Nonnegative holomorphic sectional curvature: some examples

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Outline of the talk

- Holomorphic sectional curvature.
- Examples on Hirzebruch manifolds.
- Examples on BSV tori.
Almost-Hermitian manifolds

We begin with a $2n$-dimensional Riemannian manifold $M$. We call it $(M, J)$ an *almost complex manifold* if there exists $J : TM \to TM$ such that $J^2 = -\text{Id}$.

Let $g$ be a Riemannian metric on an almost complex manifold $M$, we call $(M, J, g)$ an *almost Hermitian manifold* if $J$ is compatible with $g$ in the sense that $g(X, Y) = g(JX, JY)$.

Next we define the *Kähler form* on an almost Hermitian manifold $(M, J, g)$ by $\omega(X, Y) = g(JX, Y)$, and the *Nijenhuis tensor* $N : TM \times TM \to TM$ is defined by

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Integrability of an almost complex manifold

When an almost complex manifold is a complex manifold?
In 1957 Newlander-Nirenberg proved that if $N = 0$ on an almost complex manifold $(M, J)$, then it can be realized as a complex manifold in the following sense:
There exists a collection of open cover $(U_\alpha, z_\alpha)$ of $M$ where $z_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ is a homeomorphism onto its image, and the transition map

$$f_{\alpha\beta} = z_\beta \cdot z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\alpha) \rightarrow z_\beta(U_\alpha \cap U_\alpha)$$

is a biholomorphism onto its image. At $p \in U_\alpha$ we write $i-$th component $z_i(p) = x_i(p) + \sqrt{-1}y_i(p)$, then

$$T_p(M) = \text{Span}\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \},$$

and

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i} \quad \text{and} \quad J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}.$$
Hermitian manifolds

From now on, we will consider integrable almost Hermitian manifold \((M, g, J)\), we call it is a *Hermitian manifold*.

Basic question: How to study curvatures on Hermitian manifolds?

The first thing is to introduce connections on Hermitian manifolds. There are at least 3 connections much studied on Hermitian manifolds.
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Three connections

- **Levi-Civita (Riemannian) connection** $\nabla$, which is uniquely determined by $\nabla g = 0$ and the vanishing torsion tensor $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

- **Chern connection** $\nabla^c$, which is uniquely determined by $\nabla^c g = 0$, $\nabla^c J = 0$, and after complexification $\nabla^c \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_i} = 0$.

- **Bismut connection** $\nabla^b$. which is uniquely determined by $\nabla^b g = 0$, $\nabla^b J = 0$, and the torsion form $g(T^b(X, Y), Z)$ is skew-symmetric.
The structural equation of a Hermitian connection $\nabla$ on Hermitian manifold $(M, J, g)$. Let $\{e_1, \cdots, e_n\}$ be the frame and $\{\varphi^1, \cdots, \varphi^n\}$ coframe of $T^{1,0}M$.

\[
\nabla e_i = \theta^k_i \otimes e_k,
\]
\[
d\varphi^i = \varphi^k \wedge \theta^i_k + \tau^i,
\]
\[
d\theta^i_j = \theta^k_i \wedge \theta^j_k + \Theta^i_j.
\]

If we choose $\nabla$ as the Chern connection $\nabla^c$, then the torsion $\tau^i = T^i_{jk} \varphi^j \wedge \varphi^k$ is of $(2, 0)$ type and the curvature tensor is of $(1, 1)$ type and $R_{i\bar{j}k\bar{l}} = \Theta^p_i(e_k, \bar{e}_l)g_{p\bar{l}}$. 
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Holomorphic sectional curvature: part II

- We define the **holomorphic sectional curvature** of a real-2 plane \( \pi = \{X, JX\} \) by

\[
H(\pi) = H(V) = \frac{R_{ijkl} V^i \overline{V}^j V^k \overline{V}^l}{\|V\|^4}
\]

where \( V = X - \sqrt{-1}JX \in T^{1,0}(M) \). It makes sense to define this notion for any connection with \( \nabla g = 0 \). In this talk, we focus on the Chern holomorphic sectional curvature on Hermitian manifolds.

- We say that \((M, J, g)\) has positive **holomorphic sectional curvature** \((H > 0)\) if \(H(V) > 0\) for any \(V \in T^{1,0}(M)\).
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Kähler manifolds

- A Kähler manifold is a Hermitian manifold \((M, J, g)\) with its Kähler form \(\omega(X, Y) = g(JX, Y)\) a closed 2-form.

- The above three connections on a Kähler manifold coincide, and in this sense a Kähler manifold is the most natural Hermitian manifold to study. There has been much progress on Kähler geometry. In this talk, we are concerned with positive (nonnegative) holomorphic sectional curvature.
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Question (S.-T. Yau)

Consider a compact Kähler manifold with $H > 0$, is it unirational? Is it projective? If a projective manifold is obtained by blowing up a compact manifold with positive holomorphic sectional curvature along a subvariety, does it still carry a metric with positive holomorphic sectional curvature? In general, can we find a geometric criterion to distinguish the concept of unirationality and rationality?
Basic question on compact Kähler manifolds with $H > 0$

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Recent Progress on Yau’s question

- Tsukamoto 1957: $H > 0$ implies simply connected. A generalization of this argument (Ni-Zheng 2018) shows any holomorphic isometry must have one fixed point.
- Berger 1966: $H > 0$ implies positive scalar curvature $S > 0$. A vanishing result (Kobayashi-Wu 1970) shows $S > 0$ implies the pluri-canonical ring vanishes, in particular $Kod = -\infty$.
- Hitchin 1975: Kähler surface with $H > 0$ implies projective and rational.
- Heier-Wong 2015: (assuming projective) rationally connected.
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'Metric moduli space’ for $H > 0$?

In this talk we focus on differential-geometrics aspect of $H > 0$, let us propose the following question.

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*Given a fixed compact Kähler manifold, what can we say about the space of all Kähler metrics with $H > 0$? Is it path-connected?*

Similar questions have been posed in several contexts in differential geometry (Chen-Tian, Marques, Kreck-Stolz and etc.).

**Theorem (F. Zheng and myself, 2016)**

*Given a Hirzebruch manifold $M_{n,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^{n-1}})$, there exists a Kähler metric of $H > 0$ in each of its Kähler classes. Moreover, the space of of all $U(n)$-invariant Kähler metrics of $H > 0$ on $M_{n,k}$ is path-connected.*
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Holomorphic pinching rigidity

If Yau’s conjecture is true, then how do we study complexities of rational varieties which admit Kähler metrics with $H > 0$? A naive thought is that the global and local holomorphic pinching constants of $H$ should give a stratification among all such rational varieties.

Theorem (X. Cao and myself, 2017)

For any integer $n \geq 2$, there exists a positive constant $\epsilon(n)$ such that any compact Kähler manifold with $\frac{1}{2} - \epsilon(n) \leq H \leq 1$ of dimension $n$ is biholomorphic to one of the following:

1. $\mathbb{CP}^n$,
2. $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$,
3. An irreducible rank-2 compact Hermitian symmetric space of dimension $n$.
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Essentially not much known on $H > 0$

A natural question is to look for the next threshold for holomorphic pinching constants. Another question which shows how little we know on $H > 0$ is

**Question (A special case of Yau’s question)**

Does $\mathbb{CP}^2$ with two points blown up admit a Kähler metric with $H > 0$?

Very few is known in higher dimension. Alvarez-Heier-Zheng 2016 proved a generalization of Hitchin’s construction on Hirzebruch surfaces that any projectivization of a vector boundle on a compact Kähler manifold with $H > 0$ admits Kähler metrics with $H > 0$. 
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BSV tori: definition

Given a real 6-torus $T^6_\mathbb{R}$, we will define a complex structure (orthogonal with respect to the flat metric) in the following way:

Fix an elliptic curve $M_1 = (T^2_\mathbb{R}, J_1)$. Note that $\mathbb{CP}^1 = SO(4)/U(2)$ is the space of all orthogonal complex structures on the flat $T^4_\mathbb{R}$, let $f : M_1 \to \mathbb{CP}^1$ be a nonconstant holomorphic map, one may consider a warped almost complex structure at $(y_1, y_2) \in T^2_\mathbb{R} \times T^2_\mathbb{R}$

$$J = J_1 + J_f(y_1)$$

It can be shown $J$ is integrable since $f$ is holomorphic.

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BSV tori: some properties

It has been known that BSV tori is not Kähler. Recently we note that a BSV tori \((T^6_\mathbb{R}, J)\) admits no pluri-closed Hermitian metrics. (i.e. \(\partial \bar{\partial} \omega = 0\))

The natural projection of \((T^6_\mathbb{R}, J)\) onto the elliptic curve \(M_1\) is a holomorphic submersion, while the fibers, as complex 2-tori, are not biholomorphic to each other in general. Its Kodaira dimension is \(-\infty\). It is interesting to understand more algebro-geometric properties of such non-Kähler manifolds.
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BSV tori are Riemannian flat and of Chern $H \geq 0$

BSV tori are the only ‘nonstandard’ orthogonal complex structure on flat real 6-tori.

**Theorem (Khan, Zheng and myself, 2017)**

*Given a compact Hermitian manifold with zero Riemannian curvature, then it has a finite cover being either a complex torus or a BSV torus.*

Applying a monotonicity formula relating $H$ w.r.t. Chern connection to $H$ w.r.t. Riemannian connection (in a previous work of Zheng and myself), we have: The Chern holomorphic sectional curvature of a BSV torus is nonnegative.

One may wonder if there is a reduction theorem on compact Hermitian manifolds with nonnegative Chern holomorphic sectional curvature.
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References


4. Bo Yang, Fangyang Zheng; On curvature tensors of Hermitian manifolds. arxiv. Accepted to Communications in Analysis and Geometry.


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Thank you very much for your attention!