SOME ELEMENTARY RESULTS ON THE SIEGEL HALF-PLANE

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Contents

Notation and conventions

Let V be a finite-dimensional real vector space, and let $V^\top = \mathrm{Hom}(V, \mathbb{R})$ denote the dual space to V. If W is another real vector space and $A \in \mathrm{Hom}(V, W)$, then $A^\top : W^\top \to V^\top$ is defined by $A^\top\alpha = \alpha A$ for all $\alpha \in W^\top$. Any element $B \in \mathrm{Hom}(V, V^\top)$ induces a bilinear form $(v, w) \mapsto (Bw)v$. In this case, $B^\top$ is called the adjoint of B and also maps V to $V^\top$: V is canonically identified with its double dual $(V^\top)^\top$ via the map $v \mapsto ev_v$, where ev_v is defined by $ev_v\alpha = \alpha v$, and so $(B^\top w)v = ev_w Bv = (Bv)w$. $B \in \mathrm{Hom}(V, V^\top)$ is called symmetric (or self-adjoint) if $B^\top = B$ and skew-symmetric if $B^\top = -B$. It is called positive definite, written $B > 0$, if $(Bv)v > 0$ for all nonzero $v \in V$. If $B > 0$, then B is invertible, and $B^\top$ is also positive definite. If $G \in \mathrm{Hom}(V, V^\top)$ is symmetric and positive definite, then we call it a Euclidean structure, and the bilinear form g it induces an inner product.

To illustrate these notations and conventions, which may be unfamiliar, we prove a simple lemma and state a version of the spectral theorem.

Lemma 0.1. *If $A, B \in \mathrm{Hom}(V, V^\top)$ are both positive definite, then all of the eigenvalues of $A^{-1}B$ are positive.*

Proof. Suppose λ is an eigenvalue of $A^{-1}B$ with corresponding eigenvector v. Note that $v \neq 0$. Then $Bv = \lambda Av$. Applying this transformation to v, we get $(Bv)v = \lambda(Av)v$, and because $A > 0$ and $B > 0$, also $\lambda > 0$. □

Theorem 0.2 (Spectral theorem for symmetric maps)**.** *If $G \in \mathrm{Hom}(V, V^\top)$ is a Euclidean structure and $B \in \mathrm{Hom}(V, V^\top)$ is symmetric, then the eigenvalues of $G^{-1}B$ are real and the eigenvectors of $G^{-1}B$ span V (that is, $G^{-1}B$ is "diagonalizable").*

Note that, given any Euclidean structure G on V, a map $A \in \mathrm{Hom}(V)$ is symmetric in the usual sense if GA is symmetric in our sense. The property of being symmetric depends on an inner product, although being diagonalizable does not.

I must in these notes admit my indebtedness to the exposition of Pedro J. Freitas's thesis (available online), although I have chosen a different overall approach.

1. Symplectic and orthogonal groups

Suppose $\dim_{\mathbb{R}} V = 2n \geq 2$, and fix a symplectic structure on V, i.e., a skew-symmetric linear isomorphism $\Sigma \in \mathrm{Hom}(V, V^\top)$. In other words, the bilinear form σ induced by Σ on V is alternating and non-degenerate. The symplectic group of (V, Σ) is

$$\mathrm{Sp}(V) = \mathrm{Sp}_\Sigma(V) := \{A \in \mathrm{Hom}(V) \mid A^\top \Sigma A = \Sigma\}.$$

The following two results are completely standard.

Lemma 1.1. *If λ is an eigenvalue of $A \in \mathrm{Sp}(V)$, then so is $1/\lambda$.*

Proof. From the equation $A^\top \Sigma A = \Sigma$, we get $A^{-1} = \Sigma^{-1} A^\top \Sigma$. Because $A^\top$ has the same eigenvalues on $A^\top$ as A has on V, A and A^{-1} have the same set of eigenvalues. □

Proposition 1.2. *The determinant of any element $A \in \mathrm{Sp}(V)$ is 1.*

Proof. By Lemma 1.1, if -1 is not an eigenvalue, then every eigenvalue λ appears simultaneously with the eigenvalue $1/\lambda$, which means the determinant of A must be 1. The set of such A is a Zariski open set, thereby dense in $\mathrm{Sp}(V)$, because its complement is the set defined by the equation $\det(\mathrm{id} + A) = 0$. Because the determinant is a continuous function, it must therefore equal 1 everywhere on $\mathrm{Sp}(V)$. □

We are interested in studying the interplay between $\mathcal{C}(V) = \{J \in \mathrm{Hom}(V) \mid J^2 = -\mathrm{id}\}$ and $\mathrm{Sp}(V)$, eventually leading to a description of the Siegel half-plane. We begin with an elementary result that will prove essential to our study.

Lemma 1.3. *If $J \in \mathcal{C}(V) \cap \mathrm{Sp}(V)$, then $(\Sigma J)^\top = \Sigma J$.*

Proof. By applying the identities $J^2 = -\mathrm{id}$ and $\Sigma^\top = -\Sigma$ to the equation $J^\top \Sigma J = \Sigma$, we get $\Sigma J = -J^\top \Sigma = J^\top \Sigma^\top = (\Sigma J)^\top$. □

This lemma implies that, for all $J \in \mathcal{C}(V) \cap \mathrm{Sp}(V)$, ΣJ induces a symmetric, non-degenerate bilinear form on V. Given any G (not necessarily positive definite) that induces a symmetric, non-degenerate bilinear form on V, we obtain an orthogonal group:

$$\mathrm{O}_G(V) := \{A \in \mathrm{Hom}(V) \mid A^\top G A = G\}.$$

We also define the special orthogonal group $\mathrm{SO}_G(V)$ to be the connected subgroup of $\mathrm{O}_G(V)$ containing the identity, which for $G > 0$ is just the subgroup of orthogonal transformations with determinant 1. The Lie algebra of this group is the space of $A \in \mathrm{Hom}(V)$ such that GA is skew-symmetric:

$$\mathfrak{so}_G(V) = \{A \in \mathrm{Hom}(V) \mid (GA)^\top = -GA\}.$$

The dimension of the Lie algebra, and hence the dimension of $\mathrm{O}_G(V)$, is $2n^2 - n$.

Lemma 1.4. *If $J \in \mathcal{C}(V) \cap \mathrm{Sp}(V)$, then the tangent space $T_J\mathrm{Sp}(V)$ is the space of $A \in \mathrm{Hom}(V)$ such that $\Sigma J A$ is symmetric. We have the direct sum splitting*

$$\mathrm{Hom}(V) = T_J\mathrm{Sp}(V) \oplus \mathfrak{so}_{\Sigma J}(V).$$

Proof. The first assertion follows from differentiating the condition $A^\top \Sigma A = \Sigma$ at J and taking the kernel of the derivative. To prove the second assertion, observe first that if $A \in T_J\mathrm{Sp}(V) \cap \mathfrak{so}_{\Sigma J}(V)$, then $\Sigma JA = -\Sigma JA$, which implies $A = 0$ since ΣJ is invertible. To show that the sum spans $\mathrm{Hom}(V)$, take any $A \in \mathrm{Hom}(V)$, and set

$$A_{sym} = \frac{1}{2}\left(A + (\Sigma J)^{-1} A^\top \Sigma J\right), \quad A_{skew} = \frac{1}{2}\left(A - (\Sigma J)^{-1} A^\top \Sigma J\right).$$

Then $\Sigma J A_{sym}$ is symmetric, $\Sigma J A_{skew}$ is skew-symmetric, and $A = A_{sym} + A_{skew}$. □

Thus we can think of $\mathfrak{so}_{\Sigma J}(V)$ as the normal space to $\mathrm{Sp}(V)$ at J.

Corollary 1.5. *The dimension of* $\mathrm{Sp}(V)$ *is* $2n^2 + n$.

We define the symplectic orthogonal group to be $\mathrm{Sp}_\Sigma\mathrm{O}_J(V) := \mathrm{Sp}(V) \cap \mathrm{O}_{\Sigma J}(V)$. If Σ and J are understood, we just write $\mathrm{SpO}(V)$. If $\Sigma J > 0$, this definition is equivalent to $\mathrm{Sp}_\Sigma\mathrm{O}_J(V) = \mathrm{Sp}(V) \cap \mathrm{SO}_{\Sigma J}(V)$, since any symplectic transformation has determinant 1.

Proposition 1.6. $\mathrm{Sp}(V)$ *acts on* $\mathcal{C}(V) \cap \mathrm{Sp}(V)$ *by conjugation:*

$$A : J \mapsto AJA^{-1} \qquad \text{for all } A \in \mathrm{Sp}(V),\ J \in \mathcal{C}(V) \cap \mathrm{Sp}(V).$$

The stabilizer of J under this action is $\mathrm{Sp}_\Sigma\mathrm{O}_J(V)$.

Proof. The action of $\mathrm{Sp}(V)$ on itself by inner automorphisms clearly preserves the condition $J^2 = -\mathrm{id}$. If $J \in \mathcal{C}(V) \cap \mathrm{Sp}(V)$, $A \in \mathrm{Sp}(V)$, and $AJA^{-1} = J$, then $\Sigma J = \Sigma A^{-1} J A = A^\top \Sigma J A$, and therefore $A \in \mathrm{Sp}_\Sigma\mathrm{O}_J(V)$. By the reverse argument, if $A \in \mathrm{Sp}_\Sigma\mathrm{O}_J(V)$, then $AJA^{-1} = J$. □

Lemma 1.7. *If* $J \in \mathcal{C}(V) \cap \mathrm{Sp}(V)$*, then* $J \in \mathrm{Sp}_\Sigma\mathrm{O}_J(V)$ *and* $Jv \perp_{\Sigma J} v$ *for all* $v \in V$.

Proof. The first part follows from Proposition 1.6 and the fact that J commutes with itself. The second part comes from $(\Sigma J(Jv))v = -(\Sigma v)v = 0$. □

2. The Siegel half-plane $\mathfrak{H}$

Again fix a symplectic structure Σ on V, with induced symplectic form σ. The Siegel half-plane determined by Σ is

$$\mathfrak{H} = \mathfrak{H}_\Sigma := \{J \in \mathcal{C}(V) \cap \mathrm{Sp}_\Sigma(V) \mid \Sigma J > 0\}.$$

That is, $\mathfrak{H}_\Sigma$ comprises those elements $J \in \mathcal{C}(V) \cap \mathrm{Sp}_\Sigma(V)$ such that ΣJ is a Euclidean structure on V. We shall show that $\mathfrak{H}_\Sigma$ is in fact one connected component of $\mathcal{C}(V) \cap \mathrm{Sp}_\Sigma(V)$. (The remaining components of $\mathcal{C}(V) \cap \mathrm{Sp}_\Sigma(V)$ are likewise classified by the signature of ΣJ.)

Let $W \subset V$ be any subspace. Given a bilinear form g on V, define

$$W^{\perp_g} := \{v \in V \mid g(v, w) = 0 \ \forall\ w \in W\}.$$

If g is non-degenerate, then $\dim W + \dim W^{\perp_g} = \dim V = 2n$. If g is an inner product, we call $W^{\perp_g}$ the orthogonal complement of W in V; similarly, we call $W^{\perp_\sigma}$ the symplectic complement of W. A subspace L of V is called Lagrangian if $L = L^{\perp_\sigma}$, i.e., it has real dimension n and $\sigma(v, w) = 0$ for all $v, w \in L$. We denote the set of Lagrangian subspaces of V by $\Lambda_\Sigma(V)$.

Proposition 2.1. *Let $J \in \mathfrak{H}_\Sigma$, and let g be the associated inner product. Let $L \subset V$ be any subspace. Then $L \subset L^{\perp_\sigma}$ if and only if $JL \subset L^{\perp_g}$. In particular, if $L \in \Lambda_\Sigma(V)$, then V splits into the orthogonal sum $L \oplus JL$.*

Proof. From the definition of g, we get $g(v, Jw) = -\sigma(v, w)$. Hence $\sigma(v, w) = 0$ for all $v, w \in L$ if and only if $g(v, w') = 0$ for all $v \in L$, $w' \in JL$, which proves the inclusions. A dimension count now proves the latter statement. □

In what follows it will be useful to note that, if J_1 and J_2 are any complex structures on V, then the inverse of $J_1 J_2$ is $J_2 J_1$.

Lemma 2.2. *Let J_1 and J_2 be in $\mathfrak{H}$. Then all eigenvalues of $-J_2 J_1$ are positive, and the corresponding eigenspaces E_λ sum to V. For any eigenvalue λ, ${E_\lambda}^{\perp_\sigma}$ is the sum of all eigenspaces $E_{\lambda'}$ where $\lambda\lambda' \neq 1$. If 1 is an eigenvalue, then E_1 is invariant under both J_1 and J_2. For any eigenvalue $\lambda \neq 1$, J_1 and J_2 interchange E_λ and $E_{1/\lambda}$.*

Proof. First observe that $-J_2 J_1 = -J_2 \Sigma^{-1} \Sigma J_1 = (\Sigma J_2)^{-1}(\Sigma J_1)$, and therefore by the spectral theorem and Lemma 0.1 all the eigenvalues of $-J_2 J_1$ are positive and the eigenvectors span V. Given $v \in E_\lambda$, $w \in E_{\lambda'}$, we have

$$(\Sigma w)v = \lambda\lambda'(\Sigma J_1 J_2 w)(J_1 J_2 v) = \lambda\lambda'(\Sigma w)v$$

because J_1 and J_2 are symplectic. If $\lambda\lambda' \neq 1$, this equality implies $(\Sigma w)v = 0$. Because a subspace and its symplectic complement sum to V, this shows that ${E_\lambda}^{\perp_\sigma}$ is as claimed.

An eigenspace E_λ is the kernel of $-J_2 J_1 - \lambda \cdot \mathrm{id}$. This map factors as $J_2(\lambda J_2 - J_1)$, and because J_2 is non-singular, E_λ is also the kernel of $\lambda J_2 - J_1$. Suppose $v \in E_\lambda$. Then $(J_2 - \lambda J_1)J_1 v = -J_2(\lambda J_2 - J_1)v = 0$, and therefore J_1 maps E_λ to $E_{1/\lambda}$. Because $-J_1 J_2$ has the same eigenspaces as $-J_2 J_1$, the same argument shows that J_2 maps E_λ to $E_{1/\lambda}$. In particular, if 1 is an eigenvalue, then E_1 is invariant under J_1 and J_2. □

Lemma 2.3. *Let J_1 and J_2 be in $\mathfrak{H}$. Then $(-J_2 J_1)^t$ is in $\mathrm{Sp}(V)$ for all $t \in \mathbb{R}$.*

Proof. Let $\mathscr{E}$ be the set of eigenvalues of $-J_2 J_1$, and for each $\lambda \in \mathscr{E}$ let E_λ denote the corresponding eigenspace. By Lemma 2.2, every eigenvalue is positive and $V = \bigoplus_{\lambda \in \mathscr{E}} E_\lambda$. Hence $(-J_2 J_1)^t$ is defined on each E_λ by $w \mapsto \lambda^t w$. We need to show that this map is symplectic. Suppose $\lambda, \lambda' \in \mathscr{E}$, and $v \in E_\lambda, w \in E_{\lambda'}$. Then

$$\big(\Sigma(-J_2 J_1)^t w\big)(-J_2 J_1)^t v = \Sigma({\lambda'}^t w)\lambda v = (\lambda\lambda')^t(\Sigma w)v.$$

If $\lambda\lambda' \neq 1$, Lemma 2.2 shows that both sides of this equality are zero. If $\lambda\lambda' = 1$, then the equality shows that Σ is preserved on $E_\lambda \oplus E_{1/\lambda}$ (or on E_1, if $\lambda = \lambda' = 1$). Because the E_λs sum to V, this shows that $(-J_2 J_1)^t$ is symplectic for all $t \in \mathbb{R}$. □

Proposition 2.4. $\mathrm{Sp}(V)$ *acts transitively on $\mathfrak{H}$ by conjugation (as in Proposition 1.6).*

Proof. Let $J \in \mathfrak{H}$ and $A \in \mathrm{Sp}(V)$. Then for all $v \in V$

$$(\Sigma A^{-1} J A v)v = (A^\top \Sigma J A v)v = ((\Sigma J)Av)Av.$$

Therefore $\Sigma A^{-1} J A > 0$ because $\Sigma J > 0$ and A is nonsingular. Hence the action of Proposition 1.6 preserves $\mathfrak{H}$.

Given any pair (J_1, J_2) of points in $\mathfrak{H}$, $\sqrt{-J_2J_1} = (-J_2J_1)^{1/2}$ is an element of $\mathrm{Sp}(V)$ by Lemma 2.3. We shall show that $\sqrt{-J_2J_1}$ sends J_1 to J_2. The inverse of $\sqrt{-J_2J_1}$ is $\sqrt{-J_1J_2}$, because taking inverses of linear transformations commutes with taking square roots (when both exist). It suffices to show that J_2 equals $\sqrt{-J_2J_1}J_1\sqrt{-J_1J_2}$ on $E_\lambda \oplus E_{1/\lambda}$ for each $\lambda \neq 1$, since $J_2 = J_1$ on E_1 if 1 is an eigenvalue.

Recall from the proof of Lemma 2.2 that E_λ is the kernel of $\lambda J_1 - J_2$. Thus J_2 restricts on $E_\lambda \oplus E_{1/\lambda}$ to $\lambda J_1 \oplus (1/\lambda)J_1$. $\sqrt{-J_1J_2}$ restricts on $E_\lambda \oplus E_{1/\lambda}$ to $\lambda^{1/2}\mathrm{id} \oplus \lambda^{-1/2}\mathrm{id}$. Likewise, $\sqrt{-J_2J_1}$ restricts on $E_\lambda \oplus E_{1/\lambda}$ to $\lambda^{-1/2}\mathrm{id} \oplus \lambda^{1/2}\mathrm{id}$. J_1 interchanges E_λ and $E_{1/\lambda}$. Therefore the composition of $\sqrt{-J_1J_2}$, J_1, and $\sqrt{-J_2J_1}$ equals J_2. □

Geometrically, we see that to move from J_1 to J_2 involves, loosely speaking, a choice of a set of $\lambda_i > 1$ and some subspaces E_i such that $E_i \subset {E_i}^{\perp_\sigma}$. Then J_2 is the composition of J_1 with an expansion by λ_i in E_i and a contraction by $1/\lambda_i$ in J_1E_i. This suggests a family of natural metrics on $\mathfrak{H}$: for $J_1, J_2 \in \mathfrak{H}$, let $\mathscr{E}$ be the set of eigenvalues of $\sqrt{-J_2J_1}$. Then, given $p \in [1, \infty]$, define the Siegel p-metric d_p on $\mathfrak{H}$ by

$$d_p(J_1, J_2) = \Big(\sum_{\lambda \in \mathscr{E}} |\log \lambda|^p \Big)^{1/p} \quad (1 \le p < \infty), \qquad d_\infty(J_1, J_2) = \max\{\log \lambda_i\}.$$

Proposition 2.5. *For $1 \le p \le \infty$, d_p is a $\mathrm{Sp}(V)$-invariant metric on $\mathfrak{H}$, and $(\mathfrak{H}, d_p)$ is a geodesic metric space.*

Proof. The symmetry of d_p follows from an application of Lemma 1.1 to the equality $-J_2J_1 = (-J_1J_2)^{-1}$. If $d_p(J_1, J_2) = 0$, then $-J_2J_1 = \mathrm{id}$, i.e., $J_1 = J_2$, and so d_p is non-degenerate. For any three points J_1, J_2, J_3 in $\mathfrak{H}$, we have $-J_3J_1 = (-J_3J_2)(-J_2J_1)$. The triangle equality for d_p follows from this equation and a somewhat lengthy argument which we omit here.

Because $\mathrm{Sp}(V)$ acts by conjugation, and eigenvalues are invariant under conjugation, d_p is $\mathrm{Sp}(V)$-invariant. A path from J_1 to J_2 in $\mathfrak{H}$ is $\gamma : t \mapsto (-J_2J_1)^{t/2}J_1(-J_1J_2)^{t/2}$ for $t \in [0, 1]$. After checking that

$$d_p(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| \cdot d_p(J_1, J_2) \qquad (\text{for } t_1, t_2 \in [0, 1])$$

we conclude that the image of γ is a geodesic for d_p having J_1 and J_2 as endpoints. □

Propositions 1.6 and 2.4 imply further that $\mathfrak{H} \cong \mathrm{Sp}(V)/\mathrm{Sp}_\Sigma\mathrm{O}_J(V)$ for any choice of $J \in \mathfrak{H}$, but this description distinguishes the coset $\mathrm{Sp}_\Sigma\mathrm{O}_J(V)$, or, what is the same, J, as a base point. Indeed, d_p is the restriction to $\mathfrak{H}$ of a metric defined on the entire homogeneous space $\mathrm{GL}(V)/\mathrm{O}_{\Sigma J}(V)$, but we have chosen to exploit the very geometric description of how points in $\mathfrak{H}$ relate to each other, without reference to a base point.

3. Coordinates on $\mathfrak{H}$

3.1. **Block decompositions.** From the data (Σ, L_0, J_0), with $L_0 \in \Lambda_\Sigma(V)$ and $J_0 \in \mathfrak{H}_\Sigma$, we get a canonical splitting $V = L_0 \oplus J_0L_0$ (cf. Proposition 2.1). An $\mathbb{R}$-basis for V is $\{e_1, \ldots, e_n, J_0e_1, \ldots, J_0e_n\}$, where $\{e_1, \ldots, e_n\}$ is any basis of L_0; this latter set is therefore a $\mathbb{C}$-basis of V. This extra structure on V allows us to define, for example, complex conjugation on V: given $w \in V$, write $w = u + J_0v$. We call u the real part and v the imaginary part of w. The complex conjugate of w is $\overline{w} = u - J_0v$.

In this context, any linear transformation $A: V \to V$ can be decomposed as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} : L_0 \to L_0$, $A_{12} : J_0 L_0 \to L_0$, $A_{21} : L_0 \to J_0 L_0$, and $A_{22} : J_0 L_0 \to J_0 L_0$. J_0 itself has the form $\left[\begin{smallmatrix} 0 & -I_0^{-1} \\ I_0 & 0 \end{smallmatrix}\right]$ for some invertible map $I_0 : L_0 \to J_0 L_0$. Indeed, I_0 preserves ΣJ_0-lengths since, by Lemma 1.7, J_0 is an orthogonal map. We next determine the conditions on the $A_{\mu\nu}$ for A to be symplectic. Σ can be written as $\left[\begin{smallmatrix} 0 & \Sigma_{12} \\ -\Sigma_{12}{}^\top & 0 \end{smallmatrix}\right]$, where $\Sigma_{12} \in \mathrm{Hom}(J_0 L_0, L_0{}^\top)$ is an isomorphism, because L_0 and $J_0 L_0$ are Lagrangian and $\Sigma = -\Sigma^\top$. $A^\top$ has the form $\left[\begin{smallmatrix} A_{11}{}^\top & A_{21}{}^\top \\ A_{12}{}^\top & A_{22}{}^\top \end{smallmatrix}\right]$. Thus the condition $A^\top \Sigma A = \Sigma$ becomes the three conditions

$$\begin{cases} A_{11}{}^\top \Sigma_{12} A_{21} = (\Sigma_{12} A_{21})^\top A_{11} \\ A_{11}{}^\top \Sigma_{12} A_{22} - (\Sigma_{12} A_{21})^\top A_{12} = \Sigma_{12} \\ A_{12}{}^\top \Sigma_{12} A_{22} = (\Sigma_{12} A_{22})^\top A_{12} \end{cases} \tag{1}$$

(although there are apparently four conditions, two of them are identical). If we take $A = J_0$, the second equation yields $(\Sigma_{12} I_0)^\top = \Sigma_{12} I_0$. This is the restriction of ΣJ_0 to L_0.

If $J = \left[\begin{smallmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{smallmatrix}\right]$ is a complex structure, then the equation $J^2 = -\mathrm{id}$ translates to the conditions

$$\begin{cases} J_{11}{}^2 + J_{12} J_{21} = -\mathrm{id}_{L_0} \\ J_{22}{}^2 + J_{21} J_{12} = -\mathrm{id}_{J_0 L_0} \\ J_{11} J_{12} + J_{12} J_{22} = 0 \\ J_{21} J_{11} + J_{22} J_{21} = 0 \end{cases}. \tag{2}$$

Now we determine the conditions to ensure $J \in \mathfrak{H}$. By Proposition 2.1, $J_{21} = \mathrm{proj}_{J_0 L_0} J$ must be an isomorphism between L_0 and $J_0 L_0$, hence invertible. In this case, the system of equations (2) is equivalent to

$$J_{12} = -(J_{11}{}^2 + \mathrm{id}_{L_0}) J_{21}{}^{-1} \qquad \text{and} \qquad J_{22} = -J_{21} J_{11} J_{21}{}^{-1}. \tag{3}$$

Moreover, by Lemma 1.3 we know that ΣJ must be symmetric, which translates to

$$\begin{cases} (\Sigma_{12} J_{21})^\top = \Sigma_{12} J_{21} \\ \Sigma_{12}{}^\top J_{12} = J_{12}{}^\top \Sigma_{12} \\ \Sigma_{12} J_{22} = -J_{11}{}^\top \Sigma_{12} \end{cases}. \tag{4}$$

Combining the second equation in (3) with the first equation in (4), the final equation in (4) becomes

$$(\Sigma_{12} J_{21} J_{11})^\top = \Sigma_{12} J_{21} J_{11}. \tag{5}$$

Lastly, we need $\Sigma J > 0$. Clearly we must have $\Sigma_{12} J_{21} > 0$, because this is the restriction of ΣJ to L_0. But this condition is also sufficient: if $u \oplus v \in L_0 \oplus J_0 L_0$, then by setting

$u' = {J_{21}}^{-1} v$, we can reduce the computation of $(\Sigma J(u \oplus v))(u \oplus v)$ to a computation in L_0. We get

$$\begin{aligned}
&(\Sigma_{21} J_{21} u)u - (\Sigma_{12} v) J_{12} v + (\Sigma_{12} J_{22} v)u - (\Sigma_{12} v) J_{11} u \\
&= (\Sigma_{12} J_{21} u)u + (\Sigma_{12} J_{21} u')({J_{11}}^2 + \mathrm{id}_{L_0})u' - (\Sigma_{12} J_{21} J_{11} u')u - (\Sigma_{12} J_{21} u') J_{11} u \\
&= (\Sigma_{12} J_{21} u)u + (\Sigma_{12} J_{21} J_{11} u') J_{11} u' + (\Sigma_{12} J_{21} u')u' - 2(\Sigma_{12} J_{21} J_{11} u')u \\
&= (\Sigma_{12} J_{21} u')u' + (\Sigma_{12} J_{21}(u - J_{11} u'))(u - J_{11} u').
\end{aligned}$$

Both of these terms are non-negative. If $v \neq 0$, then the first term is positive, and if $v = 0$ but $u \neq 0$, the second term is positive. Hence we have proved:

Proposition 3.1. *If $J = \left[\begin{smallmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{smallmatrix}\right]$ is an element of $\mathrm{Hom}(V)$, then necessary and sufficient conditions for $J \in \mathfrak{H}$ are $\Sigma_{12} J_{21} > 0$, $(\Sigma_{12} J_{21})^\top = \Sigma_{12} J_{21}$, and the equations* (3) *and* (5)*.*

3.2. **Bounded complex domain.** From previous results, we know that $\mathcal{C}(V)$ has canonical charts. At $J_0 \in \mathcal{C}(V)$, the canonical chart is $k_0 : J \mapsto (\mathrm{id} + J_0 J)(\mathrm{id} - J_0 J)^{-1}$; its natural domain is all $J \in \mathcal{C}(V)$ such that 1 is not an eigenvalue of $J_0 J$, and its image is all A in $\mathrm{Hom}_{\overline{J_0}}(V) = \{A \in \mathrm{Hom}(V) \mid AJ_0 = -J_0 A\}$ for which 1 is not an eigenvalue. The inverse map is ${k_0}^{-1} : A \mapsto J_0(\mathrm{id} - A)(\mathrm{id} + A)^{-1}$.

Proposition 3.2. *Given any $J_0 \in \mathfrak{H}$, all of $\mathfrak{H}$ lies in the domain of the natural chart k_0 at J_0. The image $k_0(\mathfrak{H})$ is an open bounded domain of the complex subspace*

$$\mathrm{Hom}_{\overline{J_0},\Sigma}(V) = \{A \in \mathrm{Hom}_{\overline{J_0}} \mid (\Sigma A)^\top = \Sigma A\}.$$

Proof. There are several pieces to prove.

First, given any other $J \in \mathfrak{H}$, all eigenvalues of $J_0 J$ are negative by Lemma 2.2, hence in particular $J_0 J$ does not have 1 as an eigenvalue. Therefore all of $\mathfrak{H}$ lies in the domain of the canonical chart at J_0.

Secondly, we show that any symplectic J maps to $\mathrm{Hom}_{\overline{J_0},\Sigma}(V)$ under k_0. That is, we want to determine the condition on $A \in \mathrm{Hom}_{\overline{J_0}}(V)$ such that ${k_0}^{-1}(A)$ is symplectic. Because ${k_0}^{-1}(A)$ is a complex structure, we have

$$J_0(\mathrm{id} - A)(\mathrm{id} + A)^{-1} = (\mathrm{id} - A)^{-1}(\mathrm{id} + A) J_0.$$

(Note that power series in A commute.) The requirement $(\Sigma {k_0}^{-1}(A))^\top = \Sigma {k_0}^{-1}(A)$ is equivalent to each of the following:

$$\begin{gathered}
(\mathrm{id} + A^\top)^{-1}(\mathrm{id} - A^\top)\Sigma J_0 = \Sigma(\mathrm{id} - A)^{-1}(\mathrm{id} + A) J_0, \\
(\mathrm{id} - A^\top)\Sigma(\mathrm{id} - A) = (\mathrm{id} + A^\top)\Sigma(\mathrm{id} + A), \\
-A^\top \Sigma - \Sigma A = A^\top \Sigma + \Sigma A, \\
(\Sigma A)^\top = \Sigma A.
\end{gathered}$$

Thirdly, we show that $\mathrm{Hom}_{\overline{J_0},\Sigma}(V)$ is invariant under J_0, i.e., $\mathrm{Hom}_{\overline{J_0},\Sigma}(V)$ is a complex subspace of $\mathrm{Hom}_{\overline{J_0}}(V)$. If $A \in \mathrm{Hom}_{\overline{J_0},\Sigma}(V)$, then $(\Sigma J_0 A)^\top = A^\top \Sigma J_0 = -\Sigma A J_0 = \Sigma J_0 A$.

Fourthly, $k_0(\mathfrak{H})$ is open because it is a component of the complement in $\mathrm{Hom}_{\overline{J_0},\Sigma}(V)$ of the zero set of $\det(\mathrm{id} - A)$.

Finally, we show that $k_0(\mathfrak{H})$ is bounded. 0 is certainly in the image, and so it suffices to show that every line through the origin contains at least one point such that $\det(\mathrm{id} - A)$. Note that $A \in \mathrm{Hom}_{\overline{J_0},\Sigma}(V)$ is diagonalizable by the spectral theorem: ΣJ_0 is a Euclidean structure, and as we saw before, $(\Sigma J_0 A)^\top = \Sigma J_0 A$. In particular, all eigenvalues of A^2 are positive or zero. Note also that, because J_0 sends $\ker(\mathrm{id} - A)$ isomorphically to $\ker(\mathrm{id} + A)$, the vanishing of $\det(\mathrm{id} - A)$ is equivalent to the vanishing of $\det(\mathrm{id} + A)$, hence also of $\det(\mathrm{id} - A^2)$. These two observations imply that for any non-zero $A \in \mathrm{Hom}_{\overline{J_0},\Sigma}(V)$, there exists some $t \in \mathbb{R}$ such that $\det(\mathrm{id} - tA) = \det(\mathrm{id} - t^2 A^2) = 0$. Because the set of directions through 0 is compact, there is some uniform bound on all directions. □

A generalization of a previous result is: for any eigenvalue λ of $A \in \mathrm{Hom}_{\overline{J_0}}(V)$, $-\lambda$ is also an eigenvalue of A, and J_0 interchanges the corresponding eigenspaces. This result is highly reminiscent of Lemmas 1.1 and 2.2, particularly in the case where $(\Sigma A)^\top = \Sigma A$ and thus all the eigenvalues of A are real. If $A \in k_0(\mathfrak{H})$, then we know the leading eigenvalue of A must have absolute value less than 1. In that case, if v is an eigenvector of $A = k_0(J)$ corresponding to the eigenvalue λ,

$$-JJ_0 v = (\mathrm{id} - A)^{-1}(\mathrm{id} + A)v = (\mathrm{id} - A)^{-1}(1 + \lambda)v = \frac{1+\lambda}{1-\lambda} v,$$

and so $(1 + \lambda)/(1 - \lambda)$ is an eigenvalue of $-JJ_0$. This implies the following:

Proposition 3.3. *On $k_0(\mathfrak{H})$, the distance from 0 in the Siegel p-metric is given by*

$$d_p(0, A) = \left(\sum_{\lambda \in \mathscr{E}^+} \log^p \frac{1+\lambda}{1-\lambda} \right)^{1/p} \quad (1 \le p < \infty), \qquad d_\infty(0, A) = \max_{\lambda \in \mathscr{E}^+} \left\{ \log \frac{1+\lambda}{1-\lambda} \right\}$$

where $\mathscr{E}^+$ is the set of positive eigenvalues of A.

4. Basic examples

Example 4.1. The simplest case is $V = \mathbb{R}^2$, $\sigma(v, w) = \det(v, w)$, $L_0 = x$-axis, $J_0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. A symplectic transformation is one which preserves the determinant, hence the symplectic group is $\mathrm{SL}_2(\mathbb{R})$. All one-dimensional subspaces are Lagrangian. Any complex structure on $\mathbb{R}^2$ has the form $J = \left[\begin{smallmatrix} a & -(a^2+1)/b \\ b & -a \end{smallmatrix}\right]$, which is already in $\mathrm{SL}_2(\mathbb{R})$. For J to lie in $\mathfrak{H}$, the form $(v, w) \mapsto \sigma(v, Jw)$ must be positive definite, which implies $y > 0$. Thus the Siegel half-plane has a natural identification with the upper half-plane, given by $J \mapsto z = a/b + i/b$. The Siegel 2-metric on $\mathfrak{H}$ coincides with the Poincaré metric on $\mathbb{H}$ under this identification.

Example 4.2. Now we generalize the previous example. Identify $\mathbb{R}^{2n} = (\mathbb{R}^n)^2$ with $\mathbb{C}^n$ via the bijection $(q, p) \leftrightarrow q + ip$. The standard symplectic structure Σ and complex structure J are given by

$$\Sigma = -J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where I_n denotes the $n \times n$ identity matrix. Here we have used the standard inner product to identify $\mathbb{R}^{2n}$ with its dual. Observe that $\Sigma J = I_{2n}$ induces the standard inner product. The subspaces $\{(q, 0) \mid q \in \mathbb{R}^n\}$ and $\{(0, p) \mid p \in \mathbb{R}^n\}$ are Lagrangian and interchanged by J.

The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R}) = \mathrm{Sp}(\mathbb{R}^{2n})$ comprises those $2n \times 2n$ matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, broken into $n \times n$ blocks such that $A^\top C$ and $B^\top D$ are symmetric and $A^\top D - C^\top B = I_n$.

Using Proposition 3.1, we find that there is a one-to-one correspondence between points in the Siegel half-plane $\mathfrak{H}_n$ and $n \times n$ symmetric (*not* Hermitian) complex matrices $Z = X + iY$ with positive definite imaginary part; the maps are

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \mapsto J_{11}{J_{21}}^{-1} + i{J_{21}}^{-1} \qquad \text{and} \qquad X + iY \mapsto \begin{bmatrix} XY^{-1} & -(XY^{-1}X + Y) \\ Y^{-1} & -Y^{-1}X \end{bmatrix}.$$

Under this correspondence, the action of $\mathrm{Sp}_{2n}(\mathbb{R})$ on $\mathfrak{H}_n$ becomes an action by "generalized fractional linear transformations", i.e., $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : Z \mapsto (AZ + B)(CZ + D)^{-1}$.

Because any real even-dimensional vector space with a symplectic form can be identified with $(\mathbb{R}^{2n}, \Sigma)$ by an appropriate choice of basis, this example shows that $\mathfrak{H}$ is always simply-connected and has real dimension $n^2 + n$. Moreover, because the stabilizer of J under conjugation (or iI_n under fractional linear transformations, as can be checked directly) is $\mathrm{SpO}_{2n}(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{O}_{2n}(\mathbb{R})$, we see that $\dim \mathrm{SpO}_{2n}(\mathbb{R}) = (2n^2 + n) - (n^2 + n) = n^2$.