Applications of Delaunay triangulations to Teichmüller theory

Joshua P. Bowman
Cornell University

2 June 2008
Roscoff, France

ИЗВЕСТИЯ АКАДЕМИИ НАУК СССР. 1934
BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'URSS
Classe des sciences mathématiques et naturelles
Отделение математических и естественных наук

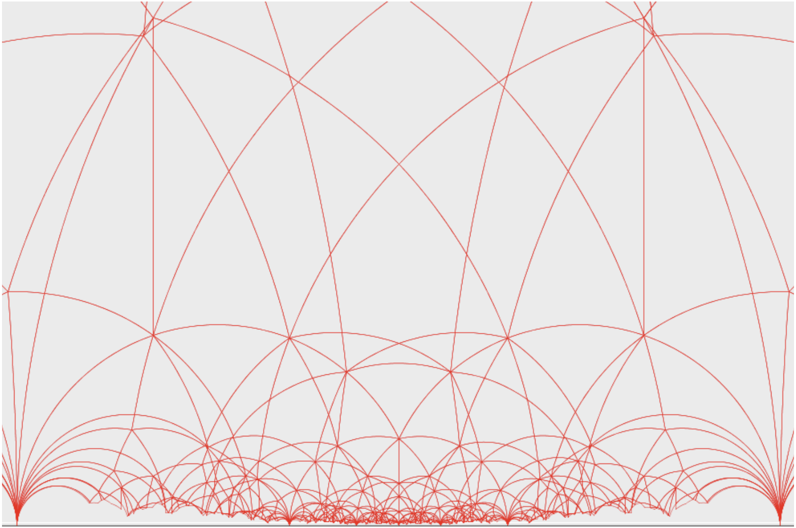
SUR LA SPHÈRE VIDE

A LA MÉMOIRE DE GEORGES VORONOÏ

Par B. DELAUNAY

(*Presenté par* I. Vinogradov, *membre de l'Académie*)

Soit donné un système quelconque de points dans l'espace à n dimensions. Je me propose de considérer une sphère se mouvant entre les points de ce système se rétrécissant et se dilatant à volonté et assujettie à la seule condition d'être «vide», c'est-à-dire de ne pas contenir dans son intérieur des points de ce système. C'est la «méthode de la sphère vide» que j'ai proposé pour la première fois dans une communication faite au Congrès de Toronto.



A tessellation of $\mathbb{H}$ determined by an unfolded billiard

Previous work on Delaunay triangulations of flat surfaces

- Thurston: “Shapes of polyhedra and triangulations of the sphere”, preprint c. 1987, published 1998
- Masur–Smillie: “Hausdorff dimension of sets of nonergodic measured foliations”, 1991
- Rivin: “Euclidean structures on simplicial surfaces and hyperbolic volume”, 1994
- Veech: “Delaunay partitions”, 1996
- Indermitte–Liebling–Troyanov–Clémençon, 2001: application to biological growth
- Bobenko–Springborn, 2007: application to discrete harmonic functions and mean curvature

Cotangents and Delaunay weights

The cotangent of the angle between an (ordered) pair of vectors $v, w \in \mathbb{R}^2$ is a rational function of the vectors' coordinates:

$$\cot \angle(v, w) = \frac{\langle v, w \rangle}{|v \; w|}$$

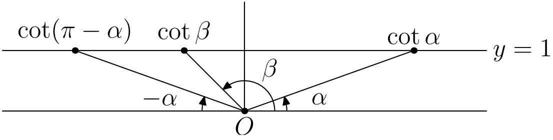
If E is an edge joining two Euclidean triangles, define the **Delaunay weight** of E to be

$$w(E) = \cot \alpha + \cot \beta,$$

where α and β are the angles opposite E.

E is **Delaunay** if $w(E) \geq 0$.

Prop. $\cot\alpha + \cot\beta \geq 0 \iff \alpha + \beta \leq \pi$
(Equality is also an iff statement.)



Cor. *E is Delaunay $\iff$ the triangles adjacent to E have empty circumcircles.*

Observe: if the triangles form a convex quadrilateral, let E' be the other diagonal. Then

$$w(E) = 0 \iff w(E') = 0 \iff E \text{ and } E' \text{ are both Delaunay.}$$

Flat surfaces

A **flat surface** is a triple (X, g, Z) such that:

- X is a surface;
- Z is a discrete subset of X;
- g is a metric on X:
  - on $X \setminus Z$, locally isometric to $\mathbb{R}^2$,
  - each pt of Z has a nbhd isometric to a Euclidean cone.

Examples:

- polyhedra in $\mathbb{R}^3$
- Riemann surface with a non-zero abelian differential
- Riemann surface with a non-zero quadratic differential
- Riemann surface with a higher-order differential

We assume that X is compact. (Could also handle “finite type” by treating punctures as points of Z.)

Triangulations

Given a flat surface (X, g, Z), a (g, Z)**-triangulation** of X is a simplicial structure on X such that:

- the vertex set is Z, and
- the edges are geodesic with respect to g.

The number of faces and edges are determined by the Euler characteristic of X and the size of Z:

$$|Z| - \#(\text{edges}) + \#(\text{faces}) = 2 - 2 \cdot \text{genus}(X)$$
$$\#(\text{edges}) = \frac{3}{2} \cdot \#(\text{faces})$$
$$\#(\text{faces}) = 4 \cdot (\text{genus}(X) - 1) + 2 \cdot |Z|$$

A (g, Z)-triangulation of X is **Delaunay** if all of its edges are Delaunay.

Two characterizing theorems

Thm. "Delaunay Lemma" for flat surfaces
(Masur–Smillie, Bobenko–Springborn)
Given a flat surface (X, g, Z)*, there exists a Delaunay* (g, Z)*-triangulation of* X*, which is unique up to exchanges of edges with Delaunay weight* 0*.*

Hence we define the **Delaunay partition** of (X, g, Z) to be the cell structure on X obtained from any Delaunay triangulation by removing edges with weight 0.

Thm. (Rivin, Indermitte et al.)
A Delaunay triangulation may be obtained from any (g, Z)*-triangulation of* X *by an "edge-flipping" algorithm.*

Curvature

The **curvature** at a point $p \in Z$ is $2\pi - \theta_p$.
($\theta_p =$ cone angle at p)

Note that the total curvature over X must be

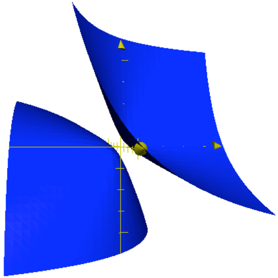
$$\sum_{p \in Z} (2\pi - \theta_p) = 4\pi \cdot (1 - \mathrm{genus}(X)),$$

following Gauss–Bonnet (or by counting triangles).

For a flat surface $(X, |\sqrt{q}|, \mathrm{zeroes}(q))$ (where X is a Riemann surface and q is a quadratic differential on X), all curvatures are multiples of π. If $q = \omega^2$ for some abelian differential ω, then all curvatures are multiples of 2π.

In any Euclidean triangle with angles $(\alpha_1, \alpha_2, \alpha_3)$, the cotangents $a_i = \cot \alpha_i$ satisfy the equation

$$a_1 a_2 + a_2 a_3 + a_3 a_1 = 1.$$



This equation defines a hyperboloid in $\mathbb{R}^3$, hence the space of marked Euclidean triangles, up to similarity, carries a canonical hyperbolic metric.

More generally, the solution set of the equation

$$\tan\left(\cot^{-1}(x_1)+\cot^{-1}(x_2)+\cdots+\cot^{-1}(x_n)\right)=0$$

is a smooth algebraic variety in $\mathbb{R}^n$ with $n-1$ components, each of which is contractible.

The kth component corresponds to n angles adding up to $k\pi$. (Therefore it is contractible, since we can follow a path to make all angles equal $k\pi/n$.)

These can be applied to give local equations for a stratum of quadratic differentials with prescribed types of curvature.

Partition of cotangent bundle to Teichmüller space

An example: Let $X = \mathbb{R}^2/\mathbb{Z}^2$ and $\omega = dz$.
Let $Z = \{p\}$, where p is the image of $\mathbb{Z}^2$ on X.

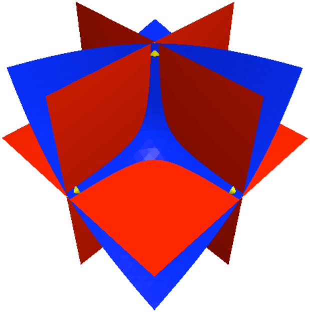
Recall that the Teichmüller space of (X, Z) is one-dimensional, i.e., it is just a copy of the hyperbolic plane.

A $(|\omega|, Z)$-triangulation of X is given by a pair of congruent triangles, with corresponding sides glued.

Mark the angles of one of the triangles by α_1, α_2, α_3;
the Delaunay weights on the three edges are then $2\cot\alpha_1$, $2\cot\alpha_2$, and $2\cot\alpha_3$.

Let (X, ω) vary under the usual $\mathrm{SL}_2(\mathbb{R})$-action, and just keep track of the Delaunay weights.

For all weights to be non-negative, we must have $\cot \alpha_i \geq 0$ for all i; that is, all the angles must be acute. This condition determines an ideal triangle in the hyperboloid.



General case: Let Y be a compact Riemann surface, and let $T^*\mathrm{Teich}(Y)$ be the cotangent bundle to the Teichmüller space of Y, whose fibers consist of quadratic differentials.
(We will ignore points of the zero section.)

Given $(X, q) \in T^*\mathrm{Teich}(Y)$, find the Delaunay partition of $(X, |\sqrt{q}|, \mathrm{zeroes}(q))$. Use the marking of Y to identify the points in $\mathrm{zeroes}(q)$ for all q in each stratum. Partition the stratum according to which edges are in the Delaunay partition.

From the uniqueness of Delaunay partitions, it follows that:

Thm. (Veech)
The above partition of $T^*\mathrm{Teich}(Y)$ is $\mathrm{Mod}(Y)$-equivariant.

Orbits of flat surfaces

We now want to consider this partition in the context of Teichmüller disks.

Let $(X, q) \in T^*\mathrm{Teich}(Y)$, and scale to assume $\mathrm{area}(q) = 1$.

The $\mathrm{PSL}_2(\mathbb{R})$-orbit of (X, q) is canonically identified with the unit tangent bundle to $\mathbb{H}$. The projection $P : \mathrm{orbit}(X, q) \to \mathbb{H}$ can be written explicitly as

$$[A] \cdot (X, q) \mapsto [A]^{-1} \cdot i,$$

where the right is defined by usual action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$.

Tessellations of $\mathbb{H}$

A **tessellation** of $\mathbb{H}$ is a set Σ of closed finite-area (not necessarily bounded) polygons such that

- $\mathbb{H} = \bigcup(\sigma \in \Sigma)$;
- for any $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

An **automorphism** of a tessellation Σ is an element $f \in \mathrm{Isom}(\mathbb{H})$ such that $f(\sigma) \in \Sigma$ for all $\sigma \in \Sigma$.

Prop. Given any tessellation Σ of $\mathbb{H}$, $\mathrm{Aut}(\Sigma)$ is a Fuchsian group.

Iso-Delaunay tessellations

Fix $(X, q) \in T^*\mathrm{Teich}(Y)$, and set $Z = \mathrm{zeroes}(q)$.

For any $(|\sqrt{q}|, Z)$-triangulation τ, define

$$\mathbb{H}_\tau = \{P([A]) \cdot i \mid \tau \text{ is Delaunay for } [A] \cdot (X, q)\}$$

and

$$\Sigma(X, q) = \{\mathbb{H}_\tau \mid \mathrm{int}(\mathbb{H}_\tau) \neq \varnothing\}.$$

Thm. (B., Veech)
$\Sigma(X, q)$ is a tessellation of $\mathbb{H}$.

$\Sigma(X, q)$ is the **iso-Delaunay tessellation** of $\mathbb{H}$.

Proof

Recall the Delaunay weight of an edge: $w(E) = \cot\alpha + \cot\beta$.

For each edge E in τ, define

$$\mathbb{H}_E = \{P([A]) \mid w(A \cdot E) \geq 0\}.$$

Observe that $\mathbb{H}_\tau = \bigcap_{E\in\tau} \mathbb{H}_E$.

Lemma. *Each $\mathbb{H}_E$ is either a Poincaré half-plane or all of $\mathbb{H}$.*

If the quadrilateral with E as its diagonal is not convex, then $w(A \cdot E) \geq 0$ for any $A \in \mathrm{SL}_2(\mathbb{R})$.

Otherwise, let v_1, v_2 and w_1, w_2 be the vectors forming the remaining sides of the triangles adjacent to E, ordered so that $|v_1\ \ v_2| > 0$ and $|w_1\ \ w_2| > 0$.

The following conditions are equivalent to $w(A \cdot E) \geq 0$:

$$\frac{\langle Av_1, Av_2 \rangle}{|v_1 \ v_2|} + \frac{\langle Aw_1, Aw_2 \rangle}{|w_1 \ w_2|} \geq 0$$

$$\langle Av_1, Av_2 \rangle |w_1 \ w_2| + |v_1 \ v_2| \langle Aw_1, Aw_2 \rangle \geq 0$$

This reduces to a quadratic inequality in the coordinates of $P([A])$, whose boundary set is a Poincaré geodesic.

$\square$ (*Lemma*)

Lemma. *There exists some triangulation* τ *such that* $\mathbb{H}_\tau$ *contains a geodesic ray limiting at* $\cot\theta \iff \theta$ *is a periodic direction on* (X, q).

Lemma. *Each* $\mathbb{H}_\tau$ *is a finite-area hyperbolic polygon.*

Every $\mathbb{H}_\tau$ is an intersection of finitely many half-planes.

Suppose some $\mathbb{H}_\tau$ has infinite area. Then $\overline{\mathbb{H}_\tau} \subset \overline{\mathbb{H}}$ must contain an interval on the boundary, hence (X, q) has uncountably many periodic directions. By a result of Vorobets, (X, q) has only countably many saddle connections (contradiction).

$\square$ (*Lemma*)

$\square$ (*Theorem*)

Conj. *For all* τ, $\mathrm{area}(\mathbb{H}_\tau) \leq \pi$.

Veech group

Let $\mathrm{Aff}(X, q)$ be the group of affine self-maps of (X, q).

Suppose $f \in \mathrm{Aff}(X, q)$. Then f must send the Delaunay cells of (X, q) to the Delaunay cells of $[\mathrm{der}(f)] \cdot (X, q)$.
Thus $\mathrm{Aff}(X, q)$ acts by automorphisms of $\Sigma(X, q)$, and so does the **Veech group** $\Gamma(X, q) := \mathrm{der}(\mathrm{Aff}(X, q)) \leq \mathrm{PSL}_2(\mathbb{R})$.

That is, $\Gamma(X, q) \leq \mathrm{Aut}(\Sigma(X, q))$.

Prop. *If (X, q) is primitive among Riemann surfaces with quadratic differentials, then $\Gamma(X, q) = \mathrm{Aut}(\Sigma(X, q))$.*

The converse is not true: the "quaternion" origami (X_W, ω_W) is not primitive, but $\Gamma(X_W, \omega_W) = \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{Aut}(\Sigma(X_W, \omega_W))$.

Other applications

- If F is any triangle in a triangulation of (X, q), then the hyperbolic metric on the space of triangles containing F coincides with the Teichmüller metric on the disk of (X, q).
- Even if θ is not a periodic direction of (X, q), it may happen that the saddle connections in the direction θ cut X into subsurfaces with boundary.

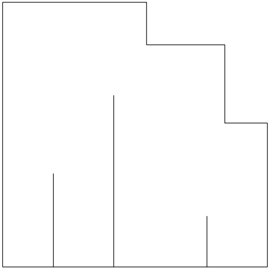
  In this case, contracting the direction θ will cause saddle connections in that direction to appear in the Delaunay triangulation. Thus the Delaunay triangulations can be used to study, e.g., minimality properties.

Genus 3 Arnoux–Yoccoz surface

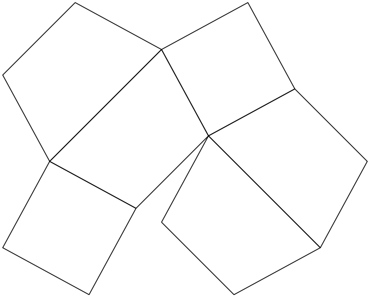
First in family of hyperelliptic surfaces, one for each genus $\gamma \geq 3$, each admitting a pseudo-Anosov diffeomorphism with an expansion constant λ whose inverse is the unique real solution to

$$x + x^2 + \cdots + x^\gamma = 1.$$

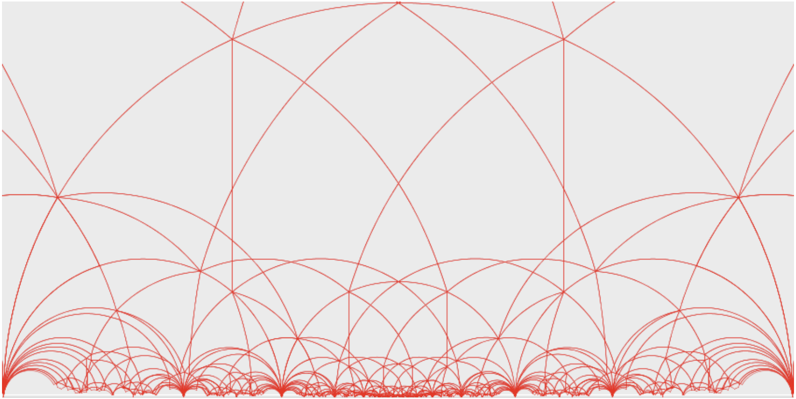
Originally constructed via interval exchange transformation.



We find another description using Delaunay cells:

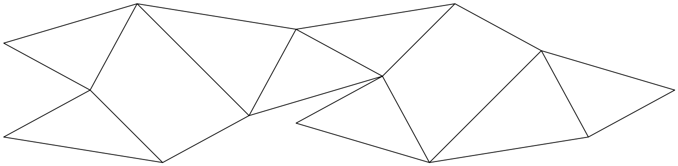


Let $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ denote this flat surface.



A portion of $\Sigma(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$

The pseudo-Anosov element is visible by scaling the horizontal direction by λ and the vertical direction by $1/\lambda$, then drawing the new Delaunay edges:



Now match trapezoids and squares between the two pictures.

Prop. (B.)
$(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$ belongs to a family of pairs $(X_{t,u}, \omega_{t,u})$ with $t > 1$ and $u > 0$, where $X_{t,u}$ has the equation

$$y^2 = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu)$$

and $\omega_{t,u} = \dfrac{x\,dx}{y}$.

For the values $(t_{\mathrm{AY}}, u_{\mathrm{AY}})$ corresponding to $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$, we find

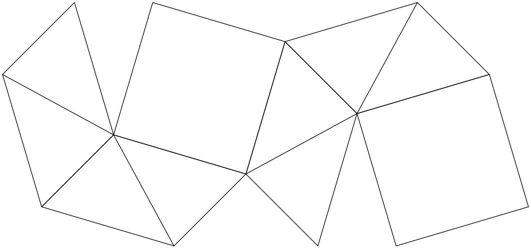
$$t_{\mathrm{AY}} \approx 1.91709843377,$$
$$u_{\mathrm{AY}} \approx 2.07067976690.$$

Conj. *t_{AY} and u_{AY} are algebraic.*

These surfaces are characterized by the following properties:

- $X_{t,u}$ is hyperelliptic ($\Upsilon =$ hyperelliptic involution)
- $\omega_{t,u}$ has two zeroes of order 2
- $X_{t,u}$ has two real structures ρ_1, ρ_2:
  - each fixes 6 Weiertrass points, including zeroes of ω_{AY}
  - exchanges 2 other Weierstrass points
  - $\rho_1 \circ \rho_2 = \rho_2 \circ \rho_1 = \Upsilon$
- $X_{t,u}$ has two other anti-holomorphic involutions σ_1, σ_2:
  - fixed-point free
  - $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \Upsilon$
- for $i, j \in \{1, 2\}$, $(\rho_i \circ \sigma_j)^2 = \Upsilon$

Scaling only the horizontal direction of $(X_{\mathrm{AY}}, \omega_{\mathrm{AY}})$, again by λ, we obtain another surface with additional real structures.



Prop. (B.)
This new surface belongs to a family of pairs $(X_{r,s}, \omega_{r,s})$ with $r > 0$ and $s \notin \mathbb{R}$, where $X_{r,s}$ has the equation

$$y^2 = x(x^2 + r)(x - s)(x - \bar{s})(x + r/s)(x + r/\bar{s})$$

and $\omega_{r,s} = \dfrac{x\,dx}{y}$.