ORIENTATION-REVERSING INVOLUTIONS OF THE GENUS 3 ARNOUX–YOCCOZ SURFACE AND RELATED SURFACES

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Abstract. We present a new description of the genus 3 Arnoux–Yoccoz translation surface in terms of Delaunay polygons and show that, with the appropriate scalings, it fits into two families of surfaces whose isometry groups include the dihedral group of the square.

1. Introduction

1.1. **Flat surfaces.** For our purposes, a *flat surface* is a pair (X, q), where X is a Riemann surface and q is a non-zero meromorphic quadratic differential of finite area on X. We will also speak of the flat surface (X, ω) instead of (X, ω^2) when $q = \omega^2$ is the global square of an abelian differential ω; in this case, (X, ω) is also called a *translation surface*. A quadratic differential determines a canonical metric structure on the underlying surface [HM, EG]; we will consider two flat surfaces to be the same when there is an isometry between them.

Recall that $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{R})$ act on the space of translation surfaces, while $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PGL}_2(\mathbb{R})$ act on the space of flat surfaces. If $A \cdot (X, \omega) = (X, \omega)$ or $[A] \cdot (X, q) = (X, q)$, then $\det A = \pm 1$. The set of all $[A] \in \mathrm{PGL}_2(\mathbb{R})$ such that $[A] \cdot (X, q) = (X, q)$ form the *projective generalized Veech group* of (X, q), which we will simply call the Veech group [Ve1] (cf. [HS]).

The results in this paper began with a study of Delaunay polygons on the surface to be described in §1.2, and so we recall their definition.

Definition 1.1. A *Delaunay triangle* on (X, q) is the image of a 2-simplex on X, embedded on its interior, whose vertices lie in the set of cone points of q, whose edges are geodesic with respect to the metric of q, and whose (possibly immersed) circumcircle contains no cone points of q in its interior. A *Delaunay triangulation* of (X, q) is a cell structure on X whose 2-cells are Delaunay triangles. A *Delaunay polygon* is a Delaunay triangle or the union of two or more adjacent Delaunay triangles that share the same circumcircle.

A generic flat surface has a unique Delaunay triangulation. When it is not unique, we can start with any Delaunay triangulation and join two or more adjacent Delaunay triangles to form a Delaunay polygon. After joining Delaunay triangles to form all of the maximal Delaunay polygons, we obtain the *Delaunay decomposition* of (X, q), which is invariant under isometries of (X, q) (cf. [MS]).

1.2. **The Arnoux–Yoccoz surface.** In 1981, P. Arnoux and J.-C. Yoccoz [AY] introduced a family of pseudo-Anosov diffeomorphisms, one for each genus greater than 2, each of which stabilizes a hyperelliptic Riemann surface. The genus 3 example has received a great deal of attention in recent years. Hubert–Lanneau–Möller [HLM] showed that the relevant abelian

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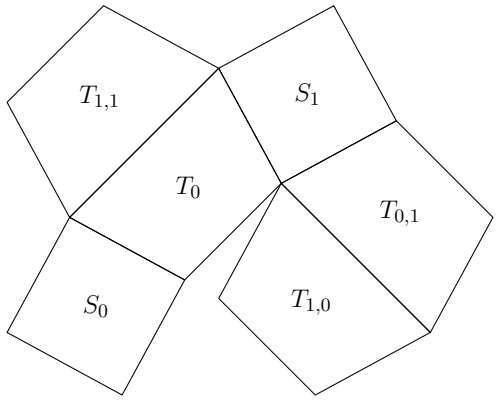

FIGURE 1. The tiling of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ by its Delaunay polygons

differential has a second, independent pseudo-Anosov element in its stabilizer, and using techniques introduced by C. McMullen [Mc] in the study of genus 2 orbits they showed that the $\mathrm{SL}_2(\mathbb{R})$-orbit of the genus 3 example is dense in the largest possible region of the moduli space of abelian differentials. Hubert–Lanneau [HL] also showed that the stabilizer contains no parabolic elements. In this section we give a new description of this translation surface in terms of its Delaunay polygons (of which there are only two kinds, up to isometry) and very simple gluing instructions.

Let $\alpha \approx 0.543689$ be the real root of the polynomial $x^3 + x^2 + x - 1$. Let S_0 be the square with vertex set $\{(0,0), (\alpha^2, \alpha), (\alpha^2 - \alpha, \alpha^2 + \alpha), (-\alpha, \alpha^2)\}$, and let T_0 be the trapezoid with vertex set $\{(0,0), (1-\alpha, 1-\alpha), (1-\alpha-\alpha^2, 1), (-\alpha, \alpha^2)\}$. We form a flat surface $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ from two copies of S_0 and four copies of T_0: reflecting S_0 across either a horizontal or vertical axis yields the same square S_1 (up to translation); we denote by $T_{1,0}$, $T_{0,1}$, and $T_{1,1}$ the reflections of T_0 across a vertical axis, across a horizontal axis, and across both, respectively. (In fact, $T_{1,0}$, $T_{0,1}$, and $T_{1,1}$ are all rotations of T_0 by multiples of $\pi/2$, but this description via reflections will be invariant under horizontal and vertical scaling, i.e., the Teichmüller geodesic flow.) Identify the long base of T_0 with the long base of $T_{1,1}$, as well as their short bases; do the same with $T_{1,0}$ and $T_{0,1}$. Each remaining side of a trapezoid is parallel to exactly one side of S_0 or S_1; identify by translations those sides which are parallel. (See Figure 1.)

The resulting flat surface $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ has genus 3 and two singularities each with cone angle 6π. The images of S_0 and T_0 are the Delaunay cells of ω_{AY}. X_{AY} is hyperelliptic; the hyperelliptic involution $\tau : X_{\mathrm{AY}} \to X_{\mathrm{AY}}$ is evident in Figure 1 as rotation by π around the centers of the squares and the midpoints of the edges joining two trapezoids; these six points together with the cone points are therefore the Weierstrass points of the surface. (See Figure 3.) Moreover, ω_{AY} is odd with respect to τ, i.e., $\tau^*\omega_{\mathrm{AY}} = -\omega_{\mathrm{AY}}$.

The pseudo-Anosov diffeomorphism ψ_{AY} constructed by Arnoux–Yoccoz scales the surface by a factor of $1/\alpha$ in the horizontal direction and by α in the vertical direction. In Figure 2 we show the result of scaling the boundary of Figure 1, along with the new Delaunay edges. Two of the trapezoids—having the orientations of $T_{1,1}$ and $T_{1,0}$—are clearly visible; the squares and the other two trapezoids are constructed from the remaining triangles.

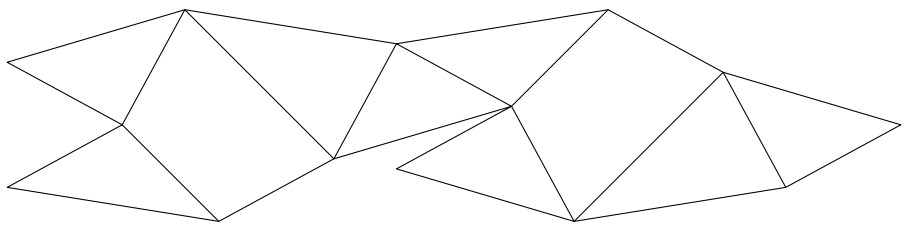

FIGURE 2. The result of applying the Arnoux–Yoccoz pseudo-Anosov diffeomorphism to ω_{AY}

2. Two families of surfaces

2.1. **Labeling the Weierstrass points of X_{AY}.** As before, we denote the hyperelliptic involution of X_{AY} by τ. The purpose of this section is to show the following.

Theorem 2.1. *The surface $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ belongs to a family $(X_{t,u}, \omega_{t,u})$, with $t > 1$ and $u > 0$, such that $X_{t,u}$ has the equation*

$$y^2 = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu), \tag{1}$$

and $\omega_{t,u}$ is a multiple of $x\,\mathrm{d}x/y$ on $X_{t,u}$.

Each of the surfaces in Theorem 2.1 has the following isometries:

- Two fixed-point-free orientation-reversing involutions σ_1 and σ_2, whose product is τ, and which do not send any point to its image by τ; they therefore descend to a single fixed-point-free orientation-reversing involution σ of $\mathbb{P}^1$.
- Two real structures (orientation-reversing involutions whose fixed-point set is 1-dimensional) ρ_1 and ρ_2 whose product is again τ, and which therefore descend to a single real structure ρ on $\mathbb{P}^1$.

Any product of the form $\rho_i\sigma_j$ $(i, j \in \{1, 2\})$ is a square root of τ, and therefore the group generated by the four elements listed above is the dihedral group of the square. We will demonstrate these isometries in our presentation of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$. In §2.3 we will look at surfaces in this family that have additional symmetries.

Let $\varpi : X_{\mathrm{AY}} \to \mathbb{P}^1$ be the degree 2 map induced by τ, i.e., $\varpi \circ \tau = \varpi$. We can normalize ϖ so that the zeroes of ω_{AY} are sent to 0 and ∞, and the midpoint of the short edge between T_0 and $T_{1,1}$ maps to 1. We wish to find the images of the remaining Weierstrass points, so that we can write an affine equation for X_{AY} in the form $y^2 = P(x)$, where P is a degree 7 polynomial with roots at 0 and 1. Hereafter we assume that ϖ is the restriction to X_{AY} of the coordinate projection $(x, y) \mapsto x$. Consequently, we may consider each Weierstrass point as either a point $(w, 0)$ that solves $y^2 = P(x)$ or simply as a point w on the x-axis.

Each of the real structures ρ_1 and ρ_2 has a fixed-point set with three components: in one case, say ρ_1, the real components are the line of symmetry shared by T_0 and $T_{1,1}$, and the two bases of $T_{1,0}$ and $T_{0,1}$. The fixed-point set of ρ_2 is then the union of the corresponding lines in the orthogonal direction. Because ρ_1 and ρ_2 fix the points 0, 1, and ∞, ρ fixes the real axis; therefore ρ is simply complex conjugation.

The involutions σ_1 and σ_2 can be visualized (as in Figure 1) as "glide-reflections", one along a horizontal axis and the other along a vertical axis. The induced map σ on $\mathbb{P}^1$ exchanges 0 and ∞ and preserves the real axis; therefore σ has the form $x \mapsto -r/\overline{x}$ for some real $r > 0$.

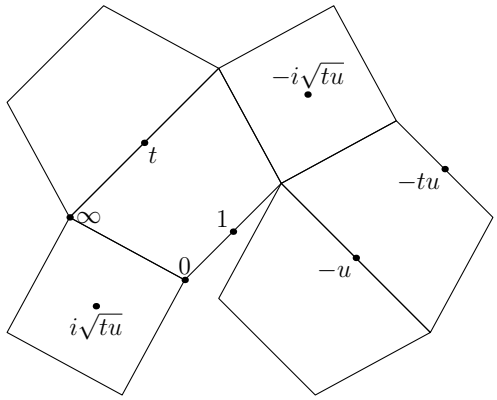

FIGURE 3. The Weierstrass points of X_{AY}, following normalization ($t > 1$, $u > 0$)

Let $s = (s, 0)$ be the center of S_0. Then $\rho_1(s) = \rho_2(s) = \sigma_1(s) = \sigma_2(s)$ is the center of S_1, which implies $\rho(s) = \sigma(s)$, i.e., $\overline{s} = -r/\overline{s}$. The solutions to this equation are $\pm i\sqrt{r}$. Topological considerations show that $s = i\sqrt{r}$; hence the center of S_1 is at $-i\sqrt{r}$.

Let t be the midpoint of the long edge of T_0. Applying σ_1 or σ_2 shows that the midpoint of the long edge of $T_{1,0}$ is at $-r/t$.

We already know that 1 is the center of the short edge of T_0. Since the short edge of $T_{0,1}$ is the image of this edge by σ_1 or σ_2, the midpoint of this edge must be at $-r$.

To simplify notation, let us make the substitution $u = r/t$, so that $r = tu$ (hence σ has the form $\sigma(x) = -tu/\overline{x}$). Thus X_{AY} has the equation (1) for some $(t, u) = (t_{\mathrm{AY}}, u_{\mathrm{AY}})$. Furthermore, ω_{AY} is the square root of a quadratic differential on $\mathbb{P}^1$ with simple zeroes at 0 and ∞ and simple poles at 1, t, $-u$, $-tu$, and $\pm i\sqrt{tu}$. There is therefore some complex constant c such that

$$\omega_{\mathrm{AY}}{}^2 = \varpi^* \left(\frac{cx}{(x-1)(x-t)(x+u)(x+tu)(x^2+tu)} \, \mathrm{d}x^2 \right),$$

i.e., $\omega_{\mathrm{AY}} = \pm\sqrt{c}\, x \, \mathrm{d}x/y$. This establishes Theorem 2.1.

2.2. **Integral equations.** To find t_{AY} and u_{AY} requires solving a system of equations involving hyperelliptic integrals, which we establish in this section using relative periods of ω_{AY}. Choose a square root of

$$f_{t,u}(x) = \frac{x}{(x-1)(x-t)(x+u)(x+tu)(x^2+tu)}$$

in the open first quadrant such that its extension $\sqrt{f_{t,u}(x)}$ to the complement of $\{1, t, i\sqrt{tu}\}$ in the closed first quadrant is positive on the open interval $(0, 1)$. Let η_0 be the Delaunay edge between S_0 and T_0; $\varpi(\eta_0)$ is then a curve from 0 to ∞ in the first quadrant. Integrating $\sqrt{cf_{t,u}(x)}\,\mathrm{d}x$ on the portion of the first quadrant below $\varpi(\eta_0)$ will then give a conformal map to half of T_0. We will be interested in integrals along the real axis.

The vector from 0 to 1 along the short side of T_0 is $\frac{1}{2}(1-\alpha)(1+i)$, while the the line of symmetry of T_0 from 1 to t gives the vector $\frac{1}{2}(\alpha+\alpha^2)(-1+i)$. Observe that

$$i \cdot (1-\alpha)(1+i) = \alpha \cdot (\alpha+\alpha^2)(-1+i),$$

and therefore

$$i\int_0^1 \sqrt{cf_{t,u}(x)}\,\mathrm{d}x = \alpha\int_1^t \sqrt{cf_{t,u}(x)}\,\mathrm{d}x. \tag{2}$$

Similarly, the vector from t to ∞ along the long side of T_0 is $\frac{1}{2}(1-\alpha^2)(-1-i)$, and because $1-\alpha^2 = (1+\alpha)(1-\alpha)$, we have

$$-(1+\alpha)\int_0^1 \sqrt{cf_{t,u}(x)}\,\mathrm{d}x = \int_t^\infty \sqrt{cf_{t,u}(x)}\,\mathrm{d}x. \tag{3}$$

In both equations we can cancel out the c, which was ever only a global (complex) scaling factor anyway. Now bring i under the square root on the right-hand side of (2) in order to make the radicand positive. We thus obtain from (2) and (3) the system of (real) integral equations

$$\begin{cases} \displaystyle\int_0^1 \sqrt{f_{t,u}(x)}\,\mathrm{d}x = \alpha\int_1^t \sqrt{-f_{t,u}(x)}\,\mathrm{d}x \\ \displaystyle(1+\alpha)\int_0^1 \sqrt{f_{t,u}(x)}\,\mathrm{d}x = -\int_t^\infty \sqrt{f_{t,u}(x)}\,\mathrm{d}x \end{cases} \tag{4}$$

whose solution is the desired pair $(t_{\mathrm{AY}}, u_{\mathrm{AY}})$. Using numeric methods, we find

$$t_{\mathrm{AY}} \approx 1.91709843377 \qquad \text{and} \qquad u_{\mathrm{AY}} \approx 2.07067976690.$$

We conjecture that t_{AY} and u_{AY} lie in some field of small degree over $\mathbb{Q}(\alpha)$.

2.3. **Other exceptional surfaces in this family.** An examination of the geometric arguments in §2.1 and an application of the principle of continuity to t and u show the following:

Theorem 2.2. *Every $(X_{t,u}, \omega_{t,u})$ can be formed by replacing T_0 in the description from §1.2 with an isosceles trapezoid T, S_0 with the square built on a leg of T, and the copies of T_0 with the rotations of T by $\pi/2$.*

The placement of t and u on $\mathbb{R}$ determines the shape of the trapezoid T, and any isosceles trapezoid may be obtained by an appropriate choice of t and u. In this section, we examine certain shapes that give $(X_{t,u}, \omega_{t,u})$ extra symmetries and determine the corresponding values of t and u. We continue to use τ to denote the hyperelliptic involution of $X_{t,u}$.

Suppose that T is a rectangle. Then there are two orthogonal closed trajectories, running parallel to the sides of T and connecting the centers $\pm i\sqrt{tu}$ of the squares, and either of these can be made into the fixed-point set of a real structure on $X_{t,u}$. The product of these two real structures is again τ, so they descend to a single real structure on $\mathbb{P}^1$. This real structure exchanges 0 with ∞ and fixes $\pm i\sqrt{tu}$, so it must be inversion in the circle $|x|^2 = tu$. It also exchanges 1 with t, which implies $1\cdot t = tu$, i.e., $u = 1$. The remaining parameter t is determined by solving the single integral equation

$$\int_0^1 \sqrt{\frac{x}{(x^2-1)(x^2-t^2)(x^2+t)}}\,\mathrm{d}x = \mu\int_1^t \sqrt{\frac{-x}{(x^2-1)(x^2-t^2)(x^2+t)}}\,\mathrm{d}x$$

where 2μ is the ratio of the width of T to its height. Recall that an *origami*, also called a *square-tiled surface*, is a flat surface that covers the square torus with at most one branch point [Sch]. By looking at rational values of μ, we have the following result:

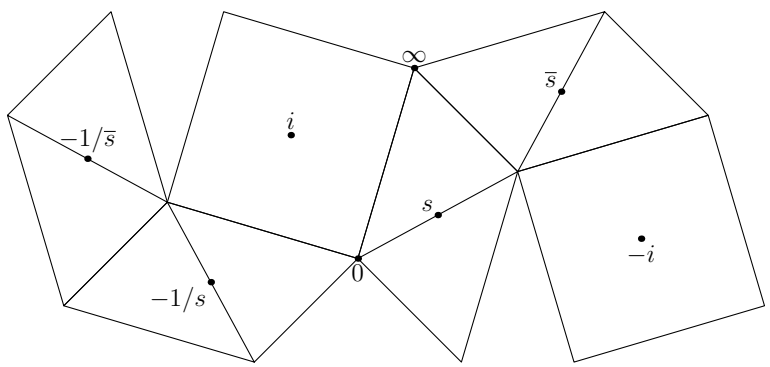

FIGURE 4. Another surface in the $\mathrm{GL}_2(\mathbb{R})$-orbit of ω_{AY} with additional real structures

Corollary 2.3. *The family $(X_{t,1}, \omega_{t,1})$ contains infinitely many origamis.*

These are not the only $(X_{t,u}, \omega_{t,u})$ that are origamis, however. If T is a trapezoid whose legs are orthogonal to each other, then $(X_{t,u}, \omega_{t,u})$ is again an origami.

2.4. **Second family of surfaces.** Conjugating ρ_1 by ψ_{AY} guarantees the existence of another orientation-reversing involution in the affine group of ω_{AY}. This element should fix a point "half-way" (in the Teichmüller metric, for instance) between ω_{AY} and its image by the pseudo-Anosov, lying in the Teichmüller disk of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$. This surface can be found either by scaling the vertical direction by $\sqrt{\alpha}$ and the horizontal direction by $1/\sqrt{\alpha}$ or, to keep our coordinates in the field $\mathbb{Q}(\alpha)$, just by scaling the horizontal by $1/\alpha$. This surface, which we will denote $(X'_{\mathrm{AY}}, \omega'_{\mathrm{AY}})$, is shown in Figure 4, along with its Delaunay polygons.

Theorem 2.4. *The surface $(X'_{\mathrm{AY}}, \omega'_{\mathrm{AY}})$ belongs to a family (X_s, ω_s), with $\operatorname{Im} s > 0$ and $s \neq i$, such that X_s has the equation*

$$y^2 = x(x^2+1)(x-s)(x-\overline{s})(x+1/s)(x+1/\overline{s}), \tag{5}$$

and ω_s is a multiple of $x\,\mathrm{d}x/y$ on X_s.

Again, we have two real structures ρ'_1 and ρ'_2 whose product is the hyperelliptic involution τ. Each of these only has one real component, however: the union of the sides of the parallelograms running parallel to the axis of reflection. The only Weierstrass points that lie on these components are 0 and ∞; the remaining Weierstrass points are the centers of the squares and of the parallelograms. We again ρ' be the induced involution of $\mathbb{P}^1$ and assume that it fixes the real axis (this we can do because we have only fixed the positions of two points on $\mathbb{P}^1$), so that the remaining Weierstrass points come in conjugate pairs.

The fixed-point free involutions σ_1 and σ_2 persist on the surface, and because they again preserve the union of the real loci of ρ'_1 and ρ'_2, they descend to a fixed-point free involution σ of the form $x \mapsto -r/\overline{x}$, with r real and positive. We have one more free real parameter for normalization, so we can assume $r = 1$. This implies that the centers of the squares are at $\pm i$. Let s be the center of one of the parallelograms; then applying ρ'_1 and σ_1 shows that the remaining Weierstrass points are $\overline{s}$, $1/s$, and $1/\overline{s}$. Using developing vectors again, we can find equations that define s, in a manner analogous to how we found (4).

As an analogue to Theorem 2.2, we have:

Theorem 2.5. *Every (X_s, ω_s) can be formed from a parallelogram P, a square built on one side of P, the rotation of P by $\pi/2$, and the images of P and its rotation by reflection across their remaining sides.*

The shape of P is determined by the value of s. If $s = \frac{1}{2}(\sqrt{3} + i)$, then P becomes a square, and we obtain one of the "escalator" surfaces in [LS]. More generally, if s is any point of the unit circle, then P is a rectangle, and inversion in the unit circle corresponds to another pair of real structures on X, which are the reflections across the axes of symmetry of P. By considering those rectangular P whose side lengths are rationally related, we have as before:

Corollary 2.6. *The family $(X_{e^{i\theta}}, \omega_{e^{i\theta}})$ (with $0 < \theta < \pi/2$) contains infinitely many origamis.*

Another origami appears when P is composed of a pair of right isosceles triangles so that s lies not on the hypotenuse, but on a leg of each.

3. Quadratic differentials and periods on genus 2 surfaces

We do not know how to compute the rest of the periods for $X_{t,u}$ or X_s, apart from those of $\omega_{t,u}$ or ω_s, respectively. In this section, however, we consider the periods of certain related genus 2 surfaces, which demonstrate remarkable relations.

Let X be any hyperelliptic genus 3 surface with an abelian differential ω that is odd with respect to the hyperelliptic involution and has two double zeroes. The pair (X, ω) has a corresponding pair (Ξ, q), where Ξ is a genus 2 surface and q is a quadratic differential on Ξ with four simple zeroes. Geometrically, the correspondence may be described as follows: two of the zeroes of ω are at Weierstrass points of X, hence (X, ω^2) covers a flat surface $(\mathbb{P}^1, \widetilde{q})$ where $\widetilde{q}$ has six poles and two simple zeroes. Then (Ξ, q) is obtained by taking the double cover of $\mathbb{P}^1$ branched at the poles of $\widetilde{q}$. In our cases, the genus 2 surface may be obtained by cutting along opposite sides of one of the squares in Figure 1 or 4, then regluing each of these via a rotation by π to the free edge provided by cutting along the other. (See [La] and [Va].)

First we consider the family $(X_{t,u}, \omega_{t,u})$ and the related genus 2 flat surfaces $(\Xi_{t,u}, q_{t,u})$. To be explicit, the defining expressions of both types of surfaces are:

$$X_{t,u} : y^2 = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu), \qquad \omega_{t,u} = \frac{x\,\mathrm{d}x}{y};$$

$$\Xi_{t,u} : y^2 = (x-1)(x-t)(x+u)(x+tu)(x^2+tu), \qquad q_{t,u} = \frac{x\,\mathrm{d}x^2}{y^2}.$$

The order 4 rotation $\rho_1\sigma_1$ of $X_{t,u}$ persists on $\Xi_{t,u}$. Following R. Silhol [Si], we find a new parameter a, depending on t and u, so that the curve

$$\Xi_a : y^2 = x(x^2-1)(x-a)(x-1/a)$$

is isomorphic to $\Xi_{t,u}$. Doing so simply requires a change of coordinates in x, namely

$$\Phi(x) = i\sqrt{tu}\frac{(x-1)}{(x+tu)}.$$

Then $\Phi(1) = 0$, $\Phi(-tu) = \infty$, and $\Phi(\pm i\sqrt{tu}) = \mp 1$. The images of t and u by Φ are

$$a = a(t,u) = i\sqrt{\frac{u}{t}}\frac{(t-1)}{(u+1)} \qquad \text{and} \qquad \frac{1}{a} = i\sqrt{\frac{t}{u}}\frac{(1+u)}{(1-t)}.$$

Because $t > 1$ and $u > 0$, a lies on the positive imaginary axis and $1/a$ lies on the negative imaginary axis. The involution ρ becomes reflection across the imaginary axis. The images of 0 and ∞ by Φ are

$$\Phi(0) = \frac{i}{\sqrt{tu}} \qquad \text{and} \qquad \Phi(\infty) = \frac{\sqrt{tu}}{i},$$

so the image of $q_{t,u}$ on Ξ_a is a scalar multiple of

$$\left(x - \frac{i}{\sqrt{tu}}\right)\left(x - \frac{\sqrt{tu}}{i}\right)\frac{\mathrm{d}x^2}{y^2} = \left(x^2 + i\left(\frac{tu-1}{\sqrt{tu}}\right)x + 1\right)\frac{\mathrm{d}x^2}{y^2}$$

Remark that, for each pair (t_0, u_0), there is a one-parameter family of surfaces $(\Xi_{t,u}, q_{t,u})$ such that $\Xi_{t,u}$ is isomorphic to Ξ_{t_0,u_0} while $q_{t,u}$ and q_{t_0,u_0} represent different differentials on the abstract Riemann surface.

Now we apply the same analysis to the second family. This time we are moving from (X_s, ω_s) to (Σ_s, q_s), as defined below:

$$X_s : y^2 = x(x^2+1)(x-s)(x-\overline{s})(x+1/s)(x+1/\overline{s}), \qquad \omega_s = \frac{x\,\mathrm{d}x}{y};$$

$$\Sigma_s : y^2 = (x^2+1)(x-s)(x-\overline{s})(x+1/s)(x+1/\overline{s}), \qquad q_s = \frac{x\,\mathrm{d}x^2}{y^2}.$$

We change coordinates in x using

$$\Psi(x) = i\left(\frac{x-s}{sx+1}\right)$$

so that $\Psi(s) = 0$, $\Psi(-1/s) = \infty$, and $\Psi(\pm i) = \mp 1$. This time we get the curve $y^2 = x(x^2-1)(x-a)(x-1/a)$, where

$$a = \Phi(\overline{s}) = \frac{2\,\mathrm{Im}\,s}{1+|s|^2} \qquad \text{and} \qquad \frac{1}{a} = \Phi\left(-\frac{1}{\overline{s}}\right) = \frac{1+|s|^2}{2\,\mathrm{Im}\,s}.$$

Here we have $0 < a < 1$ and $1/a > 1$; ρ' becomes inversion in the unit circle. The points 0 and ∞ on Σ_s become $\Phi(0) = -is$ and $\Phi(\infty) = i/s$. Again, we find just a one-parameter family of genus 2 Riemann surfaces, each carrying a one-parameter family of quadratic differentials corresponding to distinct surfaces X_a.

On any of the curves Ξ_a, let φ be the abelian differential $\frac{\mathrm{d}x}{y} - \frac{x\,\mathrm{d}x}{y}$. By [Si], the period matrix for Ξ_a is given by

$$i\begin{pmatrix} \frac{2\lambda^2-2\lambda+1}{2\lambda-1} & \frac{-2\lambda(\lambda-1)}{2\lambda-1} \\ \frac{-2\lambda(\lambda-1)}{2\lambda-1} & \frac{2\lambda^2-2\lambda+1}{2\lambda-1} \end{pmatrix} \qquad \text{where} \qquad \lambda = \frac{\int_{-1}^{0}\varphi}{\int_0^{1/a}\varphi}.$$

What is interesting, as Silhol observed, is that the fourfold symmetry permits the entire period matrix to be computed from the single parameter λ. If a lies on the positive imaginary axis, as in our first family, λ is known to be real since Ξ_a exhibits the symmetry that comes from ρ_1 and ρ_2.

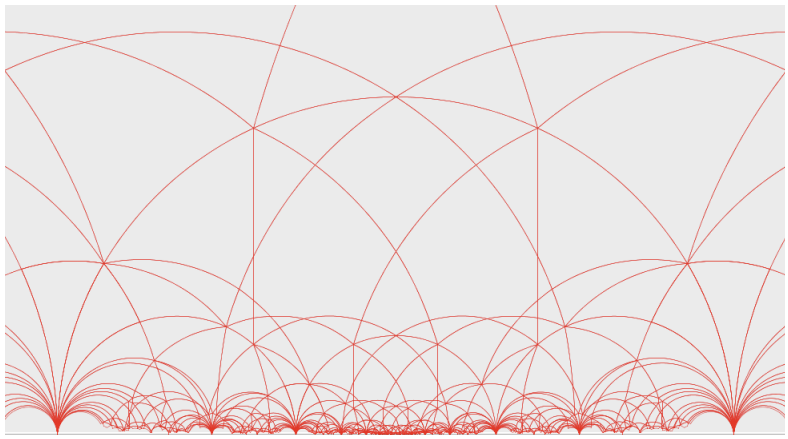

FIGURE 5. The iso-Delaunay tessellation of $\mathbb{H}$ arising from $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$

4. Final remarks

The fixed-point free isometries σ_1 and σ_2 that appear in both families we have presented are implicit in the original Arnoux–Yoccoz construction; their paper begins with a "generalized" pseudo-Anosov diffeomorphism of $\mathbb{RP}^2$, whose invariant foliations have one singular point of valence three and three "thorns", and lift it to the genus 3 example. Taking the quotient of X_{AY} by the action of the group $\langle \sigma_1, \sigma_2 \rangle$ returns us to the real projective plane, and the pseudo-Anosov map ψ_{AY} descends to a diffeomorphism of $\mathbb{RP}^2$, which is the starting point of [AY]. The real structures ρ_1, ρ_2, ρ_1', and ρ_2' do not appear to have been previously known.

Just as these isometries are evident on the surface $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$, so can their effects on the Teichmüller disk generated by this differential be seen in the *iso-Delaunay tessellation* shown in Figure 5. The open regions in this picture correspond to combinatorial classes of Delaunay triangulations of surfaces in the $\mathrm{SL}_2(\mathbb{R})$-orbit of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$; because Delaunay triangulations are not changed by any rotation of the surface, this picture can be drawn in the upper half-plane $\mathbb{H}$ rather than its unit tangent bundle. Iso-Delaunay tessellations have been described previously in [Bo] and [Ve2]. We will not define them here, but simply illustrate how elements of the Veech group Γ of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ may be seen to act on the tessellation in Figure 5.

An orientation-preserving element of Γ acts on $\mathbb{H}$ by a conformal automorphism, and an orientation-reversing element of Γ acts on $\mathbb{H}$ by a reflection in a geodesic [HS]. Figure 5 is symmetric with respect to the central axis (the imaginary axis in $\mathbb{C}$); both σ_1 and σ_2 yield elements of Γ that reflect $\mathbb{H}$ across this axis. The hyperbolic element of Γ corresponding to ψ_{AY} fixes the points 0 and ∞ in $\partial\mathbb{H}$ and translates points along the imaginary axis by $z \mapsto z/\alpha^2$. A sequence of concentric circles are visible in the tessellation; these are related by ψ_{AY}, and one is the unit circle, so their radii are all powers of $1/\alpha^2 \approx 3.38$.

There are two kinds of distinguished points on the imaginary axis: ones where two geodesics meet and ones where three geodesics meet. The latter are those whose corresponding surface is isometric to $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$, while the former correspond to $(X'_{\mathrm{AY}}, \omega'_{\mathrm{AY}})$. The real structures ρ_1 and ρ_2 (resp. ρ_1' and ρ_2') yield an element of Γ that reflects $\mathbb{H}$ across the unit circle (resp. across the circle $|z| = 1/\alpha$). The order 4 rotations of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ and

$(X'_{\mathrm{AY}}, \omega'_{\mathrm{AY}})$ are then visible as the order 2 rotations of $\mathbb{H}$ around these distinguished points. (The hyperelliptic involution acts trivally.)

If any other flat surface with additional real structures lay on the imaginary axis, then its symmetries, too, would have to induce a reflection of $\mathbb{H}$ that preserves the tessellation. No such point exists; therefore we have described all the surfaces along the orbit of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ under the geodesic flow that demonstrate additional symmetries.

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