Research statement

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1 Introduction and background

My research falls broadly within Teichmüller theory, which uses tools of geometry, topology, and analysis to study moduli spaces of flat surfaces and their underlying Riemann surfaces. I have been developing and applying tools that can be used to solve questions about the internal structure and asymptotic behavior of families of such surfaces. For the future, I plan a series of investigations that will use these tools in new ways and extend them to a larger class of affine structures.

A *Riemann surface* is a one-dimensional complex manifold—that is, a smooth topological surface with a smooth choice of multiplication by i on each tangent space, called a *complex structure* on the surface. The *moduli space* $\mathcal{M}_g$ of genus g Riemann surfaces is an algebraic variety whose points are isomorphism classes of Riemann surfaces. A refinement of the notion of isomorphism leads to the *Teichmüller space* $\mathrm{Teich}(S)$, where S is a fixed topological surface of genus g. $\mathrm{Teich}(S)$ is a complex manifold whose points are again Riemann surfaces with common underlying topological space S, but only up to isomorphisms that are homotopic to the identity map on S. The group of homeomorphisms of S taken up to homotopy is called the *mapping class group* of S, and it acts naturally and discretely on $\mathrm{Teich}(S)$. $\mathcal{M}_g$ is the quotient of $\mathrm{Teich}(S)$ by the action of the mapping class group.

If X is a point in $\mathrm{Teich}(S)$ and q is a non-zero quadratic differential on X (q has the local form $f(z)\,dz^2$), then the pair (X, q) is called a *flat surface* because it has a canonical locally Euclidean geometry (see for example [HM]). Such a surface may be thought of as being assembled from polygons in $\mathbb{R}^2$ whose edges are identified in pairs via translations or rotations by 180°. (An example is given in Figure 1.) This construction yields a metric $|\sqrt{q}|$ on X so that, on the complement of a finite set of *cone points* (the images of the polygons' vertices), $(X, |\sqrt{q}|)$ is locally isometric to $\mathbb{R}^2$. When q is the square of an ordinary holomorphic differential $\omega \in \Omega(X)$, the same construction may be carried out using only translations, and the pair (X, ω) is called a *translation surface*.

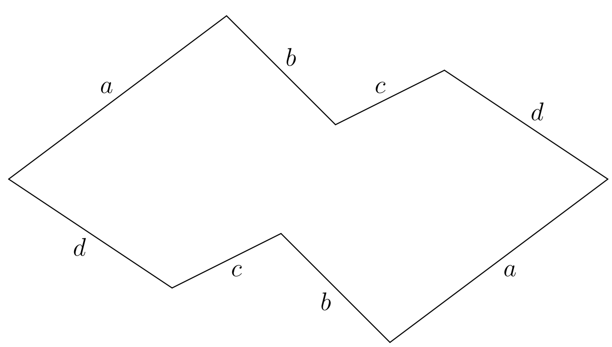

Figure 1: We can make a flat surface from this polygon by using translations to identify each pair of sides that have the same label. The resulting surface has genus 2, and all of the vertices are identified to create a single point around which there is an angle of 6π, making it a cone point.

Much of the symmetry of a flat surface (X, q) is captured by a certain discrete subgroup of $PSL_2(\mathbb{R})$ associated to it, called the *Veech group* of the surface. Each element of the Veech group is the linear part (up to $\pm$ the identity map on $\mathbb{R}^2$) of a homeomorphism $X \to X$ that is affine in the canonical coordinates of q; for a map to be affine means precisely that the linear part of the map is locally constant. An example is the map induced on the torus $\mathbb{R}^2/\mathbb{Z}^2$ by any element of $SL_2(\mathbb{Z})$. Veech showed in [Ve1] that the Veech group of a flat surface is closely linked to dynamical properties of the geodesic flow on the surface.

A flat surface (X, q), where X is in the Teichmüller space $\mathrm{Teich}(S)$, generates a family of surfaces, called a *Teichmüller disk* in $\mathrm{Teich}(S)$. Such a disk can be identified in a canonical way with the hyperbolic plane $\mathbb{H}$. Then each point of $\mathbb{H}$ represents a Riemann surface that can be obtained by using an element of $PSL_2(\mathbb{R})$ to vary the polygons that make up the geometry of (X, q). One reduces from the three dimensions of $PSL_2(\mathbb{R})$ to the two dimensions of $\mathbb{H}$ by observing that a rotation does not change the underlying complex structure of X, so the action of $SO_2(\mathbb{R})/\{\pm\mathrm{id}\}$ is trivial on the level of $\mathrm{Teich}(S)$. The Veech group of (X, q) acts on $\mathbb{H}$ in the usual way, by Möbius transformations. (A more detailed overview of this perspective can be found in [EG].)

Flat surfaces have strong connections with many different areas of mathematics. They appear in the study of dynamical systems such as billiards and interval exchange maps, algebro-geometric objects such as abelian varieties and Hilbert modular surfaces, and number-theoretic objects such as number fields and Eisenstein series (see for example [KS, Mc, Ve1]). A major problem is to understand what relations exist among the Veech group of a given flat surface, the dynamical properties of this surface, and the algebro-geometric properties of its underlying Riemann surface. In the remainder of this document, I indicate work I have done towards understanding certain cases and describe other projects that I believe will be useful and productive for collaboration.

2 Delaunay triangulations and tessellations of Teichmüller disks

The locally Euclidean geometry of a flat surface (X, q) makes it natural to consider certain canonical partitions of the surface. Let Z be the set of cone points of q. A q-*triangle* on X is the image of an isometric immersion from a Euclidean triangle to X, embedded on its interior, having its vertices in Z and containing no other points of Z. Let τ be a simplicial structure on X whose 2-cells are the interiors of q-triangles. We call τ a *Delaunay triangulation* if, for each edge E of τ, the sum of the angles opposite E is at most π. (See Figure 2 below.) Delaunay triangulations of flat surfaces were introduced in the 1980s by Thurston in the context of locally Euclidean structures on the sphere [Th2]. They were soon thereafter studied in greater generality by Masur–Smillie [MS] and Rivin [Ri], who also applied them to other geometric problems.

Figure 2: LEFT: Most quadrilaterals have a unique choice of diagonal that leads to a Delaunay triangulation. RIGHT: In a quadrilateral that is inscribable in a circle, one can choose either diagonal.

A generic flat surface has a unique Delaunay triangulation; the only possible ambiguity occurs when there is a pair of adjacent triangles such that the angles opposite their common edge add to exactly π. In these cases, we may form the *Delaunay partition* by removing all such edges.

The 2-cells of the Delaunay partition are Euclidean polygons and are permuted by isometries of the surface. The set of flat surfaces for which the Delaunay triangulation is not unique has codimension 1 in the moduli space of quadratic differentials.

The set of surfaces with a unique Delaunay triangulation is therefore open. That is, a surface with a unique Delaunay triangulation may be slightly perturbed and the same triangulation will be Delaunay. There is thus a natural partition of the $PSL_2(\mathbb{R})$-orbit of a flat surface according to the combinatorial type of the points' Delaunay triangulations. The metric on a flat surface is not affected by applying elements of $SO_2(\mathbb{R})/\{\pm \mathrm{id}\}$, and so this partition descends to the Teichmüller disk of the surface. We call this resulting partition the *iso-Delaunay tessellation* of the Teichmüller disk, or of the upper half-plane $\mathbb{H}$ when a canonical identification is made. (An example is shown in Figure 3 below.) The elements of the partition are called *tiles*. Veech and I have independently proved the following.

Theorem ([Bo3, Ve2]). *The iso-Delaunay tessellation is composed of convex polygonal tiles and is preserved by the action of the Veech group.*

The iso-Delaunay tessellation arising from a flat surface is therefore useful in studying the Veech group of the surface. Veech also shows that any tile of an iso-Delaunay tessellation has finite area; that is, the closure of a tile contains at most finitely many points of $\partial\mathbb{H}$. I have improved this result by providing an *a priori* upper bound for the area of a tile.

Theorem (B). *Any tile of an iso-Delaunay tessellation is contained in a hyperbolic triangle, possibly with vertices at infinity, and therefore has an area no greater than π.*

It should be possible to use this upper bound to control the geometry of tiles in an iso-Delaunay tessellation, as well as to get a bound for the volumes of the larger iso-Delaunay regions in the moduli space of quadratic differentials (each such region is known to be a polytope).

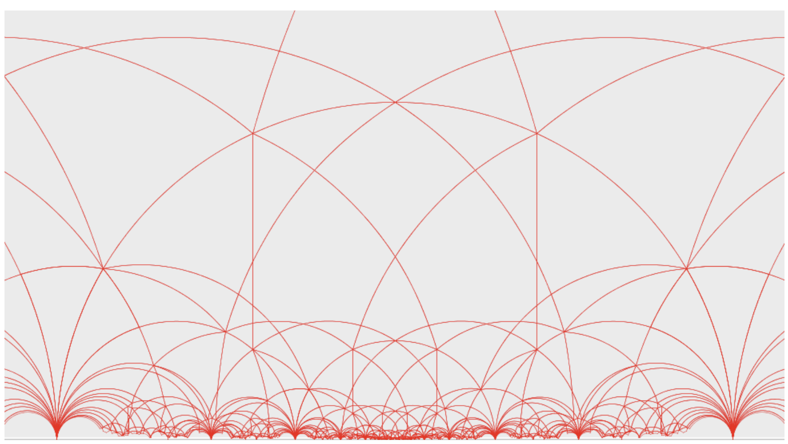

Figure 3: The iso-Delaunay tessellation arising from the genus 3 Arnoux–Yoccoz surface, described below. The Veech group of the surface includes a hyperbolic element that preserves both the imaginary axis in $\mathbb{H}$ and the tessellation.

In 1981, P. Arnoux and J.-C. Yoccoz [AY] introduced an infinite family of flat surfaces, one in each genus $g \geq 3$, each of which has a hyperbolic element (preserving a geodesic in $\mathbb{H}$) in its Veech group. Their Veech groups are known [HL] not to have any parabolic elements (which preserve a class of horocycles in $\mathbb{H}$), due to number-theoretic properties of their expansion constants. These surfaces have proved to have a rich set of properties. Using Delaunay decompositions and guided by results from my program, I was able to prove the following.

Theorem ([Bo2]). *The* $SL_2(\mathbb{R})$*-orbit of the genus* 3 *Arnoux–Yoccoz surface contains two surfaces, up to isomorphism, whose isometry groups are the dihedral group of the square.*

The additional symmetries I found lead to equations for these surfaces of the form

$$y = x(x-1)(x-t)(x+u)(x+tu)(x^2+tu) \quad \text{and} \quad y = x(x^2+1)(x-s)(x-\bar{s})(x+1/s)(x+1/\bar{s})$$

for some $t > 1$, $u > 0$, $s \in \mathbb{H} - \{i\}$. No equations were previously known for these surfaces. Finding t, u, and s requires solving systems of equations containing hyperelliptic integrals; approximate values are known, and I plan to find a closed-form expression for each.

In collaboration with Jerry Bowman, I have written a program to generate the iso-Delaunay tessellations for surfaces whose holonomy field is algebraic. The image in Figure 3 is one example; other examples may be seen at `www.math.cornell.edu/˜bowman/pictures.html`. I have computed the iso-Delaunay tessellations of several surfaces coming from triangular billiards, giving an independent proof of what their Veech groups are in several lattice cases (cf. [KS]).

Among the open questions to be addressed in this area are the following:

- Can two non-commensurable flat surfaces have the same iso-Delaunay tessellation?
- What are the volumes of the iso-Delaunay regions in the moduli space of quadratic differentials? What are the gluing maps among them?

3 Complex structures on the odd cohomology of a surface

The question of how the abelian differentials on points of a Teichmüller disk vary over the disk is a difficult one. Recall that the differentials on a Riemann surface X form a complex vector space $\Omega(X)$ whose dimension over $\mathbb{C}$ equals the genus of X. When X is considered as a point of a Teichmüller space $\mathrm{Teich}(S)$, $\Omega(X)$ is canonically identified (as a real vector space) with $H^1(S, \mathbb{R})$. Given a Teichmüller disk in $\mathrm{Teich}(S)$, the map that sends each point X to the complex structure making $H^1(S, \mathbb{R})$ into the complex space $\Omega(X)$ is a holomorphic map, and we say that such a disk in the space of complex structures on $H^1(S, \mathbb{R})$ *arises from a Teichmüller disk*. The Riemann bilinear relations imply that this disk lies in the *Siegel half-plane* $\mathfrak{H}$.

We now briefly recall the Thurston–Veech construction of affine maps [Th1, Ve1]. It begins with a choice of two *multi-curves* A and B on S—that is, two sets of curves that are homotopically nontrivial and which, within each set, are pairwise disjoint and non-homotopic. By an appropriate choice of a flat structure q on a point X in $\mathrm{Teich}(S)$, the elements of A and B become, respectively, horizontal and vertical core curves of metric cylinders on X. In particular, all geodesic paths in the horizontal and vertical directions are periodic, provided they do not meet a cone point of the flat structure. The affine group of (X, q) contains elements that fix A and B separately, and in most cases these generate a free group of rank 2 within the Veech group [Lei]. There is in fact one free real parameter in the construction, and so we obtain a family of surfaces $\{(X_t, q_t)\}_{t\in\mathbb{R}}$. The intersection matrix M of elements of A and B plays a key role in the construction.

For the moment, we assume that A and B are maximal, the characteristic polynomial of $M^\top M$ is irreducible, and all the zeroes of the q_t are simple. Then we let $\tilde{X}_t$ be the branched double cover of X_t with branch points at the zeroes of q_t, and we denote by $H^1(\tilde{X}_t, \mathbb{R})^-$ the part of $H^1(\tilde{X}_t, \mathbb{R})$ that is odd with respect to the sheet exchange. Each $H^1(\tilde{X}_t, \mathbb{R})^-$ is canonically identified with $H^1(\tilde{X}_0, \mathbb{R})^-$. The quadratic differential q_t on X_t lifts to the square of an abelian differential ω_t on $\tilde{X}_t$. From the assumptions we have made, we can define *conjugate forms* of ω_t (so called because of their relation to the other eigenvalues of $M^\top M$, which are Galois conjugates of each other) and we can show that the real and imaginary parts of ω_t and its conjugates span $H^1(\tilde{X}_t, \mathbb{R})^-$. Define a family $\{J_t\}_{t\in\mathbb{R}}$ of complex structures on $H^1(\tilde{X}_0, \mathbb{R})^-$ such that $J_t(\mathrm{Re}\,\omega'_t) = \mathrm{Im}\,\omega'_t$ for any conjugate ω'_t of ω_t. This family also lies in the Siegel half-plane $\mathfrak{H}$, but I have shown that it is essentially different from the family described in the opening paragraph of this section.

Theorem ([Bo1]). *The extension of* J_t *to a maximal holomorphically immersed disk in* $\mathfrak{H}$ *does not coincide with the disk arising from any Teichmüller disk having non-elementary Veech group.*

This theorem answers a question of J. Hubbard, which he posed by presenting J_t in matrix form and asked whether it described the complex structure on the odd part of $\Omega(\tilde{X}_t)$. The proof depends on a variational study of the family $\{J_t\}$ as a geodesic curve in $\mathfrak{H}$ and examining the limit of the disk as it approaches the boundary of $\mathfrak{H}$. This boundary has a natural stratification; these strata were introduced in [FF]. In addition to applying this stratification to the proof of the above theorem, I have shown that the strata are smooth real manifolds and computed their dimensions. I plan to investigate the following:

- How do these strata relate to boundary points of Teichmüller space (say, in the Earle–Marden bordification)?
- What kinds of extensions into this boundary do Teichmüller disks admit?

4 Directions for future research

While addressing the questions posed in the previous two sections, I also plan to extend the results of my doctoral work in new directions through the following projects.

4.1 Homothety surfaces

Flat surfaces fall naturally into a larger category of surfaces, called *homothety surfaces*. These are topological surfaces endowed with an atlas (on the complement on a finite set of points) whose transition maps are homotheties, i.e., compositions of translation and scaling. (An example is presented in Figure 4.) Just as translation surfaces appear in the study of interval exchange maps and have proved fruitful there, homothety surfaces are a natural way to study affine interval exchanges. It is also natural to consider this generalization because the space of homothety surfaces admits an action by $PSL_2(\mathbb{R})$, following the same definition given in §1. Indeed, in a certain sense they are the largest class of surfaces that contains flat surfaces and still admits such an action.

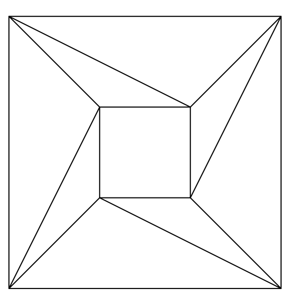

Figure 4: A triangulated homothety surface, obtained from an annulus with square boundaries. The inner boundary is identified with the outer boundary by scaling. This triangulation is Delaunay in the sense defined in §2.

Any homothety surface has a covering space which is a translation surface of (potentially) infinite area. Researchers such as P. Hubert, G. Schmithüsen, J. Smillie, F. Valdez, and B. Weiss have begun considering such non-compact "infinite-area" translation surfaces; however, compact homothety surfaces and their distinctive properties have been little explored to date. They demonstrate greater dynamical and topological flexibility than the more restricted class of flat surfaces, even as they enjoy many of the same properties. I believe the following steps are essential to initiate a proper theory of homothety surfaces:

- Find a sheaf-theoretic description of homothety structures on surfaces.
- Classify the strata of homothety surfaces, and find their dimensions.
- Describe the $PSL_2(\mathbb{R})$-orbit of a homothety surface and its image in Teichmüller space.
- Determine under what circumstances a homothety surface has a polygonal decomposition.
- Describe the possible dynamical behaviors of a linear trajectory on a homothety surface.

These are basic problems, whose analogues in the case of flat surfaces are more or less well-understood. Their answers will likely provide insight into the case of flat surfaces even as they extend into new territory.

4.2 Discrete Laplacians on flat surfaces

Graph Laplacians associated to simplicial surfaces have several applications in the fields of computational geometry and computer modeling. A natural question is what graph and what edge weights should be used to best capture the intrinsic geometry of the surface while remaining computationally feasible. This discussion has led to the proposal in [BS] that the Delaunay triangulation of the surface be used, along with weights that are determined by the angles of this triangulation. The eigenvalues of this discrete Laplace–Beltrami operator define continuous functions on the moduli space of quadratic differentials. I am interested in investigating these functions: smoothness properties, how well they separate points of the moduli space, how they reflect local and global properties of moduli space, ergodic averages over the $PSL_2(\mathbb{R})$-action, and so forth.

4.3 The Bost–Mestre dynamical system

Evaluating elliptic and hyperelliptic integrals is a source of much progress in numerical theory. The special case of computing complete periods on a surface of genus 2 was treated by Bost and Mestre in [BM]; their algorithm employs an averaging process that is an analogue of the classical arithmetic-geometric mean and had earlier incarnations in work of Richelot and Humbert. This process may be thought of as a dynamical system on sets of six points in a projective line, on a product of three copies of the Lie algebra of $PSL_2(\mathbb{R})$, or on the Teichmüller space of genus 2 surfaces. This appears to be a genuinely new complex dynamical system, which has received little attention, and all of the basic questions about dynamical systems remain open, such as the location and nature of fixed points, the existence of periodic orbits, and the structure of basins of attraction.

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