MATH 2220: SOME NOTES ON SEC 2.2

Abstract. In the following we review Sec 2.2. Key words are continuity and uniform continuity.

Without specification, all numbers and symbols correspond to the textbook (Lax-Terrell 2016).

0.1. **Continuity, discontinuous, uniform continuous. Def (2.6 p79) (Continuous functions from $D \subset \mathbb{R}^n \to \mathbb{R}^1$):** A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^1$ is continuous **at** $X \in D$ if given any tolerance $\epsilon > 0$, there exists a precision δ which usually depends on X and ϵ such that if

$$||X - Y|| < \delta \text{ and } Y \in D \qquad \text{then} \qquad |F(X) - F(Y)| < \epsilon$$

A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $X \in D$ if any $i-$th component of F, now as a function from $f^i : D \subset \mathbb{R}^n \to \mathbb{R}^1$ is continuous at $X \in D$ in the above definition.

A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is is called a continuous function on D if it is continuous at any point $X \in D$.

Def (2.16 p91) (Uniformly continuous function from $D \subset \mathbb{R}^n \to \mathbb{R}^1$): A function $F : D \subset \mathbb{R}^n \to \mathbb{R}^1$ is **uniformly continuous** on D if given any tolerance $\epsilon > 0$, there exists a precision δ which only depends on ϵ such that if

$$||X - Y|| < \delta \text{ for any given } X,\ Y \in D \qquad \text{then} \qquad |F(X) - F(Y)| < \epsilon$$

While uniformly continuous implies continuous, the inverse is only true after we impose extra assumptions on the domain of the function.

Theorem (2.12 p91) A continuous function $F : C \subset \mathbb{R}^n \to \mathbb{R}^m$ where C is a closed and bounded set is uniformly continuous on C

Now we would like to give an example to emphasize the difference of 'continuous' and 'uniformly continuous'. In view of Theorem 2.12 above, we have to work on a set C which is either not closed or not bounded in order to find such examples.

Two examples on continuous but not uniformly continuous:

(1) $F : \mathbb{R}^2 \to \mathbb{R}^1$ where $F(x, y) = x^2$. First F is continuous on $\mathbb{R}^2$ since it is product of two same linear functions $f(x, y) = x$, but why F is not uniformly continuous on $\mathbb{R}^2$?

The reason is that we can find two sequence of points $X_n, Y_n \in \mathbb{R}^2$ with $||X_n - Y_n|| \to 0$ but $|F(X_n) - F(Y_n)| \geq \epsilon_0$ where ϵ_0 is a fixed constant. This is exactly the opposite of the definition of 'uniformly continuous'! Why? Compare with Def (2.16 p91).

To make it work, just pick $\epsilon_0 = 1$ and $X_n = (n, n)$ and $Y_n = (n + \frac{1}{n}, n)$. Now check $||X_n - Y_n|| = \frac{1}{n} \to 0$ but $|F(X_n) - F(Y_n)| \geq 2 > \epsilon_0$.

(2) $F : D = \mathbb{R}^2 \to \mathbb{R}^1$ where $F(x, y) = \frac{1}{1-x^2-y^2}$ where $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$. Again F is continuous on $\mathbb{R}^2$ since it is a fraction and the denominator $1 - x^2 - y^2$ is continous (why?), but F is not uniformly continuous on D.

Why? By the similar reasoning as above, It suffices to find two sequence of points $X_n, Y_n \in \mathbb{R}^2$ with $||X_n - Y_n|| \to 0$ but $|F(X_n) - F(Y_n)| \geq \epsilon_0$ where ϵ_0 is a fixed constant.

We pick $X_n = (1 - \frac{1}{n}, 0)$ and $Y_n = (1 - \frac{2}{n}, 0)$ for $n \geq 3$. Note that both $X_n, Y_n \in D$ and $||X_n - Y_n|| = \frac{1}{n} \to 0$. Now let us check

$$\begin{aligned} |F(X_n) - F(Y_n)| &= \frac{1}{1 - (1 - \frac{1}{n})^2)} - \frac{1}{1 - (1 - \frac{2}{n})^2} \\ &= \frac{2n^3 - 3n^2}{(2n-1)(4n-4)} \end{aligned}$$

Note that $\lim_{n\to\infty} \frac{2n^3-3n^2}{(2n-1)(4n-4)} \to +\infty$. Therefore, pick some $\epsilon_0 = 10$, then we find $|F(X_n) - F(Y_n)| \geq \epsilon_0$ as n increases.