MATH 2220: SOME ADDITIONAL NOTES ON SEC 2.1

ABSTRACT. In the following we review the concept of the norm of a matrix in Sec 2.1.
Without specification, all numbers and symbols correspond to the textbook (Lax-Terrell 2016).

Def 2.4 in Lax-Terrell gives **the (Frobenius) norm** of a $m \times n$ matrix C

$$||C|| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^2}.$$

As a consequence of the Cauchy-Schwartz inequality, Theorem 2.2 in Lax-Terrell states

$$||CX|| \leq ||C|| \; |X|$$

for any $m \times n$ matrix C and a vector X in $\mathbb{R}^n$.

0.1. **Another matrix norm.** The **spectrum norm** of a $m \times n$ matrix C is defined to be

$$||C||_s = \sqrt{\lambda_{max}(C^t C)}$$

where C^t is the transpose of C, and λ_{max} the largest eigenvalue of the product matrix $C^t C$. Note that $C^t C$, as a $n \times n$ matrix, is nonnegative definite, therefore all its eigenvalues are nonnegative real numbers.

As a simple example

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

There it is an exercise to show that

$$A^t A = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and its eigenvalues are $(0, 6, 1)$, therefore

$$||A||_s = \sqrt{6}.$$

On the other hand, it is easier to see

$$||A|| = \sqrt{7}$$

0.2. **An inequality.**

Theorem 0.1. *(1) Any $m \times n$ matrix C satisfies*

$$||C||_s \leq ||C||$$

Moreover, $||C||_s = ||C||$ if and only if C is of rank 1 or $C = 0$.

(2) Combining Theorem 2.2 in Lax-Terrell, we have

$$||CX|| \leq ||C||_s \; ||X||$$

for any vector X in $\mathbb{R}^n$.

Proof of Theorem 0.1. Let $C = (c_{ij})$ where c_{ij} is the (i,j)-th entry, then $C^t = (c^t_{ij}) = (c_{ji})$ according to the rule of multiplication of matrix we see that the $i-$th diagonal element of $C^t C$ are exactly

$$\sum_{j=1}^{k} c^t_{ij} c_{ji} = \sum_{j=1}^{k} c_{ji} c_{ji}.$$

Here we use $c^t_{ij} = c_{ij}$ by the transpose. Therefore the trace (the sum of all diagonal elements of C^tC) is exactly the $||C||$. Note that the trace of C^tC equals the sum of all eigenvalues of C^tC, which by the way are all nonnegative. We conclude that the largest eigenvalue of C^tC is no more than $||C||$.

The case of $||C||_s = ||C||$ can be shown by a similar argument as in the above, it is left as an exercise. □

0.3. **When is $||CX|| = ||C||\ ||X||$?** This is a general case of the extra HW1 problem. Here we simply state the following

Proposition 0.2. *If a $m \times n$ matrix satisfies*

$$||CX|| = ||C||\ ||X||$$

for some nonzero vector X in $\mathbb{R}^n$, then either C is of rank 1 and $X \neq 0$ is a multiple any of the row vector of C, or $C = 0$ and X any vector in $\mathbb{R}^n$.

This proposition can be proved by tracing the 'equality' case of the Cauchy-Schwartz inequality. Namely any nonzero vectors U and V in $\mathbb{R}^n$ satisfies

$$||U \cdot V|| = ||U|| \cdot ||V||.$$

must be a multiple of each other.