Applications of Delaunay triangulations to Teichmüller theory

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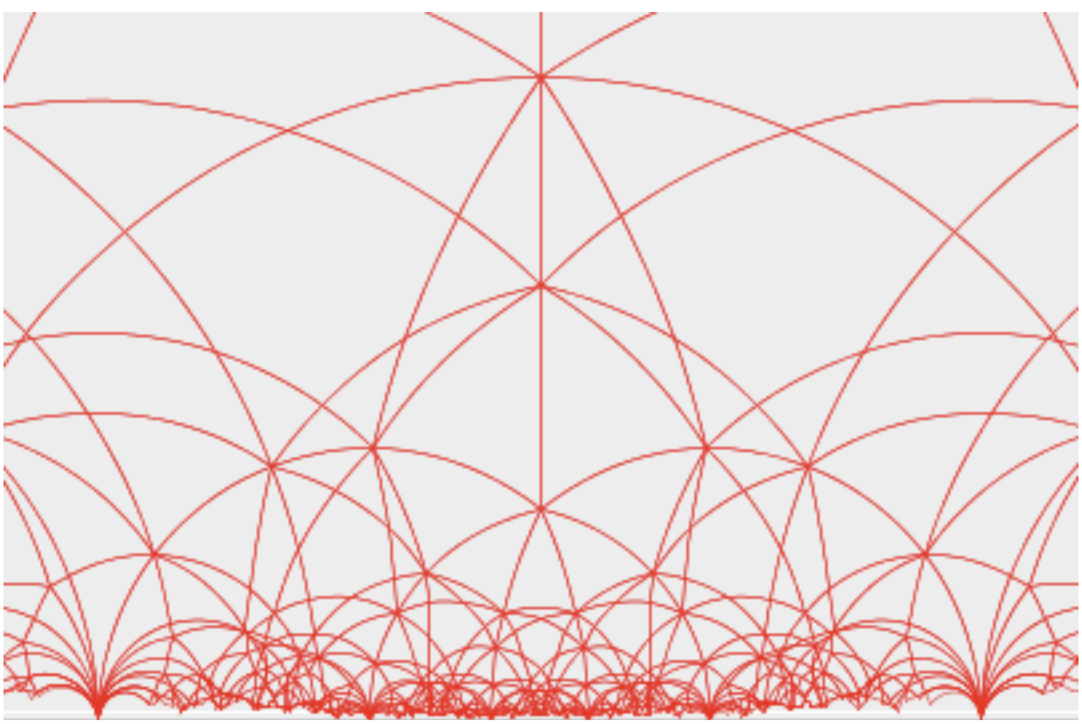


Early work on Delaunay triangulations of flat surfaces

- Named for Boris Delaunay (or Delone), who introduced such triangulations in “Sur la sphère vide”, 1934
- Thurston: “Shapes of polyhedra and triangulations of the sphere”, preprint c. 1987, published 1998
- Masur–Smillie: “Hausdorff dimension of sets of nonergodic measured foliations”, 1991
- Rivin: “Euclidean structures on simplicial surfaces and hyperbolic volume”, 1994
- Veech: “Delaunay partitions”, 1996

More recent work

- Indermitte–Liebling–Troyanov–Clémençon, 2001: application to biological growth
- Bobenko–Springborn, 2007: application to discrete harmonic functions and mean curvature

1. Cotangents and Delaunay weights

$$\cot\angle(v,w) = \frac{\langle v,w\rangle}{|v\, w|} \quad \text{for} \quad v,w \in \mathbb{R}^2$$

If E is an edge joining two Euclidean triangles, define the **Delaunay weight** of E to be

$$w(E) = \cot\alpha + \cot\beta,$$

where α and β are the angles opposite E.

E is **Delaunay** if $w(E) \geq 0$.

Prop. $\cot\alpha + \cot\beta \geq 0 \iff \alpha + \beta \leq \pi$
(Equality is also an iff statement.)

Cor. E *is Delaunay* $\iff$ *the triangles adjacent to* E *have* empty circumcircles*.*

Observe: if the triangles form a convex quadrilateral, let E' be the other diagonal. Then

$w(E) = 0 \iff w(E') = 0 \iff E$ and E' are both Delaunay.

2. TRIANGULATIONS OF FLAT SURFACES

A **flat surface** is a triple (X, g, Z) such that:

- X is a surface;
- Z is a discrete subset of X;
- g is a metric on X:
  - on $X \setminus Z$, locally isometric to $\mathbb{R}^2$,
  - each pt of Z has a nbhd isometric to a Euclidean cone.

Examples:

- polyhedra in $\mathbb{R}^3$
- Riemann surface with a non-zero abelian differential
- Riemann surface with a non-zero quadratic differential
- Riemann surface with higher-order differential

We assume hereafter that X is compact. (Could also handle "finite type" by treating punctures as points of Z.)

The **curvature** at a point $p \in Z$ is $2\pi - \theta_p$.
($\theta_p =$ cone angle at p)

Note that total curvature over X must be

$$\sum_{p \in Z} (2\pi - \theta_p) = 4\pi \cdot (1 - \text{genus}(X)),$$

following Gauss–Bonnet.

A **geodesic triangulation** of (X, g, Z) is a simplicial structure on X such that:

- the vertex set is Z, and
- the edges are geodesic with respect to g.

The number of faces and edges are determined by the Euler characteristic of X and the size of Z:

$$|Z| - \#(\text{edges}) + \#(\text{faces}) = 2 - 2 \cdot \text{genus}(X)$$

$$\#(\text{edges}) = \frac{3}{2} \cdot \#(\text{faces})$$

$$\#(\text{faces}) = 4 \cdot (\text{genus}(X) - 1) + 2 \cdot |Z|$$

A geodesic triangulation of (X, g, Z) is **Delaunay** if all of its edges are Delaunay.

Thm. (Masur–Smillie)
Delaunay triangulations exist and are unique, up to exchanges of edges with Delaunay weight 0.

Thm. (Rivin, Indermitte et al., Bobenko–Springborn)
A Delaunay triangulation may be obtained from any *geodesic triangulation of (X, g, Z) by an "edge-flipping" algorithm.*

3. TESSELLATIONS

Let Y be a compact Riemann surface, and let $T^*\mathrm{Teich}(Y)$ be the cotangent bundle to the Teichmüller space of Y, whose fibers consist of quadratic differentials.

Remark: The following statements are equivalent:

- "(X, g, Z) arises from the metric defined by q, a quadratic differential on X, and $Z = \mathrm{zeroes}(q)$."

- "X is orientable and all points of Z have cone angle $k\pi$ for some $k \in \{3, 4, 5, \ldots\}$."

Can partition $T^*\mathrm{Teich}(Y)$ according to which edges are in the Delaunay triangulation of $(X, q, \mathrm{zeroes}(q)) \in T^*\mathrm{Teich}(Y)$ (using marking from Y).

Delaunay triangulations are unique $\implies$

Thm. (Veech)
The above partition is $\mathrm{Mod}(Y)$*-equivariant.*

Given $(X, q) \in T^*\mathrm{Teich}(Y)$, scale to assume $\mathrm{area}(q) = 1$.

Define $\mathrm{orbit}(X, q) = \{[A] \cdot (X, q) \mid [A] \in \mathrm{PSL}_2(\mathbb{R})\}$, contained in the space of area 1 quadratic differentials.

Identify $\mathrm{orbit}(X, q)$ with $\mathrm{PSL}_2(\mathbb{R}) \cong T^1\mathbb{H}$, the unit tangent bundle to $\mathbb{H}$.

The projection $P : \mathrm{orbit}(X, q) \to \mathbb{H}$ can be written explicitly as

$$[A] \cdot (X, q) \mapsto [A]^{-1} \cdot i,$$

where the right is defined by usual action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$.

Thm. (B., Veech)
The partition of $\mathrm{orbit}(X, q)$ *by the combinatorial types of the points' Delaunay triangulations projects to a tessellation of* $\mathbb{H}$ *whose tiles have geodesics sides and finite area.*

This is the **iso-Delaunay tessellation** $\Sigma(X, q)$ of $\mathbb{H}$.

For example, if $X = \mathbb{R}^2/\mathbb{Z}^2$ and $q = dz^2$ (choose one point for Z), then $\Sigma(X, q)$ is the Farey tessellation of $\mathbb{H}$ by ideal triangles.

Other examples...

Proof. Let $[A] \cdot (X, q) \in \operatorname{orbit}(X, q)$, and let τ be its Delaunay triangulation; τ is also a geodesic triangulation of (X, q).

For each edge $E \in \tau$, define

$$\mathbb{H}_E = \{P([A]) \mid E \text{ is Delaunay on } [A] \cdot (X, q)\}.$$

Claim: Each $\mathbb{H}_E$ is either a Poincaré half-plane or all of $\mathbb{H}$.

> If the quadrilateral with E as its diagonal is not convex, then $A \cdot E$ is Delaunay for any $A \in SL_2(\mathbb{R})$.
>
> Otherwise, let v_1, v_2 and w_1, w_2 be the vectors forming the remaining sides of the triangles adjacent to E, ordered so that $|v_1\ v_2| > 0$ and $|w_1\ w_2| > 0$.
>
> The following conditions are equivalent to $w(A \cdot E) \geq 0$:
>
> $$\frac{\langle Av_1, Av_2\rangle}{|v_1\ v_2|} + \frac{\langle Aw_1, Aw_2\rangle}{|w_1\ w_2|} \geq 0$$
>
> $$\langle Av_1, Av_2\rangle |w_1\ w_2| + |v_1\ v_2| \langle Aw_1, Aw_2\rangle \geq 0$$
>
> This reduces to a quadratic inequality in the coordinates of $P([A])$, whose boundary set is a Poincaré geodesic.

Set $\mathbb{H}_\tau = \bigcap_{E \in \tau} \mathbb{H}_E$. $\mathbb{H}_\tau$ is non-empty, because it contains $P([A])$.

It has finitely many sides because τ has finitely many edges.

It has finite area; to assume otherwise leads to the claim that q has uncountably many saddle connections, contradicted by Vorobets.

Every point of $\mathbb{H}$ is contained in some $\mathbb{H}_\tau$. $\square$

4. Some applications

Studying isometries and affine self-maps of flat surfaces:

- Must send Delaunay cells to Delaunay cells
- Can be seen as automorphisms of $\Sigma(X, q)$

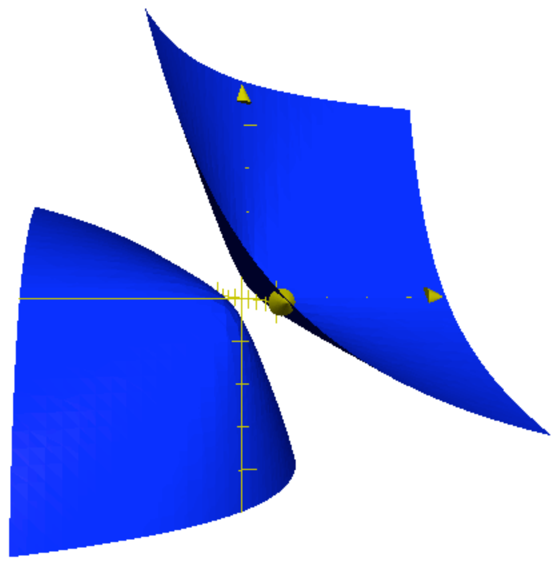
Examining properties of geodesic flow,
finding invariant subsurfaces

- If a direction contains saddle connections, contracting this direction will force these saddle connections to be edges of the Delaunay triangulation
- Periodic directions correspond to points on $\partial\mathbb{H}$ which are cusps of tiles of $\Sigma(X, q)$

Delaunay weights on edges of Delaunay triangulation can be used to define a Laplace–Beltrami operator on $\mathbb{R}^Z$, using intrinsic geometry of X, similar to finite-element approximation of Laplacian in the plane.
Spectrum of this operator defines algebraic functions on moduli space of flat surfaces.
(not much explored yet)

In any Euclidean triangle with angles $(\alpha_1, \alpha_2, \alpha_3)$,
the cotangents $a_i = \cot \alpha_i$ satisfy the equation

$$a_1 a_2 + a_2 a_3 + a_3 a_1 = 1.$$



This equation defines a hyperboloid in $\mathbb{R}^3$, hence the space of Euclidean triangles, up to scale, carries a canonical hyperbolic metric.

Prop. (B.)
If F is any triangle in a triangulation of (X, q), then the hyperbolic metric on the space of triangles containing F coincides with the Teichmüller metric on the disk of (X, q).

5. The genus 3 Arnoux–Yoccoz surface

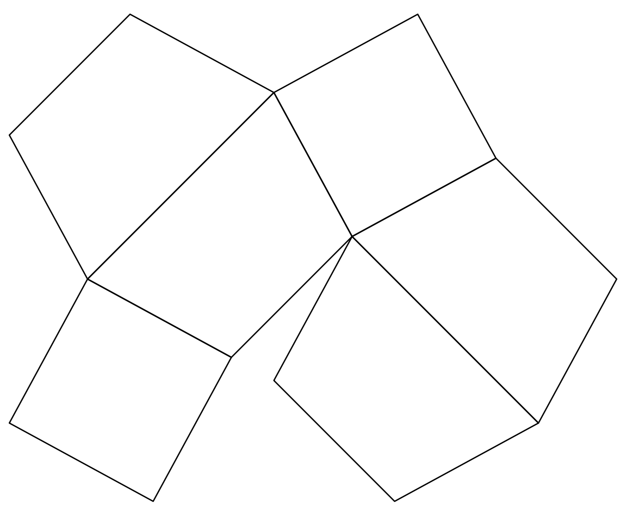
First in family of hyperelliptic surfaces, one for each genus $\gamma \geq 3$, each admitting a pseudo-Anosov diffeomorphism with an expansion constant λ whose inverse is the unique real solution to

$$x + x^2 + \cdots + x^\gamma = 1.$$

Originally constructed via interval exchange transformation:

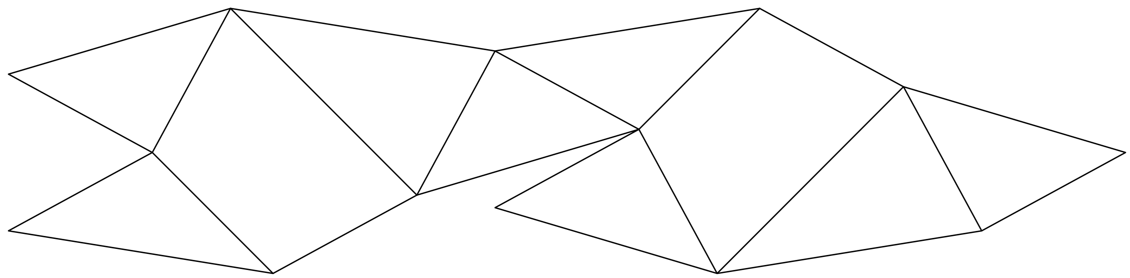
(see P. Arnoux, "Un exemple de semi-conjugaison entre un échange d'intervalles et une translation sur le tore" for a description and images)

We find a simpler description using Delaunay cells:



Let (X_{AY}, ω_{AY}) denote this flat surface.

The pseudo-Anosov element is visible by scaling the horizontal direction by λ and the vertical direction by $1/\lambda$, then drawing the new Delaunay edges:



Now match trapezoids and squares between the two pictures.

Prop. (B.)
(X_{AY}, ω_{AY}) belongs to a family of pairs $(X_{t,u}, \omega_{t,u})$ with $t > 1$ and $u > 0$, where $X_{t,u}$ has the equation

$$y^2 = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu)$$

and $\omega_{t,u} = \dfrac{x\,dx}{y}$.

These surfaces are characterized by the following properties:

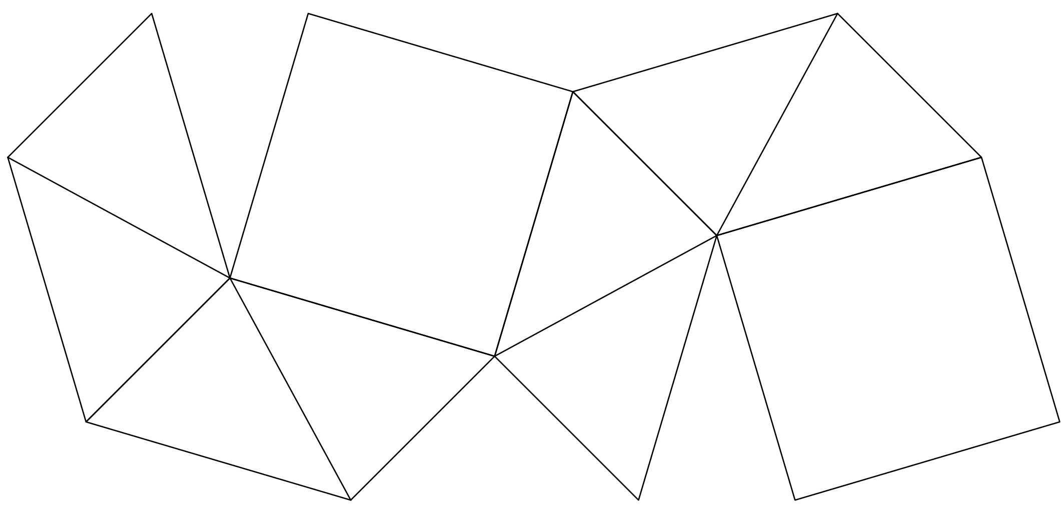
- $X_{t,u}$ is hyperelliptic (Υ = hyperelliptic involution)
- $\omega_{t,u}$ has two zeroes of order 2
- $X_{t,u}$ has two real structures ρ_1, ρ_2:
  - each fixes 6 Weiertrass points, including zeroes of ω_{AY}
  - exchanges 2 other Weierstrass points
  - $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1 = \Upsilon$
- $X_{t,u}$ has two other anti-holomorphic involutions σ_1, σ_2:
  - fixed-point free
  - $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \Upsilon$
- for $i, j \in \{1, 2\}$, $(\rho_i \circ \sigma_j)^2 = \Upsilon$

For the values (t_{AY}, u_{AY}) corresponding to (X_{AY}, ω_{AY}), we find

$$t_{AY} \approx 1.91709843377,$$
$$u_{AY} \approx 2.07067976690.$$

Conj. t_{AY} and u_{AY} are algebraic.

Scaling only the horizontal direction of (X_{AY}, ω_{AY}), again by λ, we obtain another surface with additional real structures.



Prop. (B.)
This new surface belongs to a family of pairs $(X_{r,s}, \omega_{r,s})$ *with* $r > 0$ *and* $s \notin \mathbb{R}$, *where* $X_{r,s}$ *has the equation*

$$y^2 = x(x^2 + r)(x - s)(x - \bar{s})(x + r/s)(x + r/\bar{s})$$

and $\omega_{r,s} = \dfrac{x\,dx}{y}$.