OPEN SETS, CLOSED SETS, AND CONTINUITY

Recall definition of open set: for every $\mathbf{x} \in U$, there exists $r > 0$ such that $B_r(\mathbf{x})$ is contained in U.

Recall triangle inequality: $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Proposition. The open ball $B = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| < 1\}$ is open.

Proof. Let $\mathbf{x} \in B$. Then $|\mathbf{x}| < 1$, and so $\varepsilon = 1 - |\mathbf{x}| > 0$. We want to show that $B_\varepsilon(\mathbf{x}) \subset B$. Suppose $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. Then

$$|\mathbf{y}| = |\mathbf{y} - \mathbf{x} + \mathbf{x}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x}| < \varepsilon + |\mathbf{x}| = 1,$$

and so $\mathbf{y} \in B$. This shows that $B_\varepsilon(\mathbf{x}) \subset B$, and since $\mathbf{x}$ was arbitrary, B is open. □

Recall equivalent definitions of closed set:

(1) For every convergent sequence $\{\mathbf{x}_n\}$ contained in X, the limit $\mathbf{x}_0$ is contained in X.
(2) X is closed if X^c is open.

Proposition. The closed ball $\overline{B} = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| \leq 1\}$ is closed.

Proof. We will show that $\overline{B}^c$ is open. Let $\mathbf{x} \in \overline{B}^c$. Then $|\mathbf{x}| > 1$, and so $\varepsilon = |\mathbf{x}| - 1 > 0$. We will show that $B_\varepsilon(\mathbf{x})$ is contained in $\overline{B}^c$.

To do this, we need a variant of the triangle inequality. Because we have both

$$|\mathbf{x}| = |\mathbf{x} - \mathbf{y} + \mathbf{y}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y}| \quad \text{which implies} \quad |\mathbf{x}| - |\mathbf{y}| \leq |\mathbf{x} - \mathbf{y}|, \qquad \text{and}$$
$$|\mathbf{y}| = |\mathbf{y} - \mathbf{x} + \mathbf{x}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x}| \quad \text{which implies} \quad |\mathbf{y}| - |\mathbf{x}| \leq |\mathbf{y} - \mathbf{x}| = |\mathbf{x} - \mathbf{y}|,$$

we conclude

$$\big||\mathbf{x}| - |\mathbf{y}|\big| \leq |\mathbf{x} - \mathbf{y}|.$$

This simply means that, in a triangle, the length of any one side is greater than the difference in lengths of the other two sides.

Now, let $\mathbf{y} \in B_\varepsilon(\mathbf{x})$. Then we have

$$|\mathbf{y}| = |\mathbf{x} - \mathbf{x} + \mathbf{y}| \geq \big||\mathbf{x}| - |\mathbf{x} - \mathbf{y}|\big| > \big||\mathbf{x}| - \varepsilon\big| = 1 \quad (\text{because } |\mathbf{x} - \mathbf{y}| < \varepsilon)$$

and therefore $\mathbf{y} \in \overline{B}^c$. We conclude $B_\varepsilon(\mathbf{x}) \subset \overline{B}^c$, and hence $\overline{B}^c$ is open. By definition (2) of a closed set, $\overline{B}^c$ is closed. □

Recall definition of continuity: f is continuous at $\mathbf{x}$ if either of the following equivalent conditions are satisfied:

(1) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|\mathbf{x} - \mathbf{y}| < \delta$ (i.e., $\mathbf{y} \in B_\delta(\mathbf{x})$) $\implies |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$.
(2) for all sequences $\{\mathbf{x}_n\}$ such that $\mathbf{x}_n \to \mathbf{x}$, and for all $\varepsilon > 0$, there exists $M \geq 0$ such that $n \geq M$ implies $|f(\mathbf{x}) - f(\mathbf{x}_n)| < \varepsilon$.

Proposition. The function $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ (defined by $\mathbf{x} \mapsto |\mathbf{x}|$) is continuous.

Proof. Choose $\mathbf{x} \in \mathbb{R}^n$. We want to show, using definition (1) of continuity, that $|\cdot|$ is continuous at $\mathbf{x}$. Let $\varepsilon > 0$, and set $\delta = \varepsilon$. Suppose $y \in \mathbb{R}^n$ satisfies $|\mathbf{x} - \mathbf{y}| < \delta$. Then the triangle inequality implies that

$$||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} - \mathbf{y}| < \delta = \varepsilon.$$

This shows that $|\cdot|$ is continuous at $\mathbf{x}$, and since $\mathbf{x}$ was arbitrary, $|\cdot|$ is continuous on all of $\mathbb{R}^n$. □