbf{G} = \mathbf{F}$. ($\mathbf{G}$ is called a *vector potential* for $\mathbf{F}$; $\mathbf{F}$ is said to be *solenoidal.*) Show that, if all partial derivatives of a function g are continuous, then $\mathbf{G} + \nabla g$ is also a vector potential for $\mathbf{F}$.

c. Show that the Laplacian $\nabla^2 = \nabla \cdot \nabla$ applied to each coordinate of $\mathbf{G}$ yields the corresponding coordinate of $\nabla \times \mathbf{F}$.