Math 223: Fall 2005

Final Exam

13 December, 2005
RF 231

You have 150 minutes to complete this exam. Please note: while it has been standard practice to give additional time as needed on the prelims, *no extra time can be given* on the final exam, as this room is needed by others once we are finished. Calculators are not allowed.

Point valued for problems are listed to their left (e.g. question **1** is worth [15] points). I suggest you read all problems before beginning, and do them in whatever order you are most comfortable with; however, please clearly mark which problem you are working on. Have fun! :-)

[15] **1. (a)** State the Chain Rule for differentiable functions $g\colon \mathbb{R}^n \to \mathbb{R}^m$ and $f\colon \mathbb{R}^m \to \mathbb{R}^p$.

(b) A function $h\colon \mathbb{R}^n \to \mathbb{R}$ is called *radial* if there is a (differentiable) function $g\colon \mathbb{R} \to \mathbb{R}$ such that $h(\mathbf{x}) = g(|\mathbf{x}|^2)$. If $h\colon \mathbb{R}^2 \to \mathbb{R}$ is radial, show that $x_2 D_1 h(\mathbf{x}) = x_1 D_2 h(\mathbf{x})$.

[20] **2. (a)** What is the definition of a k-dimensional manifold in $\mathbb{R}^n$? Explain how the Implicit Function Theorem can be used to show certain sets are manifolds.

(b) Show that the set M of all 2×3 matrices $\mathbf{A}$ satisfying $\mathbf{A}\mathbf{A}^\top = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is a manifold. What is its dimension? [*Hint*: Recall that $\mathbf{A}\mathbf{A}^\top$ is symmetric, so there is duplicate information. When formulating the problem to use the Implicit Function Theorem, use a 3-dimensional target space.]

(c) Find a basis for the tangent space $T_{\mathbf{A}_0}M$, where $\mathbf{A}_0 = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

[15] **3. (a)** State Kantorovich's Theorem on the convergence of Newton's method for finding roots of differentiable functions $\mathbb{R}^n \to \mathbb{R}^n$.

(b) Since $2^7 = 128$, we expect there to be a root of the polynomial $p(x) = x^7 - x - 128$ near 2. Prove that, indeed, there is a root r with $|r - 2| < 0.01$.

[20] **4.** Let V denote the vector space of polynomials of degree at most 2 (i.e., $V = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$). Define $I : V \to \mathbb{R}$ by

$$I(p) = \int_{-1}^{1} p(x)\,dx.$$

(a) Show that I is a linear function.

(b) State the Fundamental Theorem of Linear Algebra for a $k \times n$ matrix $\mathbf{A}$. Name the relevant subspaces and give their dimensions.

(c) Let $W \subset V$ be the subset of linear polynomials (i.e., $W = \{bx \mid b \in \mathbb{R}\}$). Show that W is in the kernel of I. Does $\ker I = W$? Explain your answer.

[20] **5.** Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy - x + y + z^2$. Let $\mathbf{x}_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

(a) Show that f has only one critical point, at $\mathbf{x}_0$. What kind of critical point is it?

(b) Find all constrained critical points of f on the manifold $S_{\sqrt{2}}(\mathbf{x}_0)$ (the sphere of radius $\sqrt{2}$ centred at $\mathbf{x}_0$), which is given by the equation

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x+1)^2 + (y-1)^2 + z^2 - 2 = 0.$$

(c) What are the maximum and minimum values of f on the closed ball $\overline{B_{\sqrt{2}}(\mathbf{x}_0)}$? [*Hint*: they occur either at a at a critical point in the interior $B_{\sqrt{2}}(\mathbf{x}_0)$, or at a constrained critical point on the boundary $\partial\overline{B_{\sqrt{2}}(\mathbf{x}_0)} = S_{\sqrt{2}}(\mathbf{x}_0)$.]

[10] **6.** Let H be a plane in $\mathbb{R}^3$, defined by

$$H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, ax + by + cz = d \right\},$$

where a, b, c, d are real constants and at least one of a, b, c is nonzero. Compute the Gauss map on H (remember that you have two choices, due to two orientations of H; either one is fine). Show that H has Gaussian curvature 0 everywhere; i.e. show that a plane is flat!

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