RESEARCH STATEMENT

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Introduction

My research lies at the intersection of Poisson geometry and representation theory. Beginning in the 1980's, important connections were found between these two subjects through geometric quantization; one example is Kostant's presentation [Kos09] of the Borel-Weil-Bott theorem using the symplectic geometry of coadjoint orbits. Around the same time, the discovery of quantum groups and their canonical bases gave new tools in the representation theory of Lie groups [Lus90]. The semiclassical limits of quantum groups carry a natural multiplicative Poisson structure; these are called Poisson-Lie groups.

The broad theme of my research is to continue to develop the relationship between the representation theory of a compact Lie group K and the geometry of Hamiltonian K-manifolds, using new techniques from Lie theory and algebraic geometry. The main examples are the coadjoint orbits of K and toric symplectic manifolds. These are important and well studied examples of symplectic manifolds, and both are examples of multiplicity-free spaces.

The first part of my research employs the potential cones (also called string cones) of [BK06, BZ01], which parametrize the canonical bases of irreducible K-modules as the lattice points of a polyhedral cone. These cones have been used to give formulas for the multiplicities of irreducible summands in tensor products of K-modules, in terms of the lattice points of a convex polytope. When $K = U(n)$, the potential cone coincides with the moment map image of the Gelfand-Cetlin integrable system on the linear Poisson manifold $\mathfrak{k}^* = \mathfrak{u}(n)^*$. It remains unknown if an analogous statement holds for general K.

Together with collaborators, I have previously constructed an object called a *partial tropicalization*, which gives a new and explicit connection between the Poisson geometry of $\mathfrak{k}^*$ and the potential cone of K [ABHL18a, ABHL18b]. The construction uses techniques from cluster theory as well as Poisson-Lie theory. We have used partial tropicalizations to prove new theorems in symplectic geometry [AHLL19, AHLL18]. Going forward, my first research goal is to give a stratified, monoidal version of this construction. It should be stratified in that it takes into account the boundary of the potential cone. It should be monoidal in that it relates products of coadjoint orbits with the convex polytopes describing tensor product multiplicities.

The second part of my research concerns Hamiltonian actions on a class of singular spaces called symplectic stacks [HS18]. In particular, I previously defined *toric symplectic stacks*, which are natural generalizations of toric symplectic manifolds. They admit an action by a type of singular group called a *stacky torus*. I showed that toric symplectic stacks are classified up to isomorphism by their moment polytopes (together with some combinatorial data) [Hof19]. These moment polytopes often fail to be rational.

If M is a toric symplectic manifold, its equivariant cohomology can be read from its moment polytope; if M is additionally prequantizable then its geometric quantization can be read off from the moment polytope as well. Going forward, my second research goal is to make sense of these invariants for toric symplectic stacks, and express them in terms of the moment polytope.

The results of my proposed future research have immediate applications in symplectic geometry. Partial tropicalizations have already given a new approach to providing lower bounds for the

Gromov width of regular multiplicity-free K-spaces; a stratified version of the theory would extend this approach to all multiplicity-free K-spaces. This would also be a step toward constructing new integrable systems on linear Poisson spaces, analogous to the Gelfand-Cetlin system on $\mathfrak{u}(n)^*$. Finally, achieving the first research goal would establish new connections between Poisson geometry and cluster algebras. The potential cones of Berenstein-Zelevinsky and Berenstein-Kazhdan are examples of cones built by [GHKK17], which is evidence that partial tropicalization may have further connections with mirror symmetry.

Quantization is a central theme in Poisson geometry, and achieving my second research goal would be a natural addition to the theory of quantization of Hamiltonian spaces. At the same time, stacks have found many applications in symplectic and Poisson geometry in recent years [PTVV13], and in the course of working towards this goal it would be necessary to better develop basic parts of this theory, including the connection between differentiable stacks and noncommutative geometry.

Background

Let (M, ω) be a compact connected symplectic manifold, and assume there is a Hamiltonian action of a compact Lie group K on M with moment map $\mu \colon M \to \mathfrak{k}^*$. Here $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{k}^*$ is its linear dual. If the cohomology class of ω is integral, then M is *prequantizable*. Given a prequantizable K-manifold (plus some additional data), the procedure of geometric quantization associates to M a finite dimensional unitary K representation. Often, properties of the quantization of M are reflected in its moment map image $\mu(M) \subset \mathfrak{k}^*$, which is a convex polytope. A fruitful thread of research has been the study of the relationship between the geometry of M, the representation theory of K, and the polytope $\mu(M)$.

Gelfand-Cetlin systems A first example of this thread concerns a completely integrable system discovered by Guillemin-Sternberg [GS83]. Let $K = U(n)$, and consider the linear Poisson manifold $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$; one may think of $\mathfrak{k}^*$ as the union of its coadjoint orbits together with their standard symplectic forms. Identify $\mathfrak{k}^*$ with the set $\mathcal{H}(n)$ of $n \times n$ Hermitian matrices using an invariant inner product on $\mathfrak{k}$. Given a Hermitian matrix $A \in \mathcal{H}(n)$, let $\mu_k^i(A)$ be the i^{th} largest eigenvalue of the $k \times k$ principal submatrix of A. The μ_k^i collectively determine a map $\mu \colon \mathcal{H}(n) \to \mathbb{R}^{n(n+1)/2}$, which is smooth on a dense subset of $\mathcal{H}(n)$. The map μ generates a free Hamiltonian action of $(S^1)^{n(n-1)/2}$ on its smooth locus, and its image $\mu(\mathcal{H}(n))$ is a convex polyhedral cone. The cone $C_{GC} = \mu(\mathcal{H}(n))$ is the *Gelfand-Cetlin cone*, and $(\mathcal{H}(n), \pi_{\mathfrak{k}^*}, K, \mu)$ is the *Gelfand-Cetlin system*.

There is a natural projection $\mathrm{hw} \colon C_{GC} \to \mathbb{R}^n$, which simply extracts the eigenvalues μ_n^i, $i = 1, \dots, n$. The symplectic leaves of $\mathcal{H}(n)$ are orbits under the conjugation action of K, and they coincide with fibers of $\mathrm{hw} \circ \mu$. Let T be the standard maximal torus of $U(n)$, and let $\mathfrak{t}$ be its Lie algebra. Making identifications appropriately, one may think of $\mathbb{R}^n$ as the set $\mathfrak{t}^*$ of weights of $G = GL_n(\mathbb{C})$. The prequantizable orbits of $\mathcal{H}(n)$ are the fibers $\mathscr{O}_\lambda$ over $\lambda \in \mathbb{Z}^n$. For $\lambda \in \mathbb{Z}^n$, the lattice points of $\mathrm{hw}^{-1}(\lambda)$ coincide with the Bohr-Sommerfeld set of $\mathscr{O}_\lambda$, and it turns out that they count the dimension of the irreducible G-representation V_λ with highest weight λ. There is a similar symplectic recipe to count the dimensions of weight subspaces of V_λ.

For other reductive complex algebraic groups G, there are cones analogous to C_{GC} constructed in a purely algebro-geometric fashion by [BK06, BZ01]. Beginning with an open embedding of a torus $\theta \colon (\mathbb{C}^\times)^m \hookrightarrow B$ into the positive Borel subgroup, Berenstein-Kazhdan construct an associated polyhedral cone C called a *potential cone*. The cone $C \subset \mathbb{R}^m$ is cut out by inequalities given by a tropicalized *potential function*, and its integral points count dimensions of highest weight G-representations. (The results of [BK06] are in reality much stronger; they give $C \cap \mathbb{Z}^m$ the structure of a Kashiwara crystal).

The partial tropicalization theory outlined below is an approach to unifying these two constructions. A goal of the program is to build integrable systems on the linear Poisson manifold $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$ for general K, whose moment map image is the Berenstein-Kazhdan potential cone.

Toric symplectic manifolds A compact connected symplectic manifold (M, ω) with an effective Hamiltonian action of a torus T is called *toric* if $\dim M = 2 \dim T$. A celebrated theorem of Delzant [Del88] says that a toric manifold (M, ω, T, μ) is determined up to isomorphism by its moment map image $\mu(M)$. The image is a convex polytope which has the properties of being "simple", "rational", and "smooth." Conversely, any simple, rational, and smooth polytope is the moment map image of some toric manifold.

Toric manifolds are well studied examples of Hamiltonian T-manifolds, in part due to the fact that many of their geometric properties are reflected in the combinatorics of their moment polytopes. For one, the equivariant cohomology ring of a toric manifold is isomorphic to the face ring of its moment polytope. If a toric manifold is prequantizable, then additionally the unitary irreducible T-representations which appear in the quantization of M are exactly given by the lattice points $\mu(M) \cap \mathrm{Hom}(T, S^1)$.

Previous Results

Partial tropicalizations Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$. There is a multiplicative Poisson bivector π_K on K called the *standard Poisson-Lie structure*. We use an interpolating object (K^*, π_{K^*}), which is a *dual Poisson-Lie group to* (K, π_K), to connect the potential cones of [BK06] with $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$. The idea is to introduce coordinates on K^* which depend on a parameter $s > 0$. Brackets of coordinate functions become log-canonical, on a region controlled by the potential cone of K, in the $s \to \infty$ limit.

Let G be the complexification of K, and assume a choice of positive roots has been fixed. Consider the Iwasawa decomposition $G = KAN$, where $N \subset B$ is the unipotent radical of the positive Borel subgroup and $A = \exp(\sqrt{-1}\mathfrak{t})$. Here $\mathfrak{t}$ is the Lie algebra of the maximal torus T of K. The Lie group K^* can be identified with AN. In particular, if $K = U(n)$ then K^* is the group of upper triangular $n \times n$ matrices with positive real entries along the diagonal.

The connection with $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$ comes from the surprising fact that there exists a Poisson isomorphism

$$GW_s : (\mathfrak{k}^*, \pi_{\mathfrak{k}^*}) \to (K^*, s\pi_{K^*}) \tag{1}$$

for any scaling parameter $s > 0$ [GW92]. A construction of GW_s was given in [Ale97].

The connection with the work of [BK06] is more involved. We first consider the variety $G^{e,w_0} = B \cap B_- w_0 B_- \subset G$, which is an example of a *double Bruhat cell* of G. Its coordinate algebra $\mathbb{C}[G^{e,w_0}]$ may be enriched with the structure of a *cluster algebra* [BFZ05]. There is a collection of *seeds* σ of the cluster algebra $\mathbb{C}[G^{e,w_0}]$, and each seed comes with a set of distinguished regular functions $x_1, \ldots, x_m \in \mathbb{C}[G^{e,w_0}]$ called *cluster variables*. Together they determine an open embedding $\theta_\sigma : \mathbb{C}^{\times m} \to G^{e,w_0}$. Restricting to $AN \cap G^{e,w_0}$, we have an open embedding $\mathbb{R}^r \times (\mathbb{C}^\times)^{m-r} \hookrightarrow AN = K^*$. Here r is the dimension of the maximal torus of K. Making the s-dependent change of variables $x_i = e^{s\xi_i + \sqrt{-1}\varphi_i}$, for $s \in \mathbb{R} \backslash \{0\}$, gives the map

$$L_s^\sigma : \mathbb{R}^m \times \mathbb{T} \hookrightarrow AN = K^*. \tag{2}$$

We write $\mathbb{T} = (S^1)^{m-r}$. Let $\pi_s^\sigma = (L_s^\sigma)^*(s\pi_{K^*})$ be the scaled Poisson bivector in this new coordinate system.

For any seed σ for the cluster algebra $\mathbb{C}[G^{e,w_0}]$, consider the interior $\mathring{C}_\sigma \subset \mathbb{R}^m$ of the potential cone associated to the toric chart $\theta_\sigma : \mathbb{C}^{\times m} \to G^{e,w_0} \subset B$. The first connection between the Poisson manifold $(K^*, s\pi_{K^*})$ and the potential cones of [BK06] is the following.

Theorem 3 (Alekseev-Berenstein-**Hoffman**-Li [ABHL18b]). *Let σ be any seed for the cluster algebra $\mathbb{C}[G^{e,w_0}]$. Then on $\mathring{\mathcal{C}}_\sigma \times \mathbb{T}$, the limit*

$$\pi^\sigma_\infty := \lim_{s\to\infty} \pi^\sigma_s|_{\mathring{\mathcal{C}}_\sigma\times\mathbb{T}}$$

exists. It is a constant Poisson bivector on $\mathring{\mathcal{C}}_\sigma \times \mathbb{T}$.

Theorem 3 was originally proved for specific cluster seeds σ associated to reduced words for the longest element of the Weyl group w_0; in [AHLL19] we extend it to all other cluster charts. The Poisson manifold $(\mathring{\mathcal{C}}_\sigma \times \mathbb{T}, \pi^\sigma_\infty)$ is called a *partial tropicalization of* K^*.

Previously, it has been shown that a cluster variety comes with a *compatible Poisson structure*, meaning that the Poisson brackets of cluster variables are log-canonical [FG09]. The manifold (K^*, π_{K^*}) is not a cluster variety. However the Poisson brackets of the functions x_i (and their complex conjugates $\overline{x}_i$) on $\mathring{\mathcal{C}}_\sigma \times \mathbb{T}$ are log-canonical, up to some terms which go to zero as $s \to \infty$. This generalizes the notion of a compatible Poisson structure and is the main idea behind the proof of Theorem 3.

The Poisson bracket π^σ_∞ was computed explicitly in [ABHL18a]. Properties of $(\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$ persist in the partial tropicalization: there is a linear *highest weight map* $\mathrm{hw}\colon \mathbb{R}^m \to \mathfrak{t}^*$, which is part of the Kashiwara crystal structure on $\mathcal{C}_\sigma \cap \mathbb{Z}^m$, and it cuts out the symplectic leaves of π^σ_∞.

Theorem 4 (Alekseev-Berenstein-**Hoffman**-Li [ABHL18a]). *The symplectic leaves of $(\mathring{\mathcal{C}}_\sigma \times \mathbb{T}, \pi^\sigma_\infty)$ are all of the form*

$$\mathcal{P}_\lambda := (\mathrm{hw}^{-1}(\lambda) \cap \mathring{\mathcal{C}}_\sigma) \times \mathbb{T},$$

where λ is a regular dominant weight of G. The symplectic volume of $\mathcal{P}_\lambda$ equals the symplectic volume of the coadjoint orbit $\mathcal{O}_\lambda$. For any $p \in \mathcal{P}_\lambda$, the distance from the point $GW_s^{-1} \circ L^\sigma_s(p)$ to the coadjoint orbit $\mathcal{O}_\lambda$ approaches 0 as $s \to \infty$.

Theorems 3 and 4 are evidence for the following.

Conjecture 5. *For any cluster seed σ, the limit*

$$\lim_{s\to\infty} \left(GW_s^{-1} \circ L^\sigma_s\right) \colon (\mathring{\mathcal{C}}_\sigma \times \mathbb{T}, \pi^\sigma_\infty) \to (\mathfrak{k}^*, \pi_{\mathfrak{k}^*})$$

exists, and is a smooth Poisson embedding onto an open dense subset of $\mathfrak{k}^*$.

Applications of partial tropicalizations Partial tropicalizations are a new tool for proving theorems in in symplectic geometry. The following is a slightly weaker form of Conjecture 5. It says that there is an exhaustion of any regular coadjoint orbit by toric domains. (In [AHLL19] we prove a T-equivariant version of Theorem 6 for all regular multiplicity-free K-spaces).

Theorem 6 (Alekseev-**Hoffman**-Lane-Li [AHLL19]). *Let λ be a regular dominant weight of G, and let σ be a cluster seed of $\mathbb{C}[G^{e,w_0}]$. For any $\delta > 0$, there exists a symplectic embedding $\mathcal{P}^\delta_\lambda \hookrightarrow \mathcal{O}_\lambda$, where $\mathcal{P}^\delta_\lambda \subset \mathcal{P}_\lambda$ is the product of a convex subset of $\mathrm{hw}^{-1}(\lambda) \cap \mathcal{C}_\sigma$ and the torus $\mathbb{T}$. The symplectic volume $\mathcal{O}_\lambda$ is at most δ greater than the symplectic volume of $\mathcal{P}^\delta_\lambda$.*

An important invariant of a symplectic manifold is its *Gromov width*, which measures the size of the largest ball that can be symplectically embedded in a manifold. It has been an open problem to determine the Gromov width of the coadjoint orbits of a compact group; using toric degeneration techniques the authors of [FLP18] found an answer for those coadjoint orbits $\mathcal{O}_\lambda$ with λ lying on a rational line through the origin. A consequence of Theorem 6 is the following, which avoids the rationality constraint.

Theorem 7 (Alekseev-**Hoffman**-Lane-Li [AHLL19]). *Let $\lambda \in \mathfrak{t}^*$ lie on the interior of a Weyl chamber. Then the Gromov width of $\mathcal{O}_\lambda$ equals*

$$\min\{|\langle\lambda, \alpha^\vee\rangle| \mid \alpha^\vee \textit{ is a coroot with } \langle\lambda, \alpha^\vee\rangle \neq 0\}.$$

A second application is a description of the limiting behavior of a certain family of cohomologous symplectic forms ω_s^λ on the homogenous space K/T. The forms ω_s^λ were studied in [Luo00] as examples Poisson structures coming from dynamical r-matrices. The behavior of these forms as $s \to \infty$ was partially known due to a result in [LT17], but it was unknown how the symplectic volume of the form ω_s^λ arranged itself in the limit. Using partial tropicalizations we proved the following.

Theorem 8 (Alekseev-**Hoffman**-Lane-Li [AHLL18])**.** *As $s \to \infty$, the symplectic volume of $(K/T, \omega_s^\lambda)$ concentrates around the point eT.*

Hamiltonian actions on stacks Let (M, ω, G, μ) be a Hamiltonian G-manifold. In general, the reduced space $\mu^{-1}(0)/G$ fails to be a manifold. If 0 is a regular value of μ, then it does make sense to work with the stack $[\mu^{-1}(0)/G]$ of torsors for the action Lie groupoid $G \ltimes \mu^{-1}(0)$. With Reyer Sjamaar, we built the foundations of the theory of Hamiltonian actions of étale Lie group stacks on étale differentiable stacks. In this setting, the G-invariant form $\omega|_{\mu^{-1}(0)}$ descends to a symplectic structure on $[\mu^{-1}(0)/G]$. More generally, let $H \hookrightarrow G$ be an immersed Lie subgroup with map of Lie algebras $\iota\colon \mathfrak{h} \to \mathfrak{g}$. The Lie group stack $[G/H]$ acts on the reduced space $[(\iota^* \circ \mu)^{-1}(0)/H]$. The action is Hamiltonian, with moment map $\mu\colon [(\iota^* \circ \mu)^{-1}(0)/H] \to (\mathfrak{g}/\mathfrak{h})^*$.

After establishing the foundations of the theory, we defined the symplectic reduction of a Hamiltonian stack by a Lie group stack. We proved a stacky Duistermaat-Heckman theorem, and established the following criterion for the symplectic reduction to exist.

Theorem 9 (**Hoffman**-Sjamaar [HS18]). *Let $\mathcal{G}$ be an étale Lie group stack, and assume there is a Hamiltonian action of $\mathcal{G}$ on a symplectic stack $(\mathcal{X}, \omega)$ with moment map $\mu\colon \mathcal{X} \to \mathrm{Lie}(\mathcal{G})^*$. The symplectic reduction $(\mu^{-1}(0)/\mathcal{G}, \omega^{red})$ exists (as a differentiable stack) if and only if a certain higher-order freeness condition is satisfied.*

In the course of developing this theory we proved several technical results which may be of independent interest. For instance, the appropriate notion of an action of a Lie group stack on a differentiable stack is 2-categorical, meaning that the diagrams giving the axioms of an action commute only up to a given 2-isomorphism. If these 2-isomorphism are in fact the identity, then the action is *strict*.

Theorem 10 (**Hoffman**-Sjamaar [HS18]). *If an étale Lie group stack $\mathcal{G}$ acts on an étale differentiable stack $\mathcal{X}$, then the action $\mathcal{G} \times \mathcal{X} \to \mathcal{X}$ is always 2-isomorphic to a strict action.*

As an application, I defined the notion of a *toric symplectic stack* $(\mathcal{X}, \omega, \mathbf{T}, \mu)$, which generalizes the notion of a toric symplectic manifold. Here $\mathbf{T}$ is a *stacky torus*, which is a stacky quotient of a compact torus T by any Lie group immersion $N \to T$; an example is the circle S^1 divided by the subgroup generated by an irrational rotation. In contrast to previous approaches [RZ17, Pra01], the definition of a toric symplectic stack is *intrinsic*, meaning it does not depend on the properties any particular groupoid presentation of $\mathcal{X}$. The moment image $\mu(\mathcal{X})$ is always a simple convex polytope, and may fail to be rational.

A *stacky polytope* $(\Delta, \mathbf{T}, \{v_i\}_i)$ is a simple polytope Δ, a stacky torus $\mathbf{T}$, and a normal vector v_i for each facet of Δ. The normal vectors are required to live inside the cocharacter group of $\mathbf{T}$.

Theorem 11 (**Hoffman** [Hof19])**.** *The set of toric symplectic stacks is in bijection with the set of stacky polytopes (both up to appropriately defined isomorphism).*

This generalizes results of Delzant [Del88] and Lerman-Tolman [LT97].

Research Objectives

A stratified and monoidal partial tropicalization As it currently stands, the partial tropicalization theory does not account for two important features which appear in the potential cones of [BK06] and the Gelfand-Cetlin theory. As a result, applications are restricted in scope to regular coadjoint orbits (or regular multiplicity-free spaces, more generally). My first research goal is to remedy this situation.

The first feature which needs to be accounted for is the boundary of the cone C_σ. For each open face $\tau \subset \mathfrak{t}^*_+$ of the positive Weyl chamber, let $\mathfrak{k}^*_\tau = \sqcup_{\lambda\in\tau}\mathcal{O}_\lambda$ be the Poisson manifold of coadjoint orbits through τ. The manifold $GW_s(\mathfrak{k}^*_\tau) \subset K^*$ is independent of s.

The present construction does not give enough control of the Poisson bracket on this manifold to prove, for instance, an analogue of Theorem 3. However, based on some promising initial computations I expect the following to be true: there is a family of distinguished coordinate systems $\mathbb{R}^n \times (S^1)^k \hookrightarrow GW_s(\mathfrak{k}^*_\tau)$, which are closely related to the cluster algebra structure on a double Bruhat cell $G^{e,u} = B \cap B_- u B_-$. For each such coordinate system, there is a submanifold $\mathring{C} \times (S^1)^k \subset \mathbb{R}^n \times (S^1)^k$ on which the Poisson bracket $s\pi_{K^*}$ is controlled as $s \to \infty$. The cones C are constructed in [BK06], and they are indexed by the open faces of the positive Weyl chamber. Ultimately, the aim is to prove analogues of Theorems 3 and 4 for the manifolds $\mathring{C} \times (S^1)^k$.

An immediate application would be a computation of the Gromov width of all coadjoint orbits of K, generalizing Theorem 7. This would involve the combinatorics of embedding simplices in the cones C.

The second feature is the product operation on coadjoint orbits, on one hand, and on points of the potential cone C_σ, on the other. On the coadjoint orbits side, a classical problem studied by Weyl and Horn asks: given weights λ, ν, η, is the set

$$M_{\lambda,\nu,\eta} = \{(A, B, C) \in \mathcal{O}_\lambda \times \mathcal{O}_\nu \times \mathcal{O}_\eta \mid A + B = C\}$$

nonempty? One may also study the symplectic quotient $M_{\lambda,\nu,\eta}/K$ for the diagonal action of K on $M_{\lambda,\nu,\eta}$.

An analogous problem asks for the multiplicities of irreducible summands in tensor products of K-modules. There is a piecewise-linear multiplication $C_\sigma \times C_\sigma \to C_\sigma$ on cones, which was used by [BZ01] to give an answer to this problem in terms of lattice points of convex polytopes. An elegant solution to the Weyl-Horn problem for the case $K = U(n)$ was given in [KT99] using a new description of the cone C_σ.

Using results of [ABHL18b, BK06, BZ01], one can define a product m^t on any partial tropicalization $\mathring{C}_\sigma \times \mathbb{T}$ of K^*. The *partially tropicalized Weyl-Horn problem* asks when the manifolds

$$M^t_{\lambda,\nu,\eta} = \{(x, y, z) \in \mathcal{P}_\lambda \times \mathcal{P}_\nu \times \mathcal{P}_\eta \mid m^t(x, y) = z\}$$

are nonempty. The aim is to describe the manifolds $M^t_{\lambda,\nu,\eta}$, as well as their symplectic quotients $M^t_{\lambda,\nu,\eta}/K$, using the techniques developed in [AHLL19].

The spaces $M_{\lambda,\nu,\eta}/K$ are of independent interest: for small λ, ν, η they are symplectomorphic to the moduli space of flat $\mathfrak{k}$-connections with fixed holonomy, on a sphere with three holes [AM98]. An application would be to build symplectic embeddings of balls into the manifolds $M_{\lambda,\nu,\eta}/K$.

Toric stacks, quantization, and non-commutative geometry Given the description of the equivariant cohomology and quantization of a toric manifold in terms of its moment polytope, it is natural to ask for analogous statements about toric symplectic stacks. This is my second research goal. Both questions involve careful consideration of the role of the stacky torus $\mathbf{T}$.

On the quantization side, the lattice points of the moment polytope $\mu(M)$ are not the right thing to look at; in general $\mathrm{Hom}(\mathbf{T}, S^1)$ consists of only the trivial character. A promising approach is to use the convolution C^*-algebras associated with Lie groupoids presenting the stacks in question. These are noncommutative algebras which stand in for the algebra of smooth functions on the (typically singular) coarse moduli space of a stack. C^*-algebras also arise when quantizing Hamiltonian G-manifolds with noncompact G [HL08].

The convolution algebra approach to quantization will involve developing the connection between stacks and C^*-algebras, in the spirit of [Lan01]. The group structure of a stacky torus should translate into a *Hopfish algebra* structure on its convolution algebra in [TWZ05]. Previous approaches to Hopfish algebras have been purely algebraic, and as a result C^* algebras of some stacky Lie groups are not quite Hopfish. A more analytical approach may remedy this situation, and this would be a first step toward the goals of the project.

On the cohomology side, for a stacky torus $\mathbf{T}$, the notion of $\mathbf{T}$-equivariant cohomology must be interpreted correctly. One expects that the $\mathbf{T}$-equivariant cohomology of a toric stack should be given as the face ring of the irrational moment polytope. It should be a module over the $\mathbf{T}$-equivariant cohomology of a point.

Outlook and connections with other fields The goals outlined above are part of a program which extends beyond the immediate future. One of the most interesting possible extensions is to understand how partial tropicalizations fit into the story of cluster algebras and mirror symmetry. Given a cluster variety, there is a canonical basis for its coordinate ring parametrized by the integral tropical points of a mirror variety. More generally, given a partial compactification of a cluster variety, a canonical basis is sometimes parametrized by the points of a polyhedral cone in the tropicalized mirror variety [GHKK17]. The cones of [BK06] are examples of this phenomenon [Mag15]. At the same time, the (uncompactified) cluster varieties always carry a compatible Poisson structure [FG09]. Partial tropicalizations are a first step towards extending notion of compatible Poisson structures to partial compactifications of cluster varieties. It would be interesting to find other examples of Poisson manifolds whose Poisson bracket is controlled on a potential cone as in Theorem 3. An account of the partial tropicalization of non-regular coadjoint orbits (as described above) would be a first step in this direction.

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