TEICHMÜLLER GEODESICS, DELAUNAY TRIANGULATIONS, AND VEECH GROUPS

JOSHUA P. BOWMAN

Abstract. In this article, we describe a method for computing generators of the Veech group of a flat surface (which we define as a Riemann surface with a non-zero holomorphic quadratic differential). The method employs a cell structure on the (complex) Teichmüller geodesic generated by the surface, using Delaunay triangulations, which are canonically associated to flat surfaces.

Introduction

Since the late 1980s, Veech groups have been an extremely useful tool in the study of both billiard dynamical systems and projections of "Teichmüller discs" to moduli space. It was observed as early as the 1930s [FK] that billiard paths in a rational polygon could be lifted to geodesic paths on a compact surface with a singular locally Euclidean structure. (In the case of simply connected polygons, "rational" is simply the condition that the angles be rational multiples of π.) Following the terminology of Riemann surface theory, R. Fox and R. Kerschner called this surface the "Ueberlagerungsfläche" of the polygon. Indeed, such a locally Euclidean structure is equivalent to a holomorphic quadratic differential on a Riemann surface (this equivalence is sketched in Section 1.1), and so the field of billiard dynamics becomes connected with the study of the moduli space of quadratic differentials.

W. Veech showed that when the group of affine automorphisms of such a *flat surface* (i.e., the *Veech group,* defined in Section 1.3) is "sufficiently large," then the dynamical behavior of the geodesic flow (which projects to the billiard flow in the case that the surface arises from a rational billiard table) satisfies the *Veech dichotomy:*

Theorem 0.1 ([Ve1]). *Suppose the Veech group of a flat surface is a lattice in* $\mathrm{PSL}(2,\mathbb{R})$*. Then for every direction θ on the surface, one of the two following (mutually exclusive) possibilities occurs:*

- *The surface decomposes in the direction θ into metric cylinders whose moduli are commensurable (in particular, every geodesic either connects two singular points or is closed);*
- *Every trajectory with direction θ is dense in X, and the geodesic flow in the direction θ is uniquely ergodic.*

A flat surface whose Veech group is a lattice in $\mathrm{PSL}(2,\mathbb{R})$ is called a *Veech surface* (also a *lattice surface*). A rational polygon whose Ueberlagerungfläche is a Veech surface is called a *lattice polygon.* The simplest examples of Veech surfaces are *origami*: surfaces that cover the torus with its flat metric, such that the cover has

Date: April 11, 2008.
Partially supported by NSF VIGRE grant DMS-9983660.

a single branch point. G. Schmithüsen has developed an algorithm for computing the Veech group of an origami [Sch]. She makes use of the result of E. Gutkin and C. Judge that origamis are precisely the flat surfaces whose Veech groups are finite index subgroups of $\mathrm{PSL}(2, \mathbb{Z}$ [GJ]. Veech produced the first examples of flat surfaces to have Veech groups that are lattices but not commensurable to $\mathrm{PSL}(2, \mathbb{Z})$; his examples arise from billiards in right triangles that tile regular polygons [Ve1].

There have been concerted efforts both to demonstrate the existence of Veech surfaces with specified properties and to produce examples via explicit constructions. For example, R. Kenyon and J. Smillie classified the acute lattice triangles [KS]. R. Schwartz and P. Hooper have written a Java program called McBilliards that computes periodic paths in triangular billiard tables. Using this program and similar techniques to those of Kenyon and Smillie, they discovered an obtuse triangle that produces a Veech surface of genus 4 [Ho]. C. McMullen [Mc1, Mc2, Mc3, Mc5] and K. Calta [Ca] have independently given criteria for determining when a genus 2 surface, carrying a quadratic differential which is the square of an abelian differential, is Veech.

In this article we develop a general algorithm for computing generating elements of Veech groups. The method depends on keeping track of combinatorial data that is canonically associated to a flat surface—namely, its Delaunay triangulation (Section 2). This produces a natural partition of the orbit of a flat surface under affine equivalence, whose structure is preserved by elements of the Veech group. If the Veech group is in fact a lattice, then this method is guaranteed to determine a full generating set of elements. In many of the cases of the preceding paragraph, the Veech group is known to be finitely generated, so the algorithm we shall describe can be particularly effective in studying these examples.

Figures 5 and 6 show examples of *iso-Delaunay regions,* the type of data we are seeking in the Teichmüller disc of a surface (see Definition 2.11). These images were made with a program written in MATLAB, designed to compute the boundaries of iso-Delaunay regions. A fuller program, which will compute both iso-Delaunay regions and isomorphisms between them, is under construction. More examples of these images, made with the newer program, are available at

http://www.math.cornell.edu/˜bowman/pictures.html.

During preparation of this article, the author became aware of a preprint by Veech [Ve2], which also introduces what we call the *iso-Delaunay complex* (Definition 2.14) under the name of the *tessellation of* $\mathbb{H}$ *subordinate to the quadratic differential.* In future work, we shall call this object the *iso-Delaunay tessellation.*

1. Background

1.1. **Flat surfaces.** Let X be a compact hyperbolic Riemann surface (i.e., of genus $g \geq 2$).

Definition 1.1. A *holomorphic quadratic differential* on X is a section of the tensor-square of the sheaf of holomorphic one-forms. In symbols, it is an element of $\Gamma(X, \Omega^{\otimes 2}(X))$.

Let q be a non-zero quadratic differential on X. In a local analytic coordinate z on X, q has the form $q(z) = f(z)\,\mathrm{d}z \otimes \mathrm{d}z$; often one abbreviates $\mathrm{d}z^2 = \mathrm{d}z \otimes \mathrm{d}z$. Let Z denote the set of zeroes of q on X. Given a point $x \in X - Z$ and a simply connected neighborhood $U \subset X - Z$ of x, a chart $U \to \mathbb{C}$ centered at x is

$$y \mapsto \int_\gamma \sqrt{q},$$

where $y \in U$ and $\gamma : [0,1] \to U$ is a path from x to y. In this chart, q has the form $\mathrm{d}z^2$. The choice of $\sqrt{q}$ is unique up to sign, hence q determines an analytic atlas on $X - Z$ with transition maps all of the form $z \mapsto z + c$ or $z \mapsto -z + c$. Such an atlas is called a *flat structure* on X with singularities in Z.

The Euclidean metric $|\mathrm{d}z| = |\sqrt{q}|$ is invariant under changes of coordinates in the flat structure, and so gives an elementary perspective on the geometry of the quadratic differential. The pair $(X - Z, |\sqrt{q}|)$ is locally isometric to the plane, while each point $z \in Z$ has a *cone angle* of $k\pi$ for some integer $k > 2$; $k - 2$ is called the *order* of the singular point z (this coincides with the analytic definition of the order of a zero).

The reverse construction is also possible: given a flat structure on a compact oriented surface of genus $g \geq 2$, one obtains a unique pair (X', q'), where X' is a Riemann surface and q' is a quadratic differential on X'. The conformal structure on X' is obtained by completing the flat structure to a maximal conformal atlas, and q' is obtained by taking $q'(z) = \mathrm{d}z^2$ in the charts of the flat structure.

Definition 1.2. Via the above equivalence, a *flat surface* means either a topological surface with a flat structure or a Riemann surface with a chosen non-zero quadratic differential.

Remark 1.3. One might also consider flat structures with some singularities of cone angle π, i.e., zeroes of order -1; these correspond to *meromorphic* quadratic differentials with at worst simple poles. This class of quadratic differentials is quite important, and much of the material developed in this paper applies also to these flat surfaces. All of our examples, however, will arise as holomorphic quadratic differentials.

For the remainder of the paper, (X, q) will denote a fixed flat surface, and Z the set of zeroes of q.

1.2. **Teichmüller geodesics and the Poincaré half-plane.** Let $\mathscr{T}_X$ denote the *Teichmüller space* of X, i.e., the space of Riemann surfaces marked by X, up to isomorphisms respecting the marking, and $\mathbb{H}$ the open upper half-plane in $\mathbb{C}$.

Definition 1.4. A (*complex*) *Teichmüller geodesic* (also known as a *Teichmüller disc*) is a complex-analytic embedding $f : \mathbb{H} \to \mathscr{T}_X$ whose image is geodesic for the Teichmüller metric.

The pull-back of the Teichmüller metric by f coincides with the Poincaré metric on $\mathbb{H}$. It is a fundamental result in Teichmüller theory that the bundle of quadratic differentials $\mathscr{Q}_X \to \mathscr{T}_X$ is the cotangent bundle to $\mathscr{T}_X$, and that there is a canonical pairing between $\mathscr{Q}_X$ and the tangent bundle to $\mathscr{T}_X$. Hence (X, q) generates a complex-analytic embedding of the upper half-plane $\mathbb{H}$ into $\mathscr{Q}_X$, which projects down to a Teichmüller geodesic $f : \mathbb{H} \to \mathscr{T}_X$.

For $t \in \mathbb{H}$, the flat surface $(X_t, q_t) \in \mathscr{Q}_X$ can be described explicitly in terms of the local geometry of q. Take a local coordinate z on X in which $q = \mathrm{d}z^2$, and write $z = x + iy$. Compose with the map $x + iy \mapsto x + ty$ to get a new coordinate z_t. Then X_t is the Riemann surface with local coordinate z_t, and q_t is the quadratic

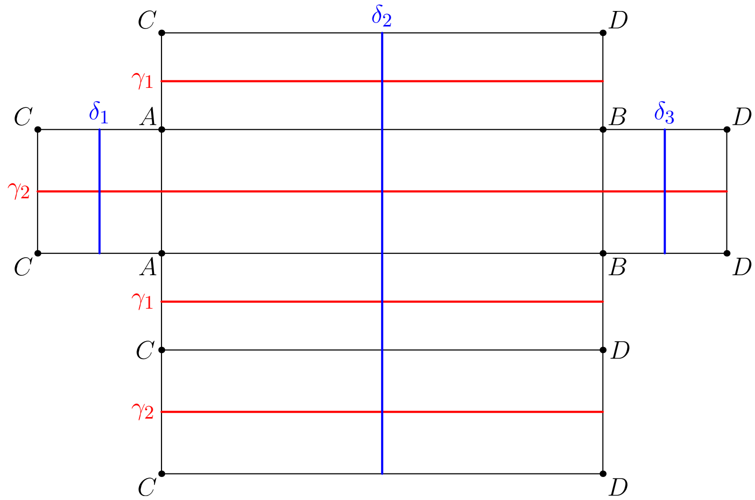

FIGURE 1. A "bouillabaisse" flat surface, $\lambda = 5 + \sqrt{17}$. (Bouillabaisse surfaces are described in Section 3.) Horizontal exterior edges are identified via vertical translations, and vertical exterior edges are identified via 180° rotations. The singular points are A, B, C, and D; they are *simple* zeroes of the quadratic differential $\mathrm{d}z^2$, each having a cone angle of 3π.

differential with local form $q_t = \mathrm{d}{z_t}^2$. The map $x + iy \mapsto x + ty$ is affine, with derivative given by the matrix $\begin{pmatrix} 1 & \operatorname{Re} t \\ 0 & \operatorname{Im} t \end{pmatrix}$.

Dividing $\begin{pmatrix} 1 & \operatorname{Re} t \\ 0 & \operatorname{Im} t \end{pmatrix}$ by the square root of its determinant $\operatorname{Im} t$, we obtain a representative element in $\mathrm{SL}(2,\mathbb{R})$ for the map $(X, q) \mapsto (X_t, (\operatorname{Im} t)^{-1/2} q_t)$. Normalizing to an element in $\mathrm{SL}(2,\mathbb{R})$ simply yields a quadratic differential having the same area on X_t as the original. As we are interested in the projection to $\mathscr{T}_X$, rather than the complex curve in $\mathscr{Q}_X$, we will in general ignore the difference between $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$.

Even further, one can consider the action of $\mathrm{SL}(2,\mathbb{R})$ directly on the flat structure of a flat surface, and obtain a picture of the Teichmüller geodesic apart from its embedding into $\mathscr{T}_X$. This description is given in [EG]; we reproduce the salient features for our computational purposes. Let $A \in \mathrm{SL}(2,\mathbb{R})$, and define $A \cdot (X, q)$ by taking each chart in the flat structure of (X, q) and post-composing it with A. Multiplying by an element of $\mathrm{SO}(2,\mathbb{R})$ is equivalent to multiplying q by a unit complex number, which changes neither the conformal nor the metric structure of (X, q). Hence we introduce:

Definition 1.5. The $\mathrm{SL}(2,\mathbb{R})$-*orbit* of (X, q) is the quotient of

$$\{A \cdot (X, q) \mid A \in \mathrm{SL}(2,\mathbb{R})\}$$

by the left action of $\mathrm{SO}(2,\mathbb{R})$.

We want to systematically identify the $\mathrm{SL}(2,\mathbb{R})$-orbit of (X,q) with the well-understood space $\mathbb{H}$, using Proposition 1.6 below. Define $P : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}$ by

$$P : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{di+b}{ci+a}$$

Let σ denote the involution on $\mathrm{SL}(2,\mathbb{R})$ given by

$$\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Recall the standard action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}$ by Möbius transformations, which, for $A \in \mathrm{SL}(2,\mathbb{R})$, we will denote by $\zeta \mapsto A\zeta$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \zeta \mapsto \frac{a\zeta+b}{c\zeta+d};$$

Then $P(A) = (\sigma(A))^{-1}i$. Note that $\mathrm{SO}(2,\mathbb{R})$ is preserved by σ, and that $Ai = i$ for all $A \in \mathrm{SO}(2,\mathbb{R})$. The following result is straightforward computation.

Proposition 1.6. *P descends to a bijection (also denoted P) from the set of right cosets of* $\mathrm{SO}(2,\mathbb{R})$ *to the upper half-plane* $\mathbb{H}$ *in* $\mathbb{C}$. (In particular, $\mathrm{SO}(2,\mathbb{R})$ itself is mapped to $i \in \mathbb{H}$.)

The group $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$ acts faithfully on this coset space via

$$B \cdot A = A(\sigma(B))^{-1},$$

yielding the standard representation of $\mathrm{PSL}(2,\mathbb{R})$ onto the group of conformal automorphisms of $\mathbb{H}$. (That is, $P(B \cdot A) = B(P(A))$.)

1.3. **Veech groups.** Recall that $\mathrm{Mod}(X)$, the Teichmüller modular group of X, (a.k.a. the mapping class group) acts on $\mathscr{T}_X$ by complex-analytic isomorphisms, which are also isometries for the Teichmüller metric, and that the quotient of $\mathscr{T}_X$ by this action is precisely $\mathcal{M}_X$, the moduli space of X.

Definition 1.7. The *Veech group* $\Gamma(X,q)$ of (X,q) is the subgroup of $\mathrm{PSL}(2,\mathbb{R})$ that corresponds to mapping classes of X, i.e., elements of $\mathrm{Mod}(X)$ that stabilize the image of the geodesic f generated by (X,q). The surface (X,q) is called *Veech* if $\Gamma(X,q)$ is a lattice in $\mathrm{PSL}(2,\mathbb{R})$, i.e., if $\mathbb{H}/\Gamma(X,q)$ has finite hyperbolic area.

If (X,q) is a Veech surface, the image of the composition $\mathbb{H} \to \mathscr{T}_X \to \mathcal{M}_X$ is an algebraic curve. Veech showed that a Veech surface also has especially "nice" geometric properties [Ve1], as stated in Theorem 0.1.

By Thurston's description [Th] of the action of elements of $\mathrm{Mod}(X)$ on $\mathscr{T}_X$, we can closely relate the topological type of $g \in \Gamma(X,q)$ and the type of isometry of $\mathbb{H}$ that g induces; see also I. Kra's study in [K]. Specifically,

- g is *periodic* if and only if the correponding isometry of $\mathbb{H}$ is *elliptic* (fixes a point in $\mathbb{H}$);
- g is *reducible* if and only if the corresponding isometry of $\mathbb{H}$ is *parabolic* (fixes a point on $\partial\mathbb{H}$ and the class of horocycles through this point);
- g is *pseudo-Anosov* if and only if the corresponding isometry of $\mathbb{H}$ is *hyperbolic* (fixes two points on $\partial\mathbb{H}$ and the Poincaré geodesic connecting them).

2. Delaunay triangulations

Our algorithm for computing generators of the Veech group depends essentially on the existence of a combinatorial structure uniquely determined by the geometry of q. The structure we will use is the Delaunay triangulation of the flat surface. The existence of Delaunay triangulations and a few basic results concerning them are proved by H. Masur and J. Smillie in [MS]. We will also use several results from I. Rivin's work in [Ri].

2.1. **Triangulations of a flat surface.** Following Vorobets's description of ω-triangles when ω is a holomorphic 1-form on a Riemann surface [Vo], we introduce:

Definition 2.1. A *q-triangle on X* is a 2-simplex in X, embedded on its interior, whose vertices lie in Z, whose sides are geodesic with respect to $|\sqrt{q}|$, and whose interior does not contain any points of Z.

On the interior of a q-triangle, the structure is purely Euclidean; for instance, the angles sum to π. Some of the vertices may coincide (for example, if Z contains only a single point). The condition that singular points have a cone angle greater than 2π prevents two edges of a single q-triangle from coinciding, however.

Definition 2.2. A *q-triangulation* of X is a simplicial cell structure on X, the closures of whose 2-cells are q-triangles. A q-triangulation on X and a q'-triangulation on X' are *combinatorially equivalent* if there is a bijective map from the q-triangles in the triangulation of X to the q'-triangles in the triangulation of X' that preserves edge identifications.

Remark 2.3. A combinatorial equivalence between (X, q) and (X', q') induces a homeomorphism from X to X', which is piecewise affine for the local coordinates of q and q'. We will use this natural homeomorphism in our algorithm, described in Section 4.

The *Whitehead move* is a natural way to change one q-triangulation into another. Suppose ABC and BCD are two triangles in some q-triangulation of X with the common edge BC. (Some or all of A, B, C, D may actually coincide as elements of Z; as far as the Euclidean structure on $X - Z$ is concerned, we can consider ABC and BCD to simply lie in the plane.) Then no points of Z are contained in the interior of the quadrilateral $ABDC$; provided $ABDC$ is convex, we can obtain a new q-triangulation of X by removing the edge BC and adding the edge AD, forming the triangles ABD and ACD.

Clearly, (X, q) admits many q-triangulations. We wish to describe a particular sort that is unique for an open dense subset of surfaces in the $\mathrm{SL}(2, \mathbb{R})$-orbit of (X, q).

2.2. **Dihedral angles and the Delaunay condition.**

Definition 2.4. Let T_1 and T_2 be a pair of Euclidean triangles with disjoint interiors and a common edge e, and let α_1 and α_2 be the (unsigned) angles opposite e in T_1 and T_2, respectively. The *dihedral angle* of e is $\alpha(e) = \alpha_1 + \alpha_2$.

Definition 2.5. A q-triangulation of X is *Delaunay* if $\alpha(e) \leq \pi$ for all edges e of the triangulation. A Delaunay triangulation is *degenerate* if $\alpha(e) = \pi$ for one or more e.

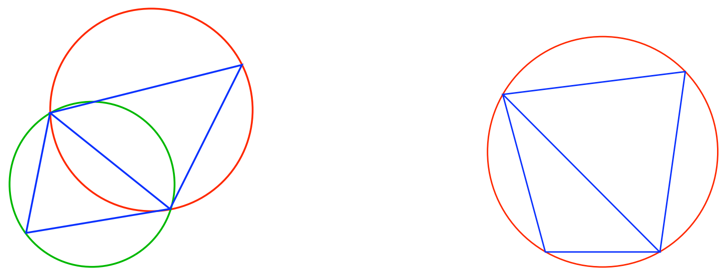

FIGURE 2. A pair of adjacent triangles in a Delaunay triangulation, and in a degenerate Delaunay triangulation

The following result is a consequence of Theorem 10.2 in [Ri]; the proof is corrected by A. Bobenko and B. Springborn in [BS].

Proposition 2.6. *Given any q-triangulation of X, one obtains a Delaunay triangulation of X via a finite sequence of Whitehead moves, each of which replaces an edge having dihedral angle $> \pi$ with its opposite diagonal.*

Remark 2.7. The Delaunay condition for a triangulation is usually phrased in terms of the circumcircles of the triangles. It is a basic fact of Euclidean geometry that if $\alpha(e) = \pi$, then the vertices of T_1 and T_2 lie on a single circle. If $\alpha(e) > \pi$, then the circumcircle of T_1 contains the remaining vertex of T_2, and vice versa. If $\alpha(e) < \pi$, then the remaining vertex of T_2 lies outside the circumcircle of T_1.

Remark 2.8. The terminology of "dihedral angles" arises from an application to hyperbolic geometry [Ri]. If we use the half-space model of hyperbolic 3-space $\mathbb{H}^3$, then $\partial\mathbb{H}^3$ is the union of the Euclidean plane $\mathbb{E}^2$ and the point ∞. A triangle ABC in $\mathbb{E}^2$ determines an ideal tetrahedron $ABC\infty$ in $\mathbb{H}^3$, with ideal vertices A, B, C, and ∞. A basic fact of hyperbolic geometry is that the opposite edges of an ideal tetrahedron have equal dihedral angles (which measure the "solid" angle between faces of the tetrahedron). The dihedral angle of $\angle ABC$ in $ABC\infty$ equals the angle $\angle ABC$ in the plane, because the faces $AB\infty$ and $BC\infty$ are contained in vertical (Euclidean) planes. Two adjacent triangles ABC and BCD then determine two ideal tetrahedra, joined along the common face $BC\infty$, and the dihedral angle of their common edge BC is the sum of the angles $\angle BAC$ and $\angle BDC$ in $\mathbb{E}^2$.

The Delaunay condition on a triangulation of (X, q) will allow us to introduce additional structure on the $\mathrm{SL}(2, \mathbb{R})$-orbit of (X, q) in the following section.

2.3. **Iso-Delaunay regions in $\mathbb{H}$.** The key observations in our algorithm are the following:

Proposition 2.9. *If a Delaunay triangulation of $(X', q') \in \mathbb{H}$ is non-degenerate, then it is unique. If a Delaunay triangulation is degenerate, then it is unique except for exchanges of edges with dihedral angle π via Whitehead moves.*

Proposition 2.10. *If $f \in \Gamma(X, q)$ and $\mathcal{T}$ is a Delaunay triangulation of (X, q), then $f^*\mathcal{T} = \{f^*(T) \mid T \in \mathcal{T}\}$ is a Delaunay triangulation of (f^*X, f^*q). I.e., The Delaunay triangulation is independent of the marking of X.*

Uniqueness is typically proved via the duality between edges of a Delaunay triangulation and those of the Voronoi cells on the surface (see, e.g., [MS]; see also [Ri] for a proof using the convexity of a volume function on the triangulation). Identify the $\mathrm{SL}(2,\mathbb{R})$-orbit of (X,q) with $\mathbb{H}$ as in Section 1.2. In this case, the openness of the non-degenerate Delaunay condition implies that (X',q') is contained in a neighborhood of surfaces with combinatorially equivalent Delaunay triangulations.

Definition 2.11. An *iso-Delaunay region* (IDR) is a maximal connected open subset of $\mathbb{H}$ such that all its points have combinatorially equivalent Delaunay triangulations.

Proposition 2.12. *With respect to the Poincaré metric on $\mathbb{H}$, each IDR is convex and has piecewise geodesic boundary.*

To prove this result, we shall use a standard test in computational geometry.

Proposition 2.13 (Incircle test)**.** *Let $P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, and $P_3 = (x_3,y_3)$ be three points in $\mathbb{R}^2$, with O, P_1, and P_2 non-colinear. Let $\mathcal{C}$ be the circumcircle of OP_1P_2. Then*

$$\text{(1)} \qquad \textit{if} \quad \det\begin{pmatrix} x_1 & y_1 & x_1^2+y_1^2 \\ x_2 & y_2 & x_2^2+y_2^2 \\ x_3 & y_3 & x_3^2+y_3^2 \end{pmatrix} \quad \begin{cases} <0, & P_3 \textit{ is in the interior of } \mathcal{C}. \\ =0, & P_3 \textit{ lies on } \mathcal{C}. \\ >0, & P_3 \textit{ is exterior to } \mathcal{C}. \end{cases}$$

Sketch of proof. Embed $\mathbb{R}^2$ into $\mathbb{R}^3$ as the (x,y)-plane. Project P_1, P_2, and P_3 vertically to Q_1, Q_2, and Q_3, respectively, on the paraboloid $z = x^2+y^2$. Let $\mathcal{P}$ be the plane determined by O, Q_1, and Q_2. Observe that $\{Q_1,Q_2,Q_3\}$ is a direct basis, an indirect basis, or a dependent set as the above determinant is greater than, less than, or equal to zero; this also corresponds to Q_3 lying above, below, or on $\mathcal{P}$ (where "above" and "below" are determined by the orientation of the basis $\{Q_1,Q_2\}$ for $\mathcal{P}$). Replacing z in the equation $ax+by+cz=0$ of $\mathcal{P}$ with x^2+y^2, one obtains the equation of a circle in the (x,y)-plane, which must pass through O, P_1, and P_2. One concludes the desired result by comparing the various geometric and algebraic conditions. □

Proof of Proposition 2.12. We will show that each IDR is an intersection of hyperbolic half-planes, which implies the desired results.

Suppose a Delaunay triangulation is given for a certain point in the $\mathrm{SL}(2,\mathbb{R})$ orbit of (X,q). For each edge in the Delaunay triangulation, choose a chart as in Section 1.1 such that one endpoint of the edge is at 0. Label the other endpoint P_2, and the remaining vertices P_1 and P_3, so that the ordered bases $\{P_1,P_2\}$ and $\{P_2,P_3\}$ are oriented counterclockwise.

Now let $A \in \mathrm{SL}(2,\mathbb{R})$ be a variable matrix, with $u+iv$ the image of A in $\mathbb{H}$. Then we can chose a representative in the same left $\mathrm{SO}(2,\mathbb{R})$-coset as A that sends x_i to x_i+uy_i and y_i to vy_i $(i=1,2,3)$. Evaluate the equality in (1) with x_i and y_i replaced by their images under the action of this representative. After some simplification (including dividing by v, which we can do because $v>0$ in the upper half-plane), this becomes

$$\text{(2)} \qquad a(u^2+v^2)+2bu+c \geq 0$$

where

$$\begin{aligned} a &= x_1y_2y_3(y_3 - y_2) + x_2y_1y_3(y_1 - y_3) + x_3y_1y_2(y_2 - y_1), \\ b &= x_1y_1(x_2y_3 - x_3y_2) + x_2y_2(x_3y_1 - x_1y_3) + x_3y_3(x_1y_2 - x_2y_1), \\ c &= x_1x_2y_3(x_1 - x_2) + x_2x_3y_1(x_2 - x_3) + x_1x_3y_2(x_3 - x_1). \end{aligned}$$

If $a \neq 0$, (2) is the equation of a circle whose center lies on the real axis. If $a = 0$ (for example, if $y_1 = 0$ and $y_2 = y_3$), then (2) is the equation of a vertical line. These are precisely the forms of geodesics in $\mathbb{H}$.

Let (X', q') be a surface in the $\mathrm{SL}(2,\mathbb{R})$-orbit of (X, q) with nondegenerate Delaunay triangulation. For each edge e, choose a chart in the corresponding flat structure whose domain contains the triangles adjacent to e. We may assume that one of the endpoints of e is at the origin. Then, applying the result of the previous paragraph, we get one inequality for each edge. Because there are only finitely many such inequalities—one for each edge of the Delaunay triangulation—the result follows. □

As noted in Veech [Ve2], the area of each IDR is finite, so it is actually a hyperbolic polygon, possibly with some vertices at infinity.

Observe that, at an edge of an IDR determined by a set of inequalities (1), by performing a Whitehead move on the degenerate edges, we reverse these particular inequalities and obtain a polygonal region on the opposite side of the edge. Hence the Delaunay triangulations of surface in the orbit of (X, q) naturally partition $\mathbb{H}$ into 2-cells (the open IDRs), 1-cells (maximally connected sets of points lying in the closures of exactly two IDRs), and 0-cells (points lying in the closures of at least three IDRs), giving $\mathbb{H}$ the structure of a cell complex.

Definition 2.14. The above cell complex on $\mathbb{H}$ is called the *iso-Delaunay complex* of (X, q).

Proposition 2.10 implies that the iso-Delaunay complex in the $\mathrm{SL}(2,\mathbb{R})$-orbit of the surface is preserved by the action of $\Gamma(X, q)$. A caveat: the iso-Delaunay complex is not a CW-complex, as some of its cells may not have compact closure in $\mathbb{H}$.

3. Bouillabaisse surfaces

Whenever a flat surface has a direction in which it decomposes into cylinders, and the moduli of these cylinders are commensurable, the Veech group of the surface contains a parabolic element, which consists of appropriate powers of Dehn twists in each of the cylinders. A *bouillabaisse surface* is a flat surface with at least two such directions; in other words, its Veech group contains two transverse parabolic elements. The construction of bouillabaisse surfaces is due to W. Thurston [Th]. In June 2003, J. Hubbard described them at the “bouillabaisse night” lecture during a conference at the Centre International de Rencontres Mathématiques in Luminy, France, whence their name.

3.1. **Basic construction.** The construction begins with an oriented compact surface S of genus $g \geq 2$. A *multi-curve* on S is a collection of pairwise disjoint, non-homotopic simple closed curves. The number of curves is bounded by the genus of the surface. In particular, any maximal multi-curve creates a decomposition of the surface into pairs of pants, and the number of pairs of pants is determined by the

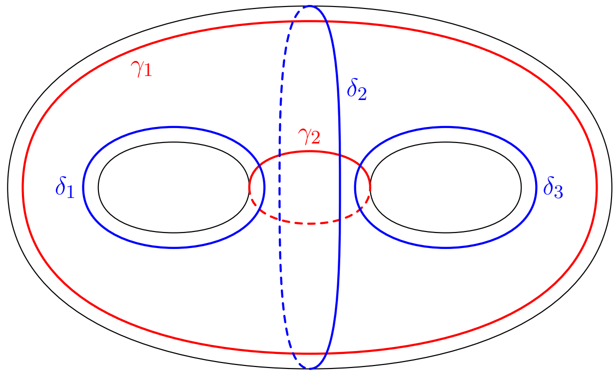

FIGURE 3. The pair of multi-curves Γ and Δ on a genus 2 surface that produce the flat surface in Figure 1.

genus. Thus the number of elements in a maximal multi-curve is $3g - 3$. A *Dehn twist* around a multi-curve is simply the composition of the Dehn twists around each of the component curves; because the curves are disjoint, all these twists commute, so the order of composition does not matter. We will always assume Dehn twists are "right-handed" with respect to the orientation of S. We will also mildly abuse notation by letting, for example, Γ represent both the set of curves in a multi-curve on S and the set of points on S contained in the images of these curves.

Let Γ and Δ be transverse, not necessarily maximal, multicurves on S such that each component of $S - (\Gamma \cup \Delta)$ is simply connected. Assume further that Γ and Δ are in minimal position, meaning that each pair $(\gamma, \delta) \in \Gamma \times \Delta$ has the minimum number of intersections as γ and δ vary in their respective homotopy classes. Define $M = [m_{\delta\gamma}] : \mathbb{R}^\Gamma \to \mathbb{R}^\Delta$ by

$$\delta \cdot M(\gamma) = m_{\delta\gamma}, \qquad \gamma \in \Gamma,\ \delta \in \Delta$$

where $m_{\delta\gamma}$ is the geometric intersection number of δ and γ. Let λ be the largest eigenvalue of $M^\top M$, and let $v \in \mathbb{R}^\Gamma$ be an eigenvector of $M^\top M$ corresponding to λ with all entries positive. (Such a v exists because $M^\top M$ is a Perron-Frobenius matrix.) Now set $u = Mv$. Then u is an eigenvector of $MM^\top$, also corresponding to the eigenvalue λ.

We use the above data to simultaneously construct a Riemann surface $X' \in \mathscr{T}_S$ and a quadratic differential q' on X'. $\Gamma \cup \Delta$ induces a cell structure on S, whose dual structure consists of rectangles $R_{\delta\gamma}$, each containing the intersection of one $\gamma \in \Gamma$ and one $\delta \in \Delta$. For each rectangle $R_{\delta\gamma}$ crossed by γ and δ in the dual cell structure to $\Gamma \cup \Delta$, take a chart on the interior to $\mathbb{C}$ so that γ is horizontal, δ is vertical, the the width of $R_{\delta\gamma}$ is u_δ and the height is v_γ. Assume further that each chart matches the orientation of S with the orientation of the plane. These charts define the complex structure on S. Take $q' = \mathrm{d}z^2$ in each of these charts.

Proposition 3.1. *In these coordinates, the Dehn twists D_Γ and D_Δ, around Γ and Δ respectively, are affine transformations, and their derivatives have the forms*

$$\operatorname{der}(D_\Gamma) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad \operatorname{der}(D_\Delta) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

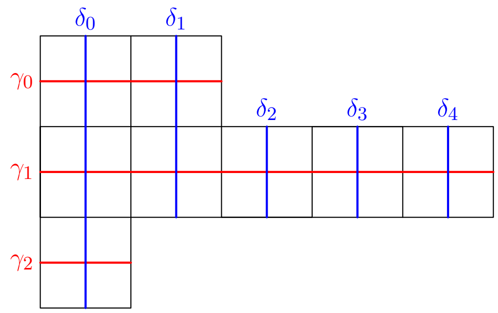

FIGURE 4. A bouillabaisse surface on which the cylinders have moduli that are rationally related, such as the square-tiled surface in this figure, can be constructed via multi-curves, using the appropriate duplication of each homotopy class of curves.

Proof. Choose a point in the cylinder crossed by $\gamma \in \Gamma$ and a rectangular chart so that one edge of the cylinder lies along the x-axis. In these coordinates, the Dehn twist around Γ maps (x, y) by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + s_\gamma y \\ y \end{pmatrix}$$

for some s_γ depending *a priori* on γ. When $y = tv_\gamma$, x is increased by

$$\sum_{\delta \in \Delta} m_{\delta\gamma} t u_\delta = (M^\top tu)_\gamma = t(M^\top M v)_\gamma = t\lambda v_\gamma.$$

Hence $s_\gamma = \lambda$ independently of γ, and $\operatorname{der}(D_\Gamma)$ has the form described. The proof for the Dehn twist around Δ is similar. $\square$

3.2. **Introducing arbitrary rational ratios of moduli.** The above construction admits several generalizations. We remark here on the principal one that applies to our study. As described above, the bouillabaisse construction creates cylinders with *equal* moduli in each of the horizontal and vertical directions. In order to create cylinders which have *commensurable* moduli, one "duplicates" the core curves of cylinders in which higher powers of Dehn twists are necessary. More precisely, fix $\gamma_0 \in \Gamma$, $\delta_0 \in \Delta$, and rational numbers r_γ, r_δ for each $\gamma \in \Gamma$, $\delta \in \Delta$, with $r_{\gamma_0} = r_{\delta_0} = 1$. We will construct a flat structure on S so that r_γ is the ratio of the modulus of the cylinder with core curve γ to that of the cylinder with core curve γ_0 (and likewise for the r_δs).

Let ρ be the least common multiple of the r_γs (i.e., ρ is the smallest rational number such that ρ/r_γ is an integer for all γ); ρ is an integer because $r_{\gamma_0} = 1$. Set $r'_\gamma = \rho/r_\gamma$, and replace each $\gamma \in \Gamma$ with r'_γ homotopic copies of γ (for example, γ_0 is replaced by ρ copies of itself); denote these copies $\alpha_{\gamma,j}$ for $1 \le j \le r'_\gamma$. Then a simultaneous Dehn twist in each $\alpha_{\gamma,j}$ for fixed γ corresponds to the r'_γth power of the Dehn twist in γ. Let $\Gamma' = \{\alpha_{\gamma,j} \mid \gamma \in \Gamma, 1 \le j \le r'_\gamma\}$. Note that the r'_γs depend only on the ratios of the moduli of the cylinders, not on the choice of γ_0.

Now carry out the same process in the vertical direction to get a collection Δ' of $\beta_{\delta,k}$s, and construct $M' : \mathbb{R}^{\Gamma'} \to \mathbb{R}^{\Delta'}$ as before. Because the geometric intersection number is well-defined on homotopy classes, we have $\beta_{\delta,k} \cdot M'(\alpha_{\gamma_j}) = \delta \cdot M(\gamma)$, i.e., M' is obtained from the ordinary $M : \mathbb{R}^\Gamma \to \mathbb{R}^\Delta$ by replacing each column $[m_{\delta\gamma}]_{\delta\in\Delta}$ with r'_γ copies, and each row $[m_{\delta\gamma}]_{\gamma\in\Gamma}$ with r'_δ copies.

By carrying out the same construction with M' as we did with M, we get a flat surface such that the cylinders containing each of $\alpha_{\gamma,j}$ all have the same modulus. The cylinder of which some γ is a core curve is comprised of r'_γ of these cylinders. For $\gamma_1, \gamma_2 \in \Gamma$, the ratio of the moduli of their respective cylinders is $r'_{\gamma_1}/r'_{\gamma_2} = r_{\gamma_2}/r_{\gamma_1}$, as desired.

3.3. **Relevance of bouillabaisse surface.** The following corollary to Theorem 0.1 demonstrates one measure of the importance of bouillabaisse surfaces:

Corollary 3.2. *Every Veech surface is bouillabaisse.*

Proof. Any direction containing a geodesic that connects two singular points decomposes the surface into cylinders whose moduli are commensurable. □

This construction has become an indispensible tool in the study of flat surfaces. For example, McMullen has used it to describe infinite families of Veech surfaces in genus 2, 3, and 4 [Mc4]. Not all flat structures with a non-trivial Veech group arise in this manner, however. The expansion constants for pseudo-Anosov elements of the subgroup $\langle D_\Gamma, D_\Delta\rangle < \Gamma(X, q)$ all lie in the field $\mathbb{Q}(\lambda)$, which is totally real since λ arises as an eigenvalue of a symmetric matrix. P. Arnoux and J.-C. Yoccoz [AY] constructed pseudo-Anosov maps of surfaces in all genera $g \geq 3$ whose expansion constants are *Pisot numbers,* some of whose Galois conjugates lie outside of $\mathbb{R}$. P. Hubert and E. Lanneau [HuLa] showed that this implies the Veech groups of the Arnoux–Yoccoz surfaces contain no parabolic elements; i.e., every stabilizing element of such a surface is either periodic or pseudo-Anosov; hence the Arnoux–Yoccoz examples are not bouillabaisse.

4. The algorithm

The IDRs and iso-Delaunay complex determined by (X, q) partition $\mathbb{H}$ into (not necessarily compact) polygons. A fundamental domain for the action of the Veech group $\Gamma(X, q)$ can be assembled from (possibly infinitely many) IDRs of the surface. In order to turn this observation into an algorithm, we need a method for detecting when two IDRs are equivalent under an element of $\Gamma(X, q)$; we will call such a pair of IDRs *Veech equivalent.*

4.1. **Veech equivalent IDRs.** Combinatorial equivalence is an obvious necessary condition for two IDRs to be Veech equivalent, by the uniqueness of non-degenerate Delaunay triangulations. It is not sufficient, however, so we must also take into account the metric structure of the surfaces. Fortunately, this structure is well-behaved for the action of $\mathrm{SL}(2, \mathbb{R})$.

Let R_1 and R_2 be two iso-Delaunay regions with the same combinatorial data, and choose a point (X', q') in one of them, say in R_2. We want to determine if (X', q') is isomorphic to a point in R_1; if it is, then there is an element of $\Gamma(X, q)$ carrying R_1 to R_2.

4.2. **Tree search in $\mathbb{H}$.** To assemble the fundamental domain of $\Gamma(X, q)$, begin with a single point in $\mathbb{H}$ whose Delaunay triangulation is not degenerate. First determine the IDR containing this point; each edge of the IDR corresponds to the degeneration of (a) particular edge(s) in the triangulation.

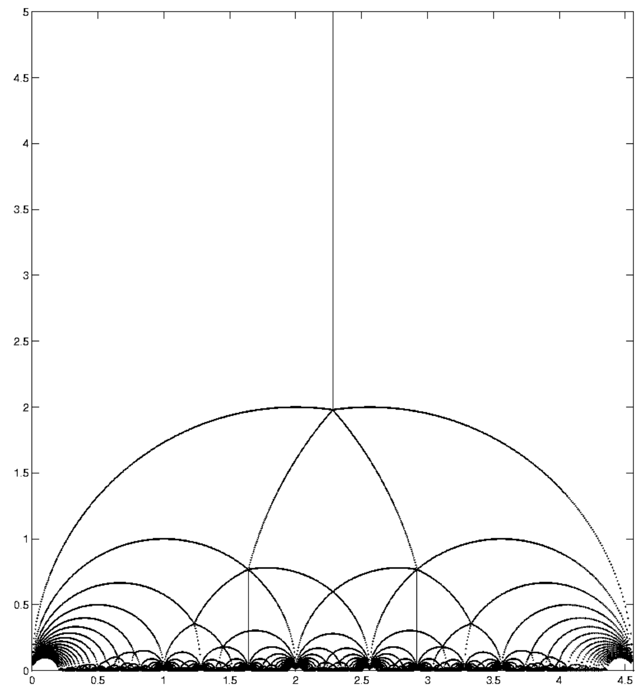

FIGURE 5. Some IDRs for the flat surface shown in Figures 1 and 3. This is a Veech surface, which explains the extreme regularity of the iso-Delaunay complex.

More systematically, we have the following algorithm:

(1) Begin with a surface in the $\mathrm{SL}(2,\mathbb{R})$-orbit of (X,q) whose Delaunay triangulation is non-degenerate, and compute its iso-Delaunay region $R_i = R_0$ using the formulas in the proof of Proposition 2.12.
(2) Along each (possibly infinite) hyperbolic segment comprising the boundary of R_i, determine which edges have degenerated, and compute the adjacent triangulation that replaces each of these edges by their opposite diagonal. This allows us to "cross" the edge into a new iso-Delaunay region.
(3) For each boundary segment crossed, compute the adjacent IDR from the new triangulation; set R_i equal to this IDR.
(4) Check if the new triangulation in R_i is combinatorially equivalent to that of a previously visited region R_j. If so:
   - check if the new surface is isomorphic to a point in R_j;
   - if such an isomorphism exists, find an isometry that carries R_i to R_j, and eliminate R_i from the tree search; add the isometry to the generating list for $\Gamma(X,q)$.
(5) For each new IDR not found to be Veech equivalent to a previous IDR, return to Step (2).

4.3. **Initial bounds for bouillabaisse surfaces.** From the construction of bouillabaisse surfaces, we automatically get part of the iso-Delaunay complex, and hence initial bounds on the fundamental domain of $\Gamma(X, q)$. The two elements of $\mathrm{SL}(2, \mathbb{R})$ described in Proposition 3.1 generate a subgroup of $\Gamma(X, q)$. This subgroup is in most cases a free group on these two generators; C. Leininger has given the precise conditions for the group to be free [Lei]. In any case, it has a fundamental domain bounded by the geodesics

$$x = 0, \quad x = \lambda, \quad y^2 + \left(x - \frac{1}{2}\right)^2 = \frac{1}{4}, \quad \text{and} \quad y^2 + \left(x - \lambda + \frac{1}{2}\right)^2 = \frac{1}{4}. \tag{3}$$

If $\lambda > 2$, F_2 is not itself a lattice in $\mathrm{SL}(2, \mathbb{R})$.

The trace field of $\Gamma(X, q)$ is exactly $\mathbb{Q}(\lambda)$. This follows from the fact that that the trace field is generated by the trace of a single hyperbolic element ([KS]), and

$$D_\Delta^{-1} \circ D_\Gamma = \begin{pmatrix} 1 & \lambda \\ 1 & 1 + \lambda \end{pmatrix}, \tag{4}$$

which has trace $2 + \lambda$, and $\mathbb{Q}(2 + \lambda) = \mathbb{Q}(\lambda)$. Because λ arises from a symmetric matrix, it and all of its Galois conjugates are real, hence the field $\mathbb{Q}(\lambda)$ is *totally real,* i.e., all of its embeddings into $\mathbb{C}$ actually have their image contained in $\mathbb{R}$.

4.4. **Other uses of IDR analysis.** By a slight modification of the surface in Figure 3, we obtain an example where $\lambda \approx 28.475$ is the largest root of the polynomial $x^3 - 30x^2 + 44x - 16$, which is irreducible over $\mathbb{Q}$. A portion of the $\mathrm{SL}(2, \mathbb{R})$-orbit of this surface with its iso-Delaunay complex is shown in Figure 6. One observes immediately that the structure of the iso-Delaunay complex is much more complicated than the one in Figure 5. Indeed, we can demonstrate that this surface is not Veech.

The IDR picture itself suggests a direction on the surface to examine: note the prominent cusp near 4.8595 on the real axis. This is in fact the positive root φ of the polynomial $x^2 + x - \lambda$, which is contained in $\mathbb{Q}(\lambda)$:

$$\varphi = \frac{-1 + \sqrt{1 + 4\lambda}}{2} = -6 + \frac{15}{2}\lambda - \frac{1}{4}\lambda^2 \approx 4.8595$$

When we apply $\begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}$ to the surface, we find that the vertical direction decomposes the surface into cylinders whose moduli are:

$$m_1 = \varphi + 1, \quad m_2 = -10 + 15\lambda - \frac{1}{2}\lambda^2 = 2(\varphi + 1)$$
$$m_3 = -\frac{21}{16} + \frac{73}{32}\lambda - \frac{5}{64}\lambda^2 = \frac{1}{16}\left(5\varphi - \lambda + 9\right),$$

These moduli are not commensurable:

$$\frac{m_2}{m_1} = 2, \quad \frac{m_3}{m_1} = \frac{1}{16}(5(\varphi + 1) - \lambda), \quad \frac{m_3}{m_2} = \frac{1}{32}(5(\varphi + 1) - \lambda).$$

Therefore the Veech group of the surface is not a lattice. Further analysis may demonstrate the existence of other cylindrical decompositions, not conjugate to these in $\mathrm{PSL}(2, \mathbb{Q}(\lambda))$.

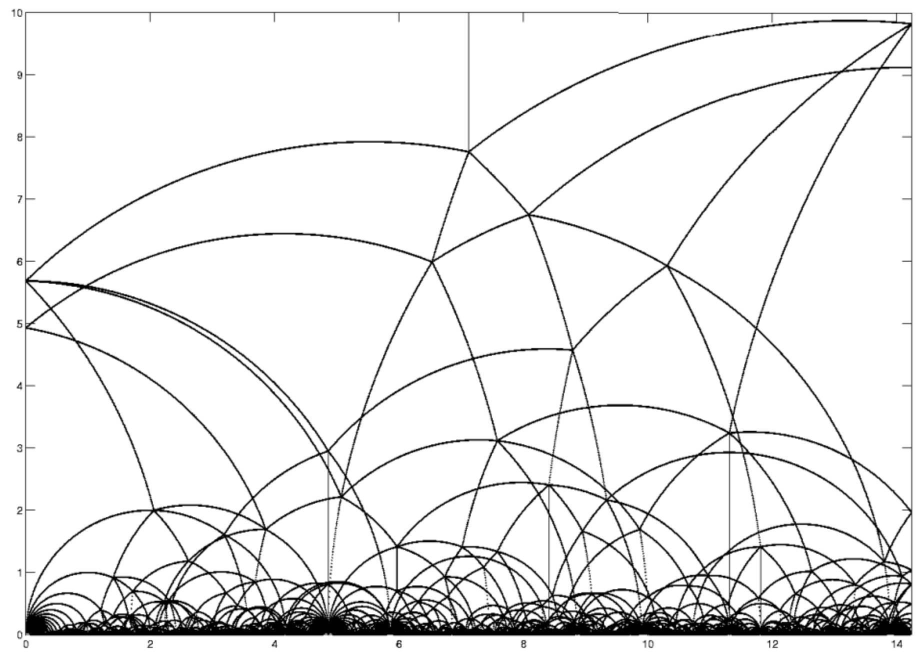

Figure 6. Some IDRs for the flat surface described in Section 4.4. Note the prominent cusp near 4.8595; this is the point φ described in the text.

5. Acknowledgements

I would like to thank my advisor, John Hubbard, for providing me with the problem of computing Veech groups, as well as John Smillie, Bill Thurston, and Cliff Earle for their great encouragement, help, and insights. Thanks to the organizers of the International Workshop on Teichmüller Theory and Moduli Problems for their invitation to speak and to the students and faculty of the Harish-Chandra Research Institute for their great welcome. Thanks also to Melanie Pivarski and my father, Jerry Bowman, who helped with programming through many conversations.

References

[AY] Pierre Arnoux and Jean-Christophe Yoccoz. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sc. Paris Sr. I Math.* **292**(1981), 75–78.

[BS] Alexander I. Bobenko and Boris A. Springborn. A discrete Laplace–Beltrami Operator for simplicial surfaces. *Discrete Comp. Geom.* **38**(2007), 740–756.

[Ca] Kariane Calta. Veech surfaces and complete periodicity in genus two. *J. Amer. Math. Soc.* **17**(2004), 871–908.

[EG] Clifford J. Earle and Frederick P. Gardiner. Teichmüller disks and Veech's $\mathcal{F}$-structures. *Contemporary Mathematics* **201**(1997), 165–189.

[FK] Ralph A. Fox and Richard B. Kerschner. Concerning the transitive properties of geodesics on a rational polyhedron. *Duke Math. J.* **2**(1936), 147–150.

[GJ] Eugene Gutkin and Chris Judge. Affine mappings of translation surfaces: Geometry and arithmetic. *Duke Math. J.* **103**(2000), 191–213.

[Ho] W. Patrick Hooper. Another Veech triangle. Preprint, 4/2007. Available online.

[HuLa] Pascal Hubert and Erwan Lanneau. Veech groups without parabolic elements. *Duke Math. J.* **133**(2006), 335–346.

[KS] Richard Kenyon and John Smillie. Billiards in rational-angled triangles. *Comment. Math. Helv.* **75**(2000), 65–108.

[K] Irwin Kra. On the Nielsen–Thurston–Bers type of some self-maps of Riemann surfaces. *Acta Math.* **146**(1981), 231–270.

[Lei] Christopher J. Leininger. On groups generated by two positive multi-twists: Teichmüller curves and Lehmer's number. *Geom. and Top.* **8**(2004), 1301–1359.

[MS] Howard Masur and John Smillie. Hausdorff dimension of sets of nonergodic measured foliations. *Ann. Math.* **134**(1991), 455–543.

[Mc1] Curtis T. McMullen. Billiards and Teichmüller curves on Hilbert modular surfaces. *J. Amer. Math. Soc.* **16**(2003), 857–885.

[Mc2] Curtis T. McMullen. Teichmüller curves in genus two: Discriminant and spin. *Math. Ann.* **333**(2005), 87–130.

[Mc3] Curtis T. McMullen. Teichmüller curves in genus two: The decagon and beyond. *J. reine angew. Math.* **582**(2005), 173–200.

[Mc4] Curtis T. McMullen. Prym varieties and Teichmüller curves. *Duke Math. J.* **133**(2006), 569–590.

[Mc5] Curtis T. McMullen. Teichmüller curves in genus two: Torsion divisors and ratios of sines. *Inv. math.* **165**(2006), 651–672.

[Ri] Igor Rivin. Euclidean structures on simplicial surfaces and hyperbolic volume. *Ann. Math.* **139**(1994), 553–580.

[Sch] Gabriela Schmithüsen. An algorithm for finding the Veech group of an origami. *Experimental Mathematics* **13**(2000), 459–472.

[Th] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. A.M.S.* **19**(1988), 417–431.

[Ve1] William A. Veech. Teichmüller curves in moduli space, Eisenstein series, and an application to triangular billiards. *Inv. Math.* **97**(1989), 553–583.

[Ve2] William A. Veech. Bicuspid F-structures and Hecke groups. Preprint.

[Vo] Yaroslav B. Vorobets. Planar structures and billiards in rational polygons: the Veech alternative. *Russ. Math. Surv* **51**(1996), 779–817.

Department of Mathematics, Cornell University, Ithaca, NY 14850