MATH 2220: SOME NOTES ON SEC 2.1

Abstract. In the following we review some important concepts in Sec 2.1. Keywords are stereographic projection and graphs of functions of several variables.

Without specification, all numbers and symbols correspond to the textbook (Lax-Terrell 2016).

Part 1: Stereographic projection

Def (2.20 p77) (Stereographic projection): The unit sphere in three-dimensional space $\mathbb{R}^3$ is the set $S = \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}$. Let $N = (0, 0, 1)$ be the "north pole", Stereographic projection is a function $S : \mathbb{R}^2 \to S \setminus \{N\}$ which maps any point $A = (x, y, 0)$ on the plane $z = 0$ to a point B on the sphere. Here $B = (X, Y, Z)$ is the intersection point of the line through B and N with the unit sphere. See the picture below.

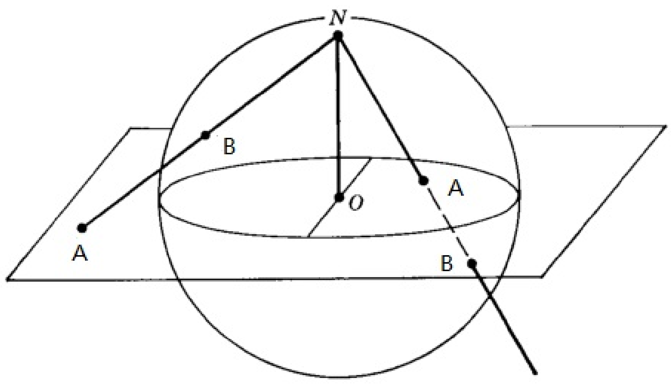

Figure 1. stereographic projection from north pole

In Lecture 3 we have shown:

Proposition (2.20 p77) $S : \mathbb{R}^2 \to S \setminus \{N\}$ has the formula:

$$S(x, y) = (X, Y, Z) = (\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1})$$

It is not hard to show that S has an inverse function $F : S \setminus \{N\} \to \mathbb{R}^2$:

$$F(X, Y, Z) = (x, y) = (\frac{X}{1 - Z}, \frac{Y}{1 - Z}).$$

Next we state two remarkable properties of $F : S \setminus \{N\} \to \mathbb{R}^2$.

Theorem 1: Any circle that does not pass through the north pole $(0, 0, 1)$ is mapped by F to a circle on the plane $z = 0$.

Intuition: It is easy to see the circle of latitude is indeed mapped by F to circles on the plane $z = 0$. In general, here is a picture: [1]

[1] Shame! Copied from http://aleph0.clarku.edu/~djoyce/java/compass/compass4.html

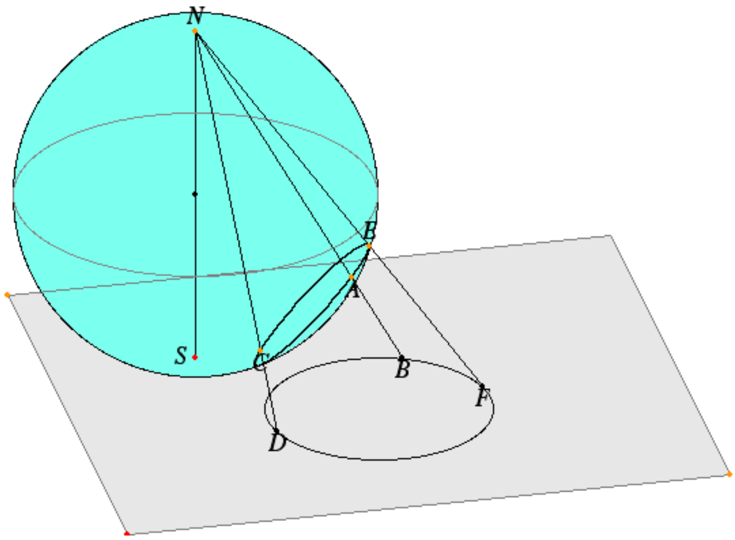

FIGURE 2. F maps a circle to a circle

Proof of Theorem 1 by an algebraic method: Any circle C on the unit sphere can be written as intersection an plane in $\mathbb{R}^3$ and the sphere.

$$\begin{cases} X^2 + Y^2 + Z^2 = 1, \\ AX + BY + CZ + D = 0. \end{cases} \tag{1}$$

According to the formula of the components of the stereographic projection S are exactly $S(x, y) = (X, Y, Z)$, we have

$$(X, Y, Z) = (\frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}). \tag{2}$$

Plug (2) back to (1) we have

$$A\frac{2x}{x^2 + y^2 + 1} + B\frac{2y}{x^2 + y^2 + 1} + C\frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} + D = 0. \tag{3}$$

Simplify it.

$$(C + D)x^2 + 2Ax + (C + D)y^2 + 2By - C + D = 0 \tag{4}$$

The crucial assumption is that the original circle C does not pass the north pole N, which N plug into (1) implies $C + D \neq 0$. Now we will see (4) is a circle on the plane, just rewrite (4) into:

$$(x + \frac{A}{C + D})^2 + (y + \frac{B}{C + D})^2 = \frac{A^2 + B^2}{(C + D)^2} + \frac{C - D}{C + D}. \tag{5}$$

This finishes the proof of Theorem 1.

Theorem 2: Stereographic projection $S : \mathbb{R}^2 \to S \setminus \{N\}$ and $F : S \setminus \{N\} \to \mathbb{R}^2$ preserves angles, in the sense that if two curves intersect at an angle θ on the sphere, so do their images curves on the plane $z = 0$ under F. [2]
Intuition: Think about two circles of longitude which meets at the south pole $(0, 0, -1)$, whose angle at south pole will be the difference of their longitudes. Under the map F, it becomes two lines passing through the origin $(0, 0, 0)$, meeting at the origin with the same angle!

[2] If you are interested, read here http://www.ams.org/samplings/feature-column/fc-2014-02 for a geometric proof.

Stereographic projection is useful because of Theorem 1 and Theorem 2. In general, it is not possible to map a portion of the sphere into the plane without introducing some distortion. Usually one cares about either the area or the angle. Stereographic projection is the latter. Angle-preserving map projections are important for navigation and it has an application in cartography. For example, one can use the map $F : S \setminus \{N\} \to \mathbb{R}^2$ to send the south hemisphere to the plane $z = 0$, here is an example: [3]

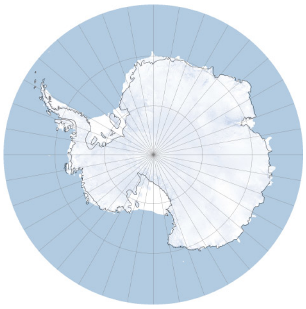

FIGURE 3. Map of the South Pole by stereographic projection

Also, in the above discussion we are using north pole for the reference point, the downside is that it is impossible to view maps near north pole. The following example use a reference point on the equator instead of north pole. [4]

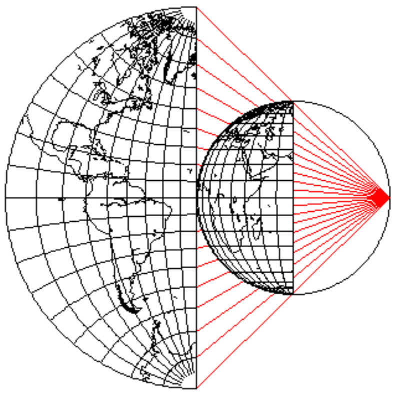

FIGURE 4. Stereographic projection from equator

Part 2: Examples of graphs of functions of several variables

We collect some examples of graphs of functions of several variables.

(1) A helix given by $F_1 : [0, 4\pi] \subset \mathbb{R}^1 \to \mathbb{R}^3$ defined by $F(t) = (\cos t, \sin t, t)$.

[3]Copied from http://earthobservatory.nasa.gov/blogs/elegantfigures/2010/09/27/g-projector/
[4]Copied from http://www.quadibloc.com/maps/maz0202.htm

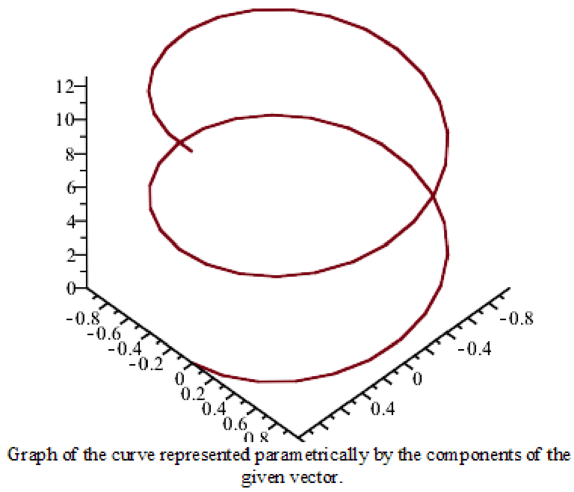

FIGURE 5. the graph of a helix

(2) A spiral given by $F_2 : [0, 4\pi] \subset \mathbb{R}^1 \to \mathbb{R}^2$ defined by $F(t) = (t \cos t, t \sin t)$.

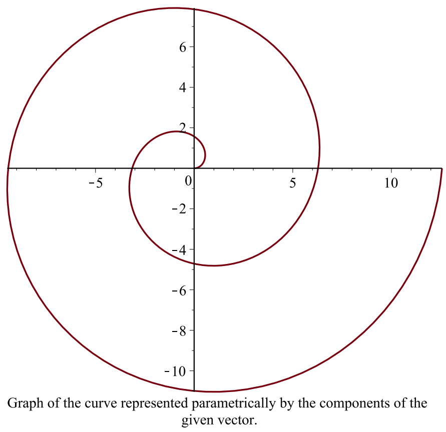

FIGURE 6. the graph of a sprial

(3) (Example 2.8 p68) $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^1$ defined by $F(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$.

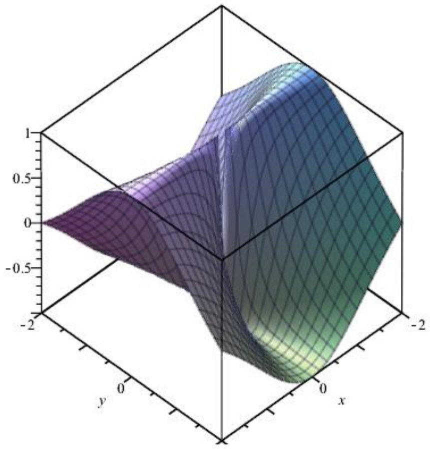

FIGURE 7. the graph of F on $[-2,-2] \times [-2,2]$, but excluding $(0,0)$

(4) (Example 2.71 p71) The plane vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $F(x,y) = (-y,x)$. We may see in the graph that it models a rotation around the origin $(0,0)$.

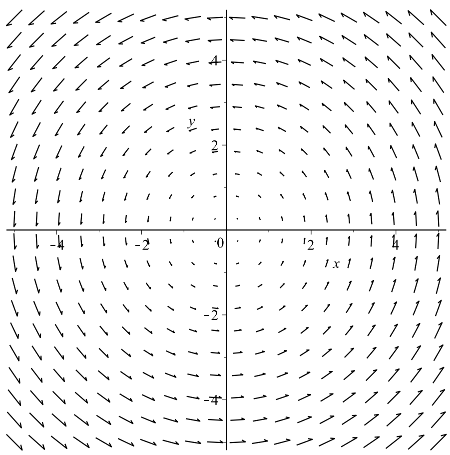

FIGURE 8. the graph of F on $[-5,-5] \times [-5,5]$