INVERSE FUNCTION THEOREM

Before we recall the exact statement of the Inverse Function Theorem, let's think about what we'd *like* for it to say. We've been talking about solving equations. Naïvely, given a function $f : \mathbb{R}^n \to \mathbb{R}^n$ and a value $\mathbf{b}$ in the range, we simply want to solve $f(\mathbf{x}) = \mathbf{b}$. Newton's method gives us a way to do this. But in the linear case, we have a much stronger situation: when f is invertible, we just have to find f^{-1} to get solutions to *all* equations $f(\mathbf{x}) = \mathbf{b}$. By examples, we know that it's generally hopeless to expect this to happen for non-linear functions. But if we know a solution exists for *some* $\mathbf{b}_0$, we might hope that solutions also exist *near* $\mathbf{b}_0$. The Inverse Function Theorem tells us that this hope is (often) justified, and that the solutions depend *differentiably* on $\mathbf{b}$ near $\mathbf{b}_0$.

Theorem 1 (Inverse Function Theorem)**.** *Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function on a neighborhood of $\mathbf{x}_0$. Suppose that $Df(\mathbf{x}_0)$ is invertible. Then f has a local C^1 inverse on a neighborhood of $\mathbf{x}_0$, i.e., $f(\mathbf{x}) = \mathbf{b}$ has a solution for $\mathbf{b}$ in some ball around $f(\mathbf{x}_0)$.*

The concept of "locally invertible" may be difficult. First, you should realize that a property being "local" on a set simply means that every point in that set is contained in a neighborhood on which the property holds. (As opposed to "pointwise", which only has to hold at each point: continuity is an example of a pointwise property.) Some examples may help explain why local invertibility is such an important concept.

Example 1 (A function that is everywhere locally invertible, but does not have a global inverse)**.** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x^2 - 1 & \text{if } x < 0. \end{cases}$$

At every point of $\mathbb{R}$, f has a local inverse. For $x > 0$, it is $y \mapsto \sqrt{y}$; for $x < 0$, it is $y \mapsto -\sqrt{y+1}$. There is also an inverse on the interval $(-1, 1)$, given by

$$y \mapsto \begin{cases} -\sqrt{y+1} & \text{if } y \in (-1, 0) \\ \sqrt{y} & \text{if } y \in [0, 1). \end{cases}$$

However, f has no global inverse, because it is not one-to-one.

Example 2 (A differentiable example)**.** Consider the exponential function $\exp : \mathbb{C} \to \mathbb{C}$. As you saw in an earlier homework, the derivative of $\exp$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$ at $z_0 = x_0 + iy_0$ is

$$\left[D \exp \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right] = e^{x_0} \begin{bmatrix} \cos y_0 & -\sin y_0 \\ \sin y_0 & \cos y_0 \end{bmatrix},$$

This matrix is *always* invertible—its determinant is $e^{2x_0} \neq 0$. Thus the Inverse Function Theorem guarantees a local inverse of $\exp$ at each point of $\mathbb{C}$, and the inverse will even be differentiable! (Aside: such a local inverse for $\exp$ is called, naturally, a *logarithm*. But as we'll see in a moment, logarithms are far from unique.)

However, $\exp$ is not one-to-one on $\mathbb{C}$: if $z_1 = x + iy_1$ and $z_2 = x + iy_2$, where y_1 and y_2 differ by a multiple of 2π, then $e^{z_1} = e^{z_2}$; $\exp$ is *periodic* in the imaginary direction. (Wow!) Any point of $\mathbb{C}$ is contained in a ball of radius π on which $\exp$ is invertible. (In your spare time, you might think about what the image of this ball would look like.)