612 CLASS LECTURE: HYPERBOLIC GEOMETRY

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1. Conformal metrics

As a vector space, $\mathbb{C}$ has a canonical norm, the same as the standard $\mathbb{R}^2$ norm. Denote this $|\mathrm{d}z|$—one should think of $\mathrm{d}z$ as the identity map on $\mathbb{C}$: $\zeta \mapsto \zeta$. This also means $\mathbb{C}$ has a standard norm as the tangent space to itself at any of its points. We will consistently use Roman letters (e.g., z) to represent a point in $\mathbb{C}$ when we're thinking of it as a topological space, and Greek letters (e.g., ζ) to represent a point in $\mathbb{C}$ thought of as a vector. Let $U \overset{\circ}{\subset} \mathbb{C}$ be a plane domain (non-empty, connected, but not necessarily simply connected).

Definition 1.1. A *conformal metric* on U is a metric of the form $\rho\,|\mathrm{d}z|$, where ρ is a smooth, positive function on U. We will also use ρ to denote the metric itself, not just the function.

Almost always, "conformal" is used with respect to something else; here, we mean that angles in a conformal metric are the same as those in the background Euclidean metric. In other words, the unit ball of the norm in the tangent space at z is invariant under (Euclidean) rotations.

Example 1.2. On $U = \mathbb{C} \setminus \{0\}$, $\rho = |\mathrm{d}z|/|z|$ is a conformal metric. U is preserved by multiplication by a non-zero complex number a; call such a map m_a. The derivative of m_a is again multiplication by a. Let $z \in U$ and let $\zeta \in T_z(U) = \mathbb{C}$. The image of z by m_a is az, and the image of ζ by the derivative is $a\zeta \in T_{az}(U)$. Hence

$$\rho(az)|a\zeta| = \frac{|a\zeta|}{|az|} = \frac{|a||\zeta|}{|a||z|} = \frac{|\zeta|}{|z|} = \rho(z)|\zeta|,$$

that is, the metric is preserved by m_a.

Definition 1.3. If ρ_U is a conformal metric on U and ρ_V is a conformal metric on V, then a conformal diffeomorphism $f : U \to V$ is called an *isometry* if it sends tangent vectors to tangent vectors of the same length, i.e.,

$$\rho_V(f(z))|f'(z)\zeta| = \rho_U(z)|\zeta| \qquad \text{for all } z \in U,\ \zeta \in \mathbb{C}.$$

Definition 1.4. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. The *Poincaré metric* (or *hyperbolic metric*) on $\mathbb{D}$ is the conformal metric

$$\rho_{\mathbb{D}} = \frac{2\,|\mathrm{d}z|}{1-|z|^2}.$$

Theorem 1.5. *Up to a global scaling factor, $\rho_{\mathbb{D}}$ is the unique conformal metric on $\mathbb{D}$ that is invariant under the action of* $\mathrm{Aut}(\mathbb{D})$.

Date: 13 March 2008.

Proof. Recall that Schwarz's Lemma implies that all automorphisms of $\mathbb{D}$ have the form

$$f_{\theta,a}(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}, \qquad \theta \in \mathbb{R},\ a \in \mathbb{D}.$$

Suppose ρ is any conformal metric on $\mathbb{D}$ invariant under $\mathrm{Aut}(\mathbb{D})$. If $z \in \mathbb{D}$ is any point and $\zeta \in T_z(\mathbb{D})$ is any tangent vector, then $f_{0,z} : w \mapsto (w-z)/(1-\overline{z}w)$ sends z to 0, and $f'_{0,z}$ sends

$$\zeta \mapsto f'_{0,z}(z)\zeta = \frac{\zeta}{1-|z|^2} \in T_0\mathbb{D}.$$

By the invariance of ρ, we must therefore have

$$\rho(z)|\zeta| = \rho(0)|f'_{0,z}(z)\zeta| = \rho(0)\frac{|\zeta|}{1-|z|^2}.$$

Which shows that ρ is a multiple of $\rho_{\mathbb{D}}$. Conversely, $\rho_{\mathbb{D}}$ has just been shown to be invariant under $f_{0,a}$ for all a, and it is clearly invariant under multiplication by $e^{i\theta}$, hence it is invariant under all of $\mathrm{Aut}(\mathbb{D})$. □

Corollary 1.6. *The isometry group of* $\rho_{\mathbb{D}}$ *is precisely* $\mathrm{Aut}(\mathbb{D})$.

Proof. Immediate, as previously observed, from Schwarz's Lemma. □

We can determine a lot about the geometry of the metric space $(\mathbb{D}, \rho_{\mathbb{D}})$ just from the our knowledge of $\rho_{\mathbb{D}}$ and $\mathrm{Aut}(\mathbb{D})$. Given any smooth (C^1) curve $\gamma : [a,b] \to \mathbb{D}$, its *Poincaré length* is

$$\ell_{\mathbb{D}}(\gamma) = \int_a^b \rho_{\mathbb{D}}(\gamma(t))\,|\gamma'(t)|\,dt.$$

Using the definition of a *geodesic* as a "locally length-minimizing" curve, we can then compute the geodesics in $\mathbb{D}$.

Let $z \in \mathbb{D} - \{0\}$; we want to find a geodesic path from 0 to z. Because $\rho_{\mathbb{D}}$ is invariant under rotations, we can assume z lies in $(0,1)$. Let $\gamma : [a,b] \to \mathbb{D}$ be any path from 0 to z. The composition of γ with the rotational projection $(r,\theta) \mapsto (r,0)$ is again a smooth path $|\gamma|$ whose radial component is the same as that of γ, but whose rotational component is zero; hence $\ell_{\mathbb{D}}(|\gamma|) \le \ell_{\mathbb{D}}(\gamma)$. Thus the geodesic from 0 to z is the segment $[0,z]$. The complete geodesics through 0 are therefore the diameters of $\mathbb{D}$.

To find the geodesic path between z_1 and z_2, we recall that we can send z_1 to 0 by an isometry, find the geodesic from 0 to the image of z_2, and reverse the isometry. Because all of the elements of $\mathrm{Aut}(\mathbb{D})$ are Möbius transformations, which are conformal and send circles to circles, we conclude that the geodesic from z_1 to z_2 is the arc of the circle passing through z_1 and z_2 and orthogonal to $\partial\mathbb{D} = S^1$. (If these properties of Möbius transformations are unfamiliar to you, then work through Exercise 30 in Chapter 14 of [4].)

Definition 1.7. Let $\mathbb{H}$ be the open upper half-plane in $\mathbb{C}$. The *Poincaré metric* (or *hyperbolic metric*) on $\mathbb{H}$ is the conformal metric

$$\rho_{\mathbb{H}} = \frac{|\mathrm{d}z|}{y}, \qquad y = \mathrm{Im}\, z.$$

As subsets of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, we know that $\mathbb{D}$ and $\mathbb{H}$ are conformally equivalent, via the Möbius transformations

$$\varphi : z \mapsto i\,\frac{1+z}{1-z} \qquad\qquad \mathbb{D} \to \mathbb{H},$$
$$\psi : w \mapsto \frac{w-i}{w+i} \qquad\qquad \mathbb{H} \to \mathbb{D}.$$

We want to show that, in fact, these maps are isometries. We'll do this by "pulling back" $\rho_{\mathbb{H}}$ via φ and showing that the result coincides with $\rho_{\mathbb{D}}$. (We won't formally define "pull-back", but it basically means cooking up a metric so that a diffeomorphism becomes an isometry. The definition should be clear after an example.) Given $z \in \mathbb{D}$, $\zeta \in \mathbb{C}$, we have

$$\rho_{\mathbb{H}}(\varphi(z))|\varphi'(z)\zeta| = \frac{1}{\operatorname{Im}\varphi(z)}\left|\frac{2i}{(1-z)^2}\zeta\right| = \frac{|1-z|^2}{1-|z|^2}\frac{2\,|\zeta|}{|1-z|^2} = \rho_{\mathbb{D}}(z)|\zeta|.$$

Definition 1.8. We call any smooth Riemannian manifold that is isometric to $(\mathbb{D}, \rho_{\mathbb{D}})$ a *model of the hyperbolic plane.*

Definition 1.9. If U is any simply connected domain in $\mathbb{C}$, not equal to all of $\mathbb{C}$, then its *Poincaré metric* ρ_U is the conformal metric obtained by pulling back the Poincaré metric on $\mathbb{D}$ via the Riemann map. (Note: this is well-defined because the Riemann map is defined up to an element of $\operatorname{Aut}(\mathbb{D})$.)

Often, pictures are easier to draw in $\mathbb{H}$ than in $\mathbb{D}$, because the geodesics are simply the circles and lines in $\mathbb{C}$ orthogonal to the real axis (again, this follows from properties of Möbius transformations). Also, the isometry group in this model simply becomes $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$, because these are the Möbius transformations that preserve the real axis. This model distinguishes a single point ∞ on the boundary, as we occasionally wish to do.

Other models are also useful. For example, if we wanted to distinguish two points on the boundary, we could use the *band model*. Let $\mathbb{B}$ be the infinite strip

$$\mathbb{B} := \{z \in \mathbb{C} \mid |\operatorname{Im} z| < \pi\},$$

and define its Poincaré metric by

$$\rho_{\mathbb{B}} = \frac{|\mathrm{d}z|}{\cos y}, \qquad y = \operatorname{Im} z.$$

Then all translations in the x-direction are isometries: moreover, they preserve the two boundary points $\pm\infty$ and the geodesic between them. The remaining geodesics in this model are much more difficult to describe, however, except to say that they meet the boundary orthogonally (or have one end limiting to $\pm\infty$).

Exercise 1.

(1) Show that the map $z \mapsto e^z$ is an isometry from $\mathbb{B}$ with the conformal metric $|\mathrm{d}z|$ to $\mathbb{C} \setminus (-\infty, 0]$ with the conformal metric $|\mathrm{d}z|/|z|$.
[Note: the same calculation shows that the covering map $z \mapsto e^z$ is a *local isometry* from $(\mathbb{C}, |\mathrm{d}z|)$ to $(\mathbb{C} \setminus \{0\}, |\mathrm{d}z|/|z|)$.]
(2) Show that $(\mathbb{B}, \rho_{\mathbb{B}})$ is isometric to $(\mathbb{D}, \rho_{\mathbb{D}})$.
(3) Find the Poincaré metric on $\mathbb{C} \setminus [0, \infty)$.
(4) Using the idea of "local isometry" from (1), find the Poincaré metric on $\mathbb{D} \setminus \{0\}$.

2. Curvature

The most geometric way to describe the qualitative difference between conformal metrics is via their curvature. Roughly speaking, we want to compare how the area of a circle grows in terms of its radius, and compare this to the Euclidean growth rate πr^2. More precisely, we define the following.

Definition 2.1. Let $\rho\,|\mathrm{d}z|$ be a conformal metric on $U \subset \mathbb{C}$. Then the *Gaussian curvature*, or simply *curvature*, of $\rho\,|\mathrm{d}z|$ at $z \in U$ is

$$K_\rho(z) = -\frac{\Delta \log \rho(z)}{\rho^2(z)},$$

where Δ is the standard Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Here's an attempt at a geometric explanation, somewhat hand-wavey for now: $\Delta/\rho^2 = \operatorname{div}_{\rho^2} \operatorname{grad}_\rho$ is the *Laplace–Beltrami operator* for (U, ρ), hence is defined intrinsically. Note that the volume form on each tangent space to points in U is given by $\rho^2\,\mathrm{d}x \wedge \mathrm{d}y$.

- $\operatorname{grad}_\rho \log \rho = \rho \begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix}$ measures the change in ρ *with respect to itself*; the ρ-length of this vector is $\rho^2\sqrt{{\rho_x}^2 + {\rho_y}^2}$, which takes into account both the current volume form and how fast it's changing.
- $\operatorname{div}_{\rho^2}\left(\rho \begin{bmatrix} \rho_x \\ \rho_y \end{bmatrix}\right)$ measures how the volume form changes when moving along flow lines of the gradient of $\log \rho$, i.e., in the direction that the metric is increasing most quickly.

More concretely, one can show that the above value $K_\rho(z)$ appears as a coefficient in the power series for the area of a circle centered at z:

$$\operatorname{Area}(B(z,r)) = \pi r^2 \left(1 - \frac{1}{12} K_\rho(z) r^2\right) + o(r^4).$$

This requires showing that $K_\rho(z)$ is invariant under conformal coordinate changes—an essential property anyway, if this is going to measure something intrinsic; see [3]—and choosing a coordinate w so that $\rho(w) = (1 + c|w|^2 + o(|w|^2))\,|\mathrm{d}w|$ (see [2], where this fact about power series is left as an exercise, but the remainder of the proof is shown).

Note that the factor $1/\rho^2$ doesn't affect whether or not a function is *harmonic*. That is, $\Delta f = 0$ if and only if $(\Delta/\rho^2) f = 0$; the property of being harmonic is a conformal property, not a metric one. So a conformal metric $\rho\,|\mathrm{d}z|$ has constant curvature 0 if and only if $\log \rho$ is harmonic.

Example 2.2. On $\mathbb{H}$, the metric $\rho_{\mathbb{H}}$ has curvature -1.

$$-\frac{\Delta \log(1/y)}{1/y^2} = y^2 \frac{\partial^2}{\partial y^2} \log y = y^2 \left(-\frac{1}{y^2}\right) = -1.$$

Exercise 2.

(1) Show that the metric $\rho = |\mathrm{d}z|/|z|$ on $\mathbb{C} \setminus \{0\}$ has zero curvature.
(2) Show directly that the metric $\rho_{\mathbb{D}}$ on $\mathbb{D}$ has constant curvature -1.
(3) Show that the conformal metric $2|\mathrm{d}z|/(1 + |z|^2)$ on all of $\mathbb{C}$ has constant curvature 1.

3. Consequences of curvature

We have geodesics in $\mathbb{D}$ and $\mathbb{H}$. Given any smooth (C^2 is all we need) curve γ in $\mathbb{H}$, we can define its curvature with respect to the Poincaré metric, i.e., the amount by which it differs from a geodesic. A careful definition requires some technical language which we won't explain here, but we'll immediately specialize to describing curves of *constant* curvature, which will obviate the need for the definition.

Definition 3.1. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{H}$ be a smooth curve parametrized by arclength with respect to $\rho_{\mathbb{H}}$. Then the *geodesic curvature* k of γ at $\gamma(0)$ is the length of the vector

$$\left[\frac{\nabla}{dt}\gamma'(t) - \frac{\nabla}{dt}\alpha'(t)\right]_{t=0}$$

where α is the unique geodesic satisfying $\alpha(0) = \gamma(0)$ and $\alpha'(0) = \gamma'(0)$, and ∇/dt is the covariant derivative of the Levi–Civita connection given by ρ.

The definition we have given is for general conformal metrics, but we only know geodesics explicitly for hyperbolic and Euclidean metrics: in the former case, we can reduce the formula to

$$k(t) = |\theta'(t) + \sin\theta(t)|,$$

where we are using the upper half-plane model and θ is the angle between the downward direction and $\gamma'(t)$ (see [2] for the proof that this formula works).

In Euclidean plane geometry, curves of constant geodesic curvature have only two kinds of behavior: either the curvature is zero, and the curve is a geodesic, or it is non-zero, and the curve is a circle with radius 1/curvature. In hyperbolic plane geometry, there are *three* (or maybe four) possible behaviors for curves of constant geodesic curvature k:

- If $k = 0$, the curve is a geodesic; in particular, it has two limit points on the boundary.
- If $k < 1$, the curve is an arc of a Euclidean circle, joining two distinct boundary points; these are represented in $\mathbb{D}$ by arcs of circles that meet the boundary transversely, or in $\mathbb{H}$ by circles that cross the real axis and non-horizontal lines.
- If $k = 1$, the curve limits in both directions to a single point of the boundary; in $\mathbb{D}$, these are circles tangent to the boundary, and in $\mathbb{H}$ they are either circles tangent to the boundary or horizontal lines (tangent to the point at infinity).
- If $k > 1$, the curve is a (closed) circle; in $\mathbb{D}$ or $\mathbb{H}$ it is both a Euclidean and a hyperbolic circle, but the Euclidean center and the hyperbolic center do not in general coincide.

Definition 3.2. In any model of hyperbolic geometry, a curve with constant geodesic curvature 1 is called a *horocycle*.

Definition 3.3. A *hyperbolic triangle* is the region enclosed by three distinct, non-colinear points and the geodesic segments between them. An *ideal triangle* is a triangle with at least one vertex on the boundary.

An ideal triangle, while unbounded, still has finite area. In fact, any triangle with three ideal vertices has the same area, because $\text{Aut}(\mathbb{D})$ acts triply transitively on $\partial\mathbb{D}$. (You can see this by using the half-plane model and showing that $\text{PSL}_2(\mathbb{R})$ acts triply transitively on $\mathbb{R} \cup \{\infty\}$.) This area is in fact a global upper bound for the area of triangles. With a bit more elementary geometry (see [1]), one can show the following form of the Gauss–Bonnet Theorem: a hyperbolic triangle with angles α, β, and γ has area $\pi - (\alpha + \beta + \gamma)$. (Ideal vertices are supposed to have zero angle.) In particular, the angles must satisfy $\alpha+\beta+\gamma < \pi$.

Exercise 3. Show that the area of a triangle in $\mathbb{H}$ with three ideal vertices is π.
[*Hint:* Assume one of the vertices is the point at infinity. Integrate the measure ${\rho_{\mathbb{H}}}^2\, dx\, dy$ over the triangle.]

More generally, the following qualitative properties hold: in positively curved spaces, parallels tend to approach each other; in negatively curved spaces, parallels tend to diverge from each other; and in zero curvature spaces, parallels maintain their distance.

4. Classification of isometries

We have already looked at three kinds of isometries of the hyperbolic plane: rotations, translations in the band model (which preserve a geodesic), and translations in the upper half-plane model (which preserve a class of horocycles). It takes very little work to show that these are the only kinds of hyperbolic isometries. We'll use the model $\mathbb{H}$ with its isometry group $\mathrm{PSL}_2(\mathbb{R})$.

Let $A : z \mapsto A \cdot z$ with $[A] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PSL}_2(\mathbb{R})$ be an isometry of $\mathbb{H}$. First, we find the fixed points of this map in $\overline{\mathbb{H}} \subset \mathbb{CP}^1$ (we know such points exist by the Brower fixed point theorem, but we will soon show it algebraically). We want to solve the equation $A \cdot z = z$ in $\mathbb{CP}^1$. That is,

$$az + b = z(cz + d), \qquad \text{or} \qquad cz^2 + (d - a)z - b = 0.$$

Now we must consider two cases: $c = 0$ and $c \neq 0$.

In the first case, we have the map $z \mapsto (a/d)z + (b/d)$, a polynomial which therefore fixes ∞. To see if any other points are fixed, we consider the equation $(d - a)z - b = 0$; if $a = d = 1$ (why is this the only possibility for $d - a$ to be zero when $c = 0$?) and $b \neq 0$, then no solutions exist in $\mathbb{C}$, so the only fixed point is ∞. If $d - a \neq 0$, then the point $b/(d - a)$ is fixed.

In the second case, ∞ is not fixed, because it is sent to a/c. We consider the discriminant $(d - a)^2 - 4bc = (a + d)^2$ (why is this equality true?). If it is 0, then there is a unique fixed point on the real axis. If it is positive, there are two fixed points on the real axis. If it is negative, there are two fixed points in $\mathbb{C}$, but they are complex conjugates, so only one lies in $\mathbb{H}$.

We have therefore shown the three possibilities: A fixes either one point in the interior of $\mathbb{H}$, or one point on the boundary of $\mathbb{H}$, or two points on the boundary of $\mathbb{H}$.

Exercise 4. Show that if A fixes:

(1) two points in $\partial\mathbb{H}$, then it preserves the geodesic between those two points;
(2) one point in $\partial\mathbb{H}$, then it preserves all horocycles tangent to that point;
(3) one point in the interior of $\mathbb{H}$, then it preserves all circles centered at that point.

These considerations lead to standard forms for each kind of isometry. Up to conjugacy in $\mathrm{PSL}_2(\mathbb{R})$, any hyperbolic isometry is induced by a matrix of one of the following types:

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \qquad \text{(geodesic translation by hyperbolic distance } 2t\text{)},$$

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \qquad \text{(horocylic translation)},$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{(rotation through angle } 2\theta\text{)}.$$

We have carried out a very algebraic analysis of the situation; one can follow a more dynamical approach instead and arrive at the same conclusions. In this case, one considers the *infimum* of the distances points in $\mathbb{H}$ are moved, and examines what happens when this number is positive, when it is zero and realized by a fixed point in $\mathbb{H}$, or when is it zero and is not realized in $\mathbb{H}$. These cases correspond, respectively, to geodesic translation, rotation, and horocylic translation.

A final word on naming: these three types of isometries are sometimes called hyperbolic, parabolic, and elliptic isometries of $\mathbb{H}$. When a matrix in $\mathrm{SL}_2(\mathbb{R})$ is applied to $\mathbb{R}^2$, it certainly preserves a hyperbola in the first case and an ellipse (a circle) in the last case. (These aren't quite the reasons for giving these names, but they're close enough.) The horocycle has no analogue in Euclidean geometry, but in some ways it resembles a parabola, in that it is a limit of ellipses, and both ends tend to the same point at infinity. It is nicer than a parabola, however, in that it has constant geodesic curvature, while a parabola flattens out.

5. Extensions and generalizations

The metric on $\mathbb{D}$ was defined in terms of maps from $\mathbb{D}$ to itself. This leads to a special kind of metric on general open subsets of $\mathbb{C}^n$.

Definition 5.1. Let M be an open subset of $\mathbb{C}^n$, $z \in M$. The *Kobayashi ball* at z is defined by

$$B_z(M) = \{f'(0) \mid f : \mathbb{D} \to M \text{ analytic}, f(0) = z\}.$$

If $B_z(M) \subset T_z(M) = \mathbb{C}^n$ is bounded for all $z \in M$, then the Kobayashi balls define a metric on M, called the *Kobayashi metric*, and M is said to be *Kobayashi hyperbolic.*

Proposition 5.2. *If M and N are open subsets of $\mathbb{C}^n$ and $f : M \to N$ is analytic, then $Df(z)$ sends $B_z(M)$ into $B_{f(z)}(N)$.*

Proposition 5.3. *The Kobayashi metric on $\mathbb{D}$ is $\rho_{\mathbb{D}}/2$.*

Corollary 5.4 (Schwarz–Pick Theorem)**.** *If $f : \mathbb{D} \to \mathbb{D}$ is analytic, then it does not increase Poincaré lengths. If f' preserves Poincaré lengths at a single point, then it is an isometry.*

The Kobayashi metric is a way to define the hyperbolic metric of a plane domain U not equal to $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ "intrinsically", without using uniformization: it is given at a point $z \in U$ by the analytic maps $\mathbb{D} \to U$ that send 0 to z. It is in general hard to compute, however, and in any case the extremal maps are uniformizing anyway.

In higher dimensions, the Kobayashi metric can play an important classifying role: for example, in $\mathbb{C}^2$, the unit ball $B^2 = \{|z_1|^2 + |z_2|^2 < 1\}$ and the bidisk $D^2 = \mathbb{D} \times \mathbb{D}$ are both open and contractible, hence homeomorphic. They are *not* biholomorphic, however; each is its own Kobayashi ball at $(0,0)$, and any biholomorphism $B^2 \to D^2$ would have to induce a linear isomorphism between the two, which is impossible because ∂D^2 contains segments while ∂B^2 does not. (This argument requires a little more knowledge about the automorphism group of B^2, but not much, just that $\mathrm{Aut}(B^2)$ is transitive on B^2. See [3] for details.) Thus, no equivalent to the Riemann Mapping Theorem exists for domains in $\mathbb{C}^n$, $n > 1$.

In Teichmüller theory, the Kobayashi metric is used to show that the isometry group of the Teichmüller space for a surface S coincides with the mapping class group of S.

Hyperbolic geometry admits generalizations to higher dimensions: the half-plane and disk models generalize to the *half-space* and *ball* models: in $\mathbb{R}^n$, the Poincaré half-space $\mathbb{H}^n$ is

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \mid x_n > 0\} \quad \text{with the metric} \quad ds = \frac{|\mathrm{d}\mathbf{x}|}{x_n},$$

and the Poincaré ball is

$$\mathbb{B}^n = \{(x_1, \ldots, x_n) \mid \sum |x_n|^2 < 1\} \quad \text{with the metric} \quad ds = \frac{|\mathrm{d}\mathbf{x}|}{1 - |\mathbf{x}|^2}.$$

Geodesics, as before, are arcs of circles orthogonal to the boundary. Hyperbolic planes (i.e., embeddings of $\mathbb{H} = \mathbb{H}^2$) are sectors of spheres orthogonal to the boundary. Horospheres, the analogue of horocycles, are spheres tangent to the boundary. The half-space and the ball, along with their boundaries, are contained in the one-point compactification $\mathbb{S}^n$ of $\mathbb{R}^n$, and as before, there is a Möbius transformation (in the sense of a diffeomorphism sending spheres to spheres) that converts $\mathbb{H}^n$ into $\mathbb{B}^n$.

By a strange coincidence, the isometry group of $\mathbb{H}^2$ (as we have seen) is $\mathrm{PSL}_2(\mathbb{R})$, while the isometry group of $\mathbb{H}^3$ is $\mathrm{PSL}_2(\mathbb{C})$. This is simply because $\partial\mathbb{H}^2 = S^1 \cong \mathbb{RP}^1$ and $\partial\mathbb{H}^3 = S^2 \cong \mathbb{CP}^1$, and does not seem to admit any higher generalizations in terms of similar Lie groups (except, of course, the groups of Möbius transformations).

Another useful model of the hyperbolic plane that immediately generalizes to higher dimensions is given by a *hyperboloid* lying in *Minkowski space* $\mathbb{R}^{2,1}$—that is, $\mathbb{R}^3$ endowed with the quadratic form $Q(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 - {x_3}^2$. Let $\mathbb{H}^2$ be a single sheet of the two-sheeted hyperboloid $Q(\mathbf{x}) = -1$, i.e.,

$$\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 - {x_3}^2 = -1,\ x_3 > 0\}.$$

A quick computation, or a quick sketch, shows that for any tangent vector $\mathbf{v} \in T_{\mathbf{x}}\mathbb{H}^2$, $Q(\mathbf{v}) > 0$, so Q restricts to a Riemannian metric on $\mathbb{H}^2$. By construction, the isometries of $\mathbb{H}^2$ are precisely the orientation-preserving *Lorentz transformations*—i.e., linear maps of $\mathbb{R}^3$ that preserve the Minkowski form Q and the sign of x_3. Geodesics in this model are simply the intersections of 2-dimensional planes with $\mathbb{H}^2$, i.e., sets of the form

$$\{\mathbf{x} \in \mathbb{R}^3 \mid Q(\mathbf{x}) = -1,\ \langle \mathbf{x}, \mathbf{g} \rangle_Q = 0\} \qquad \text{for some } \mathbf{g} \in \mathbb{R}^3 \text{ with } Q(\mathbf{g}) > 1.$$

An entire section in [2] addresses "hyperbolic trigonometry" using this model, including the analogues of the sine law and cosine law from Euclidean geometry.

The hyperboloid model and the disk model are related by projection: embed $\mathbb{D}$ as the unit disk in the (x_1, x_2)-plane of $\mathbb{R}^3$, and let $\mathbf{a} = (0, 0, -1)$. Then, given any point $\mathbf{x} \in \mathbb{H}^2$, the line from $\mathbf{a}$ to $\mathbf{x}$ intersects $\mathbb{D}$ at exactly one point. This projection is an isometry (see [2]).

References

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[4] Walter Rudin, *Real and Complex Analysis*. McGraw-Hill, 1987.