MANIFOLDS (AND THE IMPLICIT FUNCTION THEOREM)

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and that, for every point $x \in f^{-1}\{\mathbf{0}\}$, $Df(\mathbf{x})$ is onto. Then $\mathbf{0}$ is called a *regular value* of the function. The implicit function theorem tells us, almost directly, that $f^{-1}\{\mathbf{0}\}$ is a manifold if $\mathbf{0}$ is a regular value of f. This is not the only way to obtain manifolds, but it is an extremely useful way.

The *dimension* of a manifold tells you, loosely speaking, how much freedom you have to move around. It is entirely analogous to the dimension of a vector space. Surfaces, such as the sphere, the torus, and the Klein bottle, are locally graphs of functions of two variables. Those two variables give you two "degrees of freedom," so the dimension of each of these manifolds is 2.

If we're using the implicit function theorem to tell us that something is a manifold, then it also tells us the dimension of the manifold. Recall that the statement of this theorem says, "If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, $\mathbf{x}_0 \in f^{-1}\{\mathbf{0}\}$, and $Df(\mathbf{x}_0)$ is onto, then $f^{-1}\{\mathbf{0}\}$, near $\mathbf{x}_0$, is the graph of a function of $k = n - m$ variables."

Example 0.1. The *sphere in* $\mathbb{R}^n$,

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| = 1\},$$

is an $(n-1)$-dimensional manifold. Use $f(\mathbf{x}) = x_1^2 + \cdots + x_n^2 - 1$.

Example 0.2. The 2×2 orthogonal matrices form a 1-dimensional manifold $O(2)$. We can describe $O(2)$ by equations in terms of the entries: $\mathbf{X}^\top \mathbf{X} = \mathbf{I}$ becomes

$$\begin{matrix} x_{11}^2 + x_{21}^2 = 1 & x_{11}x_{12} + x_{21}x_{22} = 0 \\ x_{12}x_{11} + x_{22}x_{21} = 0 & x_{12}^2 + x_{22}^2 = 1 \end{matrix}.$$

Two of these equations are identical, so we only need three of them. Define $f : \mathrm{Mat}_{2\times 2} \to \mathbb{R}^3$ by

$$f\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = f\begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}^2 + x_{21}^2 - 1 \\ x_{12}^2 + x_{22}^2 - 1 \\ x_{11}x_{12} + x_{21}x_{22} \end{pmatrix}.$$

The Jacobian of f is

$$[Df(\mathbf{X})] = \begin{bmatrix} 2x_{11} & 0 & 2x_{21} & 0 \\ 0 & 2x_{12} & 0 & 2x_{22} \\ x_{12} & x_{11} & x_{22} & x_{21} \end{bmatrix}.$$

The first two rows of this matrix are linearly independent as long neither column of the original matrix is zero, which is certainly true of an orthogonal matrix. To show that the third row is also linearly independent, suppose there exist a and b such that

$$x_{12} = ax_{11}, \quad x_{11} = bx_{12}, \quad x_{22} = ax_{21}, \quad x_{21} = bx_{22}.$$

These imply $x_{11} = bx_{12} = abx_{11}$ and $x_{12} = ax_{11} = abx_{12}$, so either $x_{11} = x_{12} = 0$ or $ab = 1$. In either case, $\det(\mathbf{X}) = x_{11}x_{22} - x_{12}x_{21} = x_{11}x_{22} - ax_{11}bx_{22} = 0$, which is impossible for an orthogonal matrix. Therefore the third row is independent of the first two, and the derivative of f is onto.

Because $O(2)$ is the preimage of $\mathbf{0}$, where $\mathbf{0}$ is a regular value of a map $\mathbb{R}^4 \to \mathbb{R}^3$, the dimension of $O(2)$ is $4 - 3 = 1$.

We can even get a clear picture of what $O(2)$ looks like, despite it being a manifold in $\mathrm{Mat}_{2\times 2} = \mathbb{R}^4$. By previous homework, a 2×2 orthogonal matrix is determined by its first column and whether it's a rotation or a reflection matrix. Since the first column is simply a point of S^1, $O(2)$ is two copies of S^1, i.e., two circles—one composed of rotation matrices, the other of reflection matrices. This is interesting: we have a manifold whose points can be multiplied. We say that the orthogonal matrices form a *Lie group*, which very loosely means a "manifold with multiplication." Lie groups are not remotely part of the course material, but I've included this mention in case you come across the term elsewhere.

TOPICS FROM CHAPTER 2

Remember, in Chapter 1 we moved from linear maps to derivatives of non-linear maps. Again, we have made a progression from linear to non-linear in this chapter.

- Row reduction
- Solving linear systems of equations
- Inverting matrices
- Linear combinations, linear independence, span, basis, dimension
- Rank-nullity theorem and the Fundamental Theorem of Linear Algebra
- Axioms for vector spaces
- Newton's Method: the algorithm, and Kantorovich's Theorem to guarantee convergence
- Inverse Function Theorem
- Implicit Function Theorem
- Manifolds (not in chapter 2, but will be on prelim)