ČECH COHOMOLOGY AND DE RHAM'S THEOREM: CLASS LECTURE FOR MATH 758, SPRING 2008

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Much of these notes is almost verbatim from notes I took during John Hubbard's course on complex manifolds, hence there is a lot of overlap with Appendix A7 in [3]. I also used Chern's text [1] and Appendix D in Conlon's book [2] as references. When approaching Weil's proof of de Rham's Theorem, one can either:

- develop a fair amount of theory regarding sheaves and cohomology, then attack de Rham's Theorem as a special case; or
- build up a proof of de Rham's Theorem from scratch, using more elementary but possibly more notationally heavy arguments.

Roughly, Hubbard does the first while Conlon does the second. I'll try to work somewhere in the middle, developing just enough sheaf theory to handle this particular application.

Throughout, we will assume that M is a smooth paracompact manifold (recall: paracompact means every open cover has a locally finite refinement; we'll review refinements).

1. Abstract simplicial complexes and Čech covers

Definition 1.1. Let $\mathcal{I}$ be a totally ordered set. An *abstract simplicial complex* modeled on $\mathcal{I}$ is a collection Δ of finite subsets of $\mathcal{I}$, closed under taking subsets. Each $F \in \Delta$ is called a *face* of Δ; F is called a *k-face* if $|F| = k + 1$.

Definition 1.2. If $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ is a locally finite open cover of M, then there is naturally associated an abstract simplicial complex $\mathcal{N}(\mathcal{U})$, called the *nerve* of $\mathcal{U}$. Its faces are the collections of U_is with non-empty intersection. That is, $\{U_{i_1}, \dots, U_{i_k}\}$ is a face of $\mathcal{N}(\mathcal{U})$ if $U_{i_1} \cap \cdots \cap U_{i_k} \neq \varnothing$.

Example 1.3. Cover S^2 by six open hemispheres, say, north, south, east, west, front, and back ($\mathcal{U} = \{U_N, U_S, U_E, U_W, U_F, U_B\}$). Any collection of these open sets containing one of the pairs $\{U_N, U_S\}$, $\{U_E, U_W\}$, or $\{U_F, U_B\}$ has empty intersection. The remaining subsets of $\mathcal{U}$ determine an abstract simplicial complex that can be realized as an octahedron.

Definition 1.4. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of M. Then $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if every element of $\mathcal{V}$ is contained in an element of $\mathcal{U}$. Note that some elements of $\mathcal{V}$ may lie in multiple elements of $\mathcal{U}$: a choice of inclusions $\tau : \mathcal{V} \to \mathcal{U}$ is called a *refining map*.

The idea is that by taking a refinement, we're looking more locally on M, hence getting more of its structure. This is the apparent paradox that we'll be building on throughout this lecture: we can get global information (via cohomology) from sufficient amounts of local data (which is what sheaves encode). In fact, Čech showed the following.

Theorem 1.5. *Let $\mathcal{U}$ be a locally finite open cover of M. If all intersections of sets in $\mathcal{U}$ (including all elements of $\mathcal{U}$) are contractible, then $\mathcal{N}(\mathcal{U})$ is homotopy equivalent to M.*

Such a cover, with all intersections contractible, is called a *Čech cover* of M. Not all topological spaces admit Čech covers, but all manifolds do. We won't prove this result, but we give it as an example of the kind of intuition to use.

Exercise 1.

- Find a Čech cover of S^2 whose nerve can be realized as a tetrahedron.
- Find a Čech cover for the compact orientable surface of genus $g \geq 1$. (*Hint:* use the standard presentation of this surface as a quotient of the $4g$-gon.)
- Find a Čech cover for the n-dimensional torus $(S^1)^n = \mathbb{R}^n/\mathbb{Z}^n$.
- Prove Theorem 1.5. (I haven't tried this, and have no idea how hard it is.)

2. Sheaves and maps of sheaves

We already know, from courses on manifolds and algebraic topology, about two kinds of cohomology on M: de Rham and singular. Soon we'll introduce Čech cohomology on M, with various "coefficients". The content of de Rham's Theorem is that, with the appropriate choice of sheaf coefficients, these three cohomologies coincide, i.e.,

$$H^\bullet_{\mathrm{dR}}(M) \cong H^\bullet_{\mathrm{sing}}(M;\mathbb{R}) \cong \check{H}^\bullet(M, \mathbb{R}_M).$$

(The same is true with complex-valued differential forms and complex coefficients, but for concreteness we'll stick with real coefficients.) First, we need to introduce the notion of sheaves. This topic will connect well with the open covers we studied in the previous section.

Definition 2.1. A *sheaf* $\mathcal{F}$ (*of abelian groups*) on M is an assignment, to each open set $U \subset M$, of an abelian group $\mathcal{F}(U)$, that is "local". This means precisely that we need one additional kind of structure and two additional axioms. If $V \subset U$ are both open, then we should be able to "restrict" data from U to V, i.e., we have a homomorphism $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$. Of course, if $W \subset V \subset U$, then $\rho^U_W = \rho^V_W \circ \rho^U_V$ ($\mathcal{F}$ is a *functor*, if you like that language). Here are the axioms we need, along with the ideas they capture:

(1) If $a, b \in \mathcal{F}(U)$ are locally the same, then they are globally the same; that is, if $\mathcal{U}$ is an open cover of U, and $\rho^U_V(a) = \rho^U_V(b)$ for all $V \in \mathcal{U}$, then $a = b$ in $\mathcal{F}(U)$.

(2) If something can be defined locally in a way that coincides on overlapping domains, then it can be defined globally; that is, if $\mathcal{U}$ is an open cover of $U \subset M$ and $a_V \in \mathcal{F}(V)$ are chosen for all $V \in \mathcal{U}$ in such a way that $\rho^V_{V\cap W}(a_V) = \rho^W_{V\cap W}(a_W)$ whenever $V, W \in \mathcal{U}$, then there exists $a \in \mathcal{F}(U)$ such that all $a_V = \rho^U_V(a)$.

Elements of $\mathcal{F}(U)$ are called *sections over* U; elements of $\mathcal{F}(M)$ are called *global sections.*

Example 2.2. Here are the examples that will be most important to us:

- C^∞_M assigns to each open $U \subset M$ the group $C^\infty_M(U)$ of smooth functions on U.
- Ω^k_M assigns to U the group $\Omega^k_M(U)$ of smooth k-forms on U.
- $\mathbb{R}_M$, $\mathbb{C}_M$, $\mathbb{Z}_M$, etc., what are often called the *constant* sheaves, but be careful—"constant" is *not a local property!* Hence the elements of $\mathbb{R}_M(U)$, e.g., are *locally constant* $\mathbb{R}$-valued functions; they may take different values on different components.

Definition 2.3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on M. A *map of sheaves* (or *sheaf morphism*) $f : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphisms $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ that commute with restrictions; that is, whenever $V \subset U \subset M$, the following square commutes:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f_U"] \arrow[d, "\rho^U_V"'] & \mathcal{G}(U) \arrow[d, "\rho^U_V"] \\ \mathcal{F}(V) \arrow[r, "f_V"'] & \mathcal{G}(V) \end{tikzcd}$$

Definition 2.4. A sequence $\mathcal{E} \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{G}$ of sheaf maps is *exact* if $g \circ f$ is the zero map and, locally, $\ker(g) = \operatorname{im}(f)$: that is, if $g(a) = 0$ with $a \in \mathcal{F}(U)$, then every $x \in U$ has a neighborhood $V \subset U$ such that $\rho^U_V(a) = f(b)$ for some $b \in \mathcal{E}(V)$.

Example 2.5. Let $C^\infty_{M,\mathbb{C}}$ be the sheaf of smooth $\mathbb{C}$-valued functions on M, $(C^\infty_{M,\mathbb{C}})^*$ the sheaf of nowhere-vanishing smooth $\mathbb{C}$-valued functions, and $\mathbb{Z}_M$ the constant integer sheaf. The *exponential sequence*

$$0 \longrightarrow \mathbb{Z}_M \xrightarrow{\text{incl.}} C^\infty_{M,\mathbb{C}} \xrightarrow{\exp(2\pi i\cdot)} (C^\infty_{M,\mathbb{C}})^* \longrightarrow 1$$

is an exact sequence of sheaves. That is, every nowhere-vanishing function *locally* has a logarithm, well-defined up to a term in $2\pi i\mathbb{Z}$. (The function z on $\mathbb{C} - \{0\}$ shows that finding a global logarithm may not be possible; $\log z$ is a "multi-valued" function.) Note that the group operation in the first three terms is addition, while in the last two terms it is multiplication; in particular, the first and last terms are both the trivial sheaf on M.

Example 2.6. Because the exterior derivative operator can be computed locally, it induces sheaf maps $d : \Omega^k_M \to \Omega^{k+1}_M$. From $d^2 = 0$ we have the following cochain complex of sheaves:

$$0 \longrightarrow \mathbb{R}_M \xrightarrow{\text{incl.}} C^\infty_M = \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \cdots.$$

But more is true: *Poincaré's Lemma* (see any textbook on manifolds and differential forms; Chapter 10 of [4] has a good presentation) implies that closed forms are locally exact, so the above sequence is an *exact* sequence of sheaves. One further property of this sequence, the *fineness* of the sheaves involved (see §4), will clinch de Rham's theorem for us.

Exercise 2. Look up the proof of Poincaré's Lemma.

3. Čech cohomology

The purpose of sheaves is to localize data. The purpose of cohomology is to assemble this data and extract global information. The first step is to compute cohomology with respect to a fixed open cover $\mathcal{U}$ of M (which, as we saw earlier, provides an *approximation* of M that can be improved by taking refinements, which will be the second step).

Definition 3.1. Let $\mathcal{F}$ be a sheaf on M, and let $\mathcal{U} = \{U_i\}_{i\in\mathcal{I}}$ be a locally finite open cover of M. A *Čech k-cochain* is a function α on the k-faces of $\mathcal{N}(\mathcal{U})$ such that the value on $\{U_{i_0}, \dots, U_{i_k}\} \subset \mathcal{U}$ lies in $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_k})$. Thus the group of k-cochains is

$$\check{C}^k(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0<\cdots<i_k} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_k}).$$

The *coboundary operator* d sends k-cochains to $(k+1)$-cochains by the formula

$$d\alpha\left(U_{i_0}\cap\cdots\cap U_{i_k}\cap U_{i_{k+1}}\right)=\sum_{j=0}^{k+1}(-1)^j\alpha\left(U_{i_0}\cap\cdots\cap\widehat{U_{i_j}}\cap\cdots\cap U_{i_{k+1}}\right),$$

where the hat indicates omission (this is where we use the total ordering on $\mathcal{I}$). Strictly speaking, each term should also include the restriction of α, i.e.,

$$\rho^{U_{i_0}\cap\cdots\cap\widehat{U_{i_j}}\cap\cdots\cap U_{i_k}}_{U_{i_0}\cap\cdots\cap U_{i_k}},$$

but this merely adds to an already cumbersome expression. We will henceforth suppress these restriction maps and assume that their presence is understood, unless there is clarity to be gained by including them.

The image of d in $\check{C}^k(\mathcal{U},\mathcal{F})$ is the group of *k-coboundaries*, and the kernel of d is the group of *k-cocycles*:

$$\begin{aligned}\check{B}^k(\mathcal{U},\mathcal{F})&=\operatorname{im}\left(d:\check{C}^{k-1}(\mathcal{U},\mathcal{F})\to\check{C}^k(\mathcal{U},\mathcal{F})\right),\\ \check{Z}^k(\mathcal{U},\mathcal{F})&=\ker\left(d:\check{C}^{k}(\mathcal{U},\mathcal{F})\to\check{C}^{k+1}(\mathcal{U},\mathcal{F})\right).\end{aligned}$$

And finally, the *kth Čech cohomology group* of $\mathcal{F}$ with respect to the cover $\mathcal{U}$ is the quotient

$$\check{H}^k(\mathcal{U},\mathcal{F})=\check{Z}^k(\mathcal{U},\mathcal{F})/\check{B}^k(\mathcal{U},\mathcal{F}).$$

For this last definition to make sense, we must have $\check{B}^k(\mathcal{U},\mathcal{F})\subseteq\check{Z}^k(\mathcal{U},\mathcal{F})$, which is the same as $d^2=0$. This property is checked as usual.

Exercise 3.

- Compute the cohomology of the constant sheaf $\mathbb{R}_{S^1}$ with respect to a Čech cover of the circle. (*Hint:* you can make the nerve of the cover just a triangle.)
- Compute the cohomology of the constant sheaf $\mathbb{R}_{S^2}$ on S^2 with respect to the open cover whose nerve is a tetrahedron; you should only get non-zero cohomology groups for $k=0$ and $k=2$.

It will follow from the rest of the lecture that these cohomology groups are the same as the de Rham cohomology, but they make good concrete practice cases.

Next, we need to be able to relate the cohomology groups with respect to different open covers $\mathcal{U}$ and $\mathcal{V}$; any two covers have a common refinement, so we may assume without loss of generality that $\mathcal{V}$ refines $\mathcal{U}$. Given a choice of refining map $\tau:\mathcal{V}\to\mathcal{U}$, we can move from cochains with respect to $\mathcal{U}$ to cochains with respect to $\mathcal{V}$, by the adjoint

$$\tau^*:(\alpha:U\mapsto a\in\mathcal{F}(U))\mapsto\left(\tau^*\alpha:V\mapsto\rho^{\tau(V)}_V\alpha(\tau(V))\right).$$

But how much does the resulting homomorphism depend on τ?

Lemma 3.2. *If τ_1 and τ_2 are both refining maps $\mathcal{V}\to\mathcal{U}$, then they induce the same map on cohomology $\tau_1^*=\tau_2^*:\check{H}^\bullet(\mathcal{U},\mathcal{F})\to\check{H}^\bullet(\mathcal{V},\mathcal{F})$.*

Proof. We recall that it is sufficient to find a homotopy of chain complexes, i.e., a collection of homomorphisms $h : \check{C}^{k+1}(\mathcal{U}, \mathcal{F}) \to \check{C}^k(\mathcal{V}, \mathcal{F})$ in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \check{C}^{k-1}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] \arrow[dl, "h"'] \arrow[d, shift right, "\tau_1^*"'] \arrow[d, shift left, "\tau_2^*"] & \check{C}^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & \check{C}^{k-1}(\mathcal{V},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k}(\mathcal{V},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k+1}(\mathcal{V},\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

such that $dh + hd = \tau_1^* - \tau_2^*$, because this shows that the difference of τ_1^* and τ_2^* on a cocycle (a representative of a cohomology class) is a coboundary (which is zero in cohomology). Set

$$h(\alpha)\big(V_{i_0} \cap \cdots \cap V_{i_k}\big) = \sum_{j=0}^{k} \alpha\big(\tau_1(V_{i_0}) \cap \cdots \cap \tau_1(V_{i_j}) \cap \tau_2(V_{i_j}) \cap \cdots \cap \tau_2(V_{i_k})\big).$$

This h works, by a standard telescoping argument. ☐

This lemma allows us to define, unambiguously, the Čech cohomology groups of a sheaf.

Definition 3.3. Let $\mathcal{F}$ be a sheaf on M. The *kth Čech cohomology group* of $\mathcal{F}$ is

$$\check{H}^k(M, \mathcal{F}) = \varinjlim_{\mathcal{U},\text{ ordered by refinement}} \check{H}^k(\mathcal{U}, \mathcal{F}).$$

The direct limit means, roughly, that for something to appear in the final cohomology, it only needs to appear for "sufficiently refined covers." As previously observed, Čech covers exist for manifolds, and so we'll be able later to drop any concerns about the direct limit by applying a theorem of Leray.

Here we will wave our hands a bit at the question of equivalence between $\check{H}^k(M, \mathbb{R}_M)$ and $H^k_{\text{sing}}(M; \mathbb{R})$, then not concern ourselves further with it: a singular cochain $\alpha \in C^k_{\text{sing}}(M, \mathbb{R})$ is a function on maps from a k-simplex into M. A Čech k-cochain is a function on the k-faces associated to an open cover $\mathcal{U}$ of M. In fact, the nerve $\mathcal{N}(\mathcal{U})$ can be realized, at least locally, as a cellular decomposition of M when $\mathcal{U}$ is sufficiently refined, and by taking further refinements we can approximate any map from a k-simplex into M by a face of $\mathcal{N}(\mathcal{U})$. So the singular cochain α can be approximated by a Čech cochain. A homotopy argument shows that this induces maps on the cohomology groups, and taking the direct limit shows that, for sufficiently refined covers, the maps are isomorphisms.

4. Fine resolutions

Definition 4.1. A sheaf $\mathcal{F}$ on M is *fine* if its sections can be glued by partitions of unity, i.e., for every open $U \subset M$ and for every locally finite cover $\mathcal{V}$ of U, there exist *extension maps* $\varphi_U^V : \mathcal{F}(V) \to \mathcal{F}(U)$ such that

$$\sum_{V \in \mathcal{V}} \varphi_U^V \circ \rho_V^U = \mathrm{id}_{\mathcal{F}(U)}.$$

In our context of manifolds, the most natural kind of partition of unity in the sense of the foregoing definition is one given by a partition of unity in the manifold-theoretic sense, i.e., a collection of smooth functions $\{\varphi_U^V\}_{V \in \mathcal{V}}$, so that φ_U^V is only non-zero on a relatively compact subset of V and $\sum \varphi_U^V = 1$ at every $x \in U$. Thus we can get global sections from local sections simply by multiplying by these functions and summing. For the sum to be finite at every point, we need a locally finite cover, however; this is where we use paracompactness.

Example 4.2.

- As previously observed, all Ω_M^k are fine sheaves. C_M^∞ is a special case.
- Constant sheaves are *not* fine; trying to multiply by some arbitrary smooth function defined locally will most of the time kill the property of being locally constant.

The extreme flexibility of fine sheaves means that the problem of solving their cohomology is trivial:

Lemma 4.3. *If $\mathcal{F}$ is a fine sheaf on M, then $\check{H}^k(M, \mathcal{F}) = 0$ for all $k > 0$.*

Proof. It suffices to consider a locally finite cover $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ on M and a partition of unity $\{\varphi_M^U\}_{U \in \mathcal{U}}$ subordinate to $\mathcal{U}$, because every open cover has a locally finite refinement. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \check{C}^{k-1}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] \arrow[d, equal] & \check{C}^{k}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] \arrow[d, equal] \arrow[dl, "h"'] & \check{C}^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, equal] \arrow[dl, "h"'] & \cdots \\
\cdots \arrow[r] & \check{C}^{k-1}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k}(\mathcal{U},\mathcal{F}) \arrow[r, "d"] & \check{C}^{k+1}(\mathcal{U},\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

with $h : \check{C}^{k+1}(\mathcal{U}, \mathcal{F}) \to \check{C}^k(\mathcal{U}, \mathcal{F})$ defined by

$$h(\alpha)\left(U_{i_0} \cap \cdots \cap U_{i_k}\right) = \sum_{U \in \mathcal{U}} \left(\rho^U_{U_{i_0} \cap \cdots \cap U_{i_k}} \circ \varphi_U^{U \cap U_{i_0} \cap \cdots \cap U_{i_k}}\right) \left(\alpha\left(U \cap U_{i_0} \cap \cdots \cap U_{i_k}\right)\right).$$

(The φs are key; the ρs are simply there to restrict to the appropriate open subset corresponding to the face $\{U_{i_0}, \ldots, U_{i_k}\}$.) We must take care to add in an appropriate sign when we have to shuffle U among the U_{i_j}s according to the order of $\mathcal{I}$. Then $dh + hd = \mathrm{id} = \mathrm{id} - 0$, which means we have given a homotopy from the identity map to the zero map. This proves the result. ☐

At some point in the course of proving de Rham's theorem, one has to set up a double complex and show a canonical correspondence between the top row and the leftmost column. The advantage of the sheaf theory we've developed is that we can do this once and for all in a somewhat broader context than just "de Rham cohomology and sheaf cohomology with constant coefficients"; e.g., Dolbeault's Theorem will also be covered by this argument. The point is, even though fine sheaves have no cohomology, an exact sequence of such sheaves leads to a *complex* of their global sections (no restricting to smaller sets allowed!) which can be used to compute the cohomology of another sheaf.

Definition 4.4. Let $\mathcal{F}$ be a sheaf on M. A *fine resolution* of $\mathcal{F}$ is an exact sequence of sheaf maps

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots \tag{1}$$

where every $\mathcal{F}^k$ is fine.

Theorem 4.5. *Let $\mathcal{F}$ be a sheaf on M, and let* (1) *be a fine resolution of $\mathcal{F}$. Suppose $\mathcal{U}$ is an open cover of M such that the sequence of homomorphisms*

$$\mathcal{F}^{j-1}(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^{j}(U_{i_0} \cap \cdots \cap U_{i_k}) \to \mathcal{F}^{j+1}(U_{i_0} \cap \cdots \cap U_{i_k})$$

is exact for every face $\{U_{i_0}, \ldots, U_{i_k}\}$ of $\mathcal{N}(\mathcal{U})$. Then

$$\check{H}^k(\mathcal{U}, \mathcal{F}) \cong \frac{\ker\left(d : \mathcal{F}^k(M) \to \mathcal{F}^{k+1}(M)\right)}{\operatorname{im}\left(d : \mathcal{F}^{k-1}(M) \to \mathcal{F}^k(M)\right)}$$

canonically.

Proof. Set up the commuting double complex below; maps to the right between groups of cochains are coboundary maps, while maps down are induced by the sheaf maps of the resolution. The first map in each row below the first is given by restriction to elements of the open cover.

$$\begin{tikzcd}
 & \check{C}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \cdots \\
\mathcal{F}^0(M) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^0) \arrow[r] \arrow[d] & \cdots \\
\mathcal{F}^1(M) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^1) \arrow[r] \arrow[d] & \cdots \\
\mathcal{F}^2(M) \arrow[r] \arrow[d] & \check{C}^0(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \check{C}^1(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \check{C}^2(\mathcal{U},\mathcal{F}^2) \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Our hypothesis on $\mathcal{U}$ states that the columns from the second column over are exact, because they are exact on each intersection. The rows from the second row down are exact because the $\mathcal{F}^k$ are fine.

We commence the diagram chase: fortunately we only have to move on broken diagonals, the shortest possible paths between the terms we want to connect. Take a Čech cocycle $\alpha \in \check{Z}^k(\mathcal{U}, \mathcal{F}) \subset \check{C}^k(\mathcal{U}, \mathcal{F})$. Then its image to the right is zero, so it moves down to an element $\beta_0 \in \check{C}^k(\mathcal{U}, \mathcal{F}^0)$ that is also a cocycle. But the rows below the first are exact, so β_0 is a coboundary, i.e., the image of some $\alpha_0 \in \check{C}^{k-1}(\mathcal{U}, \mathcal{F}^0)$. Because the columns after the first are exact and β_0 came from an element one level up, it descends to 0 and thus α_0 descends to a cocycle β_1 in $\check{C}^{k-1}(\mathcal{U}, \mathcal{F}^1)$. Continue by finding $\alpha_1 \in \check{C}^{k-2}(\mathcal{U}, \mathcal{F}^1)$, β_2, etc., until you arrive at $\alpha_k \in \mathcal{F}^k(M)$. The cohomology class $[\alpha_k]$ in the complex of global sections is independent of the choices α, α_1, …, α_k.

The situation is symmetric, so we have an analogous map from cohomology of the complex of global sections to the cohomology of the Čech complex. These maps are inverses. $\square$

Exercise 4. Check the assertions that conclude the last two paragraphs above.

5. Final results

Theorem 4.5 can be strengthened in a couple of ways. First we observe that covers of the type described in the statement of the theorem are cofinal among covers; that is, any refinement of $\mathcal{U}$ can be refined again, if necessary, to a new cover that also satisfies the conditions of the theorem. Because the maps involved are canonical, the groups do not depend either on the particular open cover $\mathcal{U}$ nor on the particular fine resolution of $\mathcal{F}$. Thus we have the following corollaries:

Corollary 5.1 (Leray's Theorem)**.** *Let $\mathcal{F}$ be a sheaf on M, and suppose $\mathcal{U}$ is an open cover of M such that $\check{H}^k(U_{i_0}\cap\cdots\cap U_{i_j},\mathcal{F})=0$ for all finite intersections $U_{i_0}\cap\cdots\cap U_{i_j}$ of elements in $\mathcal{U}$. Then the canonical map*

$$\check{H}^k(\mathcal{U},\mathcal{F})\to\check{H}^k(M,\mathcal{F})$$

is an isomorphism for all k.

Proof. Choose a fine resolution of $\mathcal{F}$. By Theorem 4.5, the cohomology of any such cover coincides with the cohomology of the global sections in the fine resolution of $\mathcal{F}$, which implies that they are all identical. Hence the direct limit $\check{H}^k(M,\mathcal{F})$ is isomorphic to any of them. □

Corollary 5.2. *Let $\mathcal{F}$ be a sheaf on M, and let* (1) *be any fine resolution of $\mathcal{F}$. Then*

$$\check{H}^k(M,\mathcal{F})\cong\frac{\ker\big(d:\mathcal{F}^k(M)\to\mathcal{F}^{k+1}(M)\big)}{\operatorname{im}\big(d:\mathcal{F}^{k-1}(M)\to\mathcal{F}^{k}(M)\big)}\qquad\text{for all } k.$$

Proof. Recall that because M is a manifold, it has a Čech cover, which satisfies the conditions of the previous corollary. (The result is true for any paracompact space, and in general only needs an argument about independence from the covers of the given type.) □

These are highly convenient tools for computing the cohomology of a sheaf. Our goal of proving de Rham's Theorem, which we accomplish next, will turn this technique around, giving us a way to express de Rham cohomology, defined by quotients of infinite-dimensional groups, in terms of the structurally simpler Čech cohomology with real coefficients.

Theorem 5.3 (de Rham's Theorem)**.** *Let M be a paracompact C^∞-manifold. Then*

$$\check{H}^k(M,\mathbb{R}_M)\cong H^k_{\mathrm{dR}}(M)\qquad\text{for all } k.$$

Proof à la Weil. By definition, $H^k_{\mathrm{dR}}(M)$ is the quotient $\ker(d)/\operatorname{im}(d)$ in $\Omega^k(M)$. We have already seen that the sequence

$$0\longrightarrow\mathbb{R}_M\xrightarrow{\text{incl.}}\Omega^0_M\xrightarrow{d}\Omega^1_M\xrightarrow{d}\cdots\xrightarrow{d}\Omega^{\dim M}_M\longrightarrow 0$$

is a fine resolution of $\mathbb{R}_M$. Therefore the result follows by Corollary 5.2. □

To conclude, we'll prove Dolbeault's Theorem, modulo the proof of the (analytic) analogue to Poincaré's Lemma. First, we fix some notation: if M is a complex manifold, then $\Omega^{p,q}_M$ is the sheaf of smooth (p,q)-forms on M, and Ω^p_M is the sheaf of $(p,0)$-forms on M that are $\overline{\partial}$-closed (which is the same as having holomorphic coefficients in any chart). The Dolbeault operator $\overline{\partial}$ maps $\Omega^{p,q}_M$ to $\Omega^{p,q+1}_M$ and satisfies $\overline{\partial}^2=0$, giving rise to the Dolbeault complex, whose cohomology groups are denoted $H^{p,q}_{\mathrm{Dol}}(M)$.

Theorem 5.4 (Dolbeault's Theorem)**.** *Let M be a complex manifold. Then*

$$\check{H}^q(M, \Omega^p_M) \cong H^{p,q}_{\mathrm{Dol}}(M) \qquad \textit{for all } (p,q).$$

Proof. The Dolbeault–Grothendieck Lemma states that every $\overline{\partial}$-closed form in $\Omega^{p,q}_M(U)$ is locally $\overline{\partial}$-exact; this is an analytic result about complex manifolds that is somewhat harder to prove than the analogous Poincaré Lemma of differential topology. Apart from this particular difference, the form of the proof here exactly mirrors that of de Rham's Theorem.

By definition, $H^{p,q}_{\mathrm{Dol}}(M)$ is the quotient $\ker(\overline{\partial})/\mathrm{im}(\overline{\partial})$ in $\Omega^{p,q}(M)$. The kernel of $\overline{\partial}$ in $\Omega^{p,0}_M(U)$ is precisely Ω^p_M. Therefore the Dolbeault–Grothendieck Lemma implies that the sequence

$$0 \longrightarrow \Omega^p_M \xrightarrow{\text{incl.}} \Omega^{p,0}_M \xrightarrow{\overline{\partial}} \Omega^{p,1}_M \xrightarrow{\overline{\partial}} \Omega^{p,2}_M \xrightarrow{\overline{\partial}} \cdots$$

is a fine resolution of Ω^p_M, and the result follows by Corollary 5.2. $\square$

In particular, $\Omega^0_M = \mathcal{O}_M$, the sheaf of holomorphic functions, and so we get a relationship between the cohomology of the sheaf $\mathcal{O}_M$ and the anti-holomorphic forms on M. Because $\mathcal{O}_M$ also fits into the exact (exponential) sequence $0 \to \mathbb{Z}_M \to \mathcal{O}_M \to \mathcal{O}^*_M \to 1$, the interplay of these different contexts can yield a lot of information about M. For more results on sheaf cohomology and applications to the study of complex manifolds, see [1] and [3].

References

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