SPECTRA

(Unstable) Algebraic Topology has lots of rough edges:

- fiber sequences are not cofiber sequences, and cofiber sequences are not fiber sequences
- homotopy $\{\pi_n(-)\}_{n\geq 0}$ is almost, but not quite, a homology theory
- π_0 is not a group, and π_1 is not abelian

self-explanatory

axioms for a homology theory:

- homotopy: $f \simeq g \Rightarrow f_* = g_*$
- additivity: $E_n(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha E_n(X_\alpha)$
- exactness: for a cofiber seq $A \longrightarrow X \longrightarrow Y$, get LES in homology
- excision: if $X = A \cup B$, then $E_*(A/A\cap B) \cong E_*(X/B)$

homotopy has LES for fiber seqs

homotopy almost has excision

Theorem: Let $X = A \cup B$. Suppose that $A/A\cap B$ is m-connected and X/B is n-connected. Then $\pi_i(X/B) \longleftarrow \pi_i(A/A\cap B)$ induced by inclusion is an iso for $i < m+n$ and surjection for $i = m+n$.

Corollary (Freudenthal Suspension): Let X be an n-connected space. Then $\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$ is an iso for $i < 2n$.

Two important consequences:

$\Longrightarrow$ If X is m-connected, then ΣX is $(m+1)$-connected.

$\Longrightarrow$ The sequence of groups

$$\pi_i X \longrightarrow \pi_{i+1} \Sigma X \longrightarrow \pi_{i+2} \Sigma^2 X \longrightarrow \cdots$$

is eventually constant.

Proof: If X is k-connected, then $\Sigma^j X$ is $(k+j)$-connected for j sufficiently large, $2(k+j) > i+j$ and we apply Freudenthal to see that

$$\pi_{i+j}(\Sigma^j X) \cong \pi_{i+j+1}(\Sigma^{j+1}(X)) \cong \cdots$$

Exercise: $\pi_n(S^n) \cong \mathbb{Z}$.

This eventual constant is called the i^{th} stable homotopy group of X.

$$\pi_i^S(X) = \operatorname*{colim}_j \pi_{i+j}(\Sigma^j X)$$

<u>Fact:</u> stable homotopy groups form a homology theory.

So instead of studying π_n, use π_n^S.

Spectra is one step further.

<u>Def:</u> A <u>spectrum</u> X is a sequence of spaces $X_0, X_1, X_2, \_\_\_$ together with maps $\sigma_n : \Sigma X_n \longrightarrow X_{n+1}$.

A morphism of spectra is a sequence of maps $f_i : X_i \to Y_i$ commuting with the σ_n for all n.

The homotopy groups of X are the direct limit of the system

$$\pi_n(X_0) \longrightarrow \pi_{n+1}(\Sigma X_0) \xrightarrow{(\sigma_0)_*} \pi_{n+1}(X_1) \longrightarrow \pi_{n+2}(\Sigma X_1) \xrightarrow{(\sigma_1)_*} \cdots$$

Note that this makes sense for $n < 0$ too!

It is <u>always</u> an abelian group.

Examples:

Sphere spectrum $\mathbb{S}$

Suspension spectrum $\Sigma^{\infty} X$

Thom spectrum MO w/ n^{th} space $Th(\gamma^n)$

$\pi_n MO$ are cobordism groups of manifolds

Complex K-theory KU is sequence

$$\mathbb{Z} \times BU,\ U,\ \mathbb{Z} \times BU,\ U,\ \ldots$$

$$\pi_i U = \pi_{i+1}(\mathbb{Z} \times BU)$$

$$\pi_{i+2} U = \pi_i U \qquad \text{(Bott Periodicity)}$$

Eilenberg-MacLane spectrum HA

w/ n^{th} space $HA_n = K(A, n)$

Where do these examples come from?

K-theory, cobordism, etc

__Theorem (Brown Representability)__: Suppose that a sequence of functors $h^n : \text{Spaces} \longrightarrow \text{Ab}$ is a generalized cohomology theory. Then there is a spectrum E such that $h^n(X) = [X, E_n]$.

Ω-spectrum

Conversely, every Ω-spectrum E defines a generalized cohomology theory $E^n(X) := [X, E_n]$.

Example: $H^n(X;A) = [X, K(A,n)]$

The Eilenberg-Maclane spectrum represents ordinary cohomology.

Next time:

- the stable homotopy category
- algebra using spectra: rings, modules, smash product

SPECTRA

Unstable Algebraic topology has a lot of rough edges:

- π_0 is not a group, π_1 is not abelian
- fiber sequences are not cofiber sequences
  cofiber seqs are not fiber seqs

  Algebra: SES $0 \to A \to B \to C \to 0$
- homotopy is almost, but not quite a homology theory

  axioms:
  - homotopy: $f \simeq g \Rightarrow f_* = g_*$
  - additivity: $E_n(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha E_n(X_\alpha)$
  - exactness: for a cofiber seq
    $$A \to X \to C$$
    get LES in homology
  - excision: if $X = A \cup B$, then
    $$E_*(A/A \cap B) \cong E_*(X/B)$$

homotopy has LES for fiber seqs

homotopy almost has excision

Theorem: Let $X = A \cup B$.

Excision for htpy

Assume
- $A/A\cap B$ is m-connected $\quad \pi_i(A/A\cap B) = 0$, $i \le m$
- X/B is n-connected $\quad \pi_i(X/B) = 0$, $i \le n$

Then $\pi_i(A/A\cap B) \xrightarrow{i_*} \pi_i(X/B)$

is an isomorphism for $i < m+n$
a surjection for $i = m+n$

Corollary (Freudenthal Suspension): Let X be an n-connected space. Then

$$\begin{array}{ccc} \pi_i(X) & \longrightarrow & \pi_{i+1}(\Sigma X) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ [S^i, X] & \longrightarrow & [S^{i+1}, \Sigma X] \\ f & \longmapsto & \Sigma f \end{array}$$

$X = \Sigma X$
$A = C_+ X$
$B = C_- X$

is an isomorphism for $i < 2n$
a surjection for $i = 2n$

Two important consequences:
- If X is n-connected, ΣX is $(n+1)$-connected
- The sequence of groups

$$\pi_i X \longrightarrow \pi_{i+1} \Sigma X \longrightarrow \pi_{i+2} \Sigma^2 X \longrightarrow \cdots$$

is eventually constant.

Proof: X k-connected, then $\Sigma^j X$ is $(k+j)$-connected

the connectivity grows with j

the bound grows with $2j$

for $j \gg 0$, $\pi_{i+j}(\Sigma^j X) \longrightarrow \pi_{i+j+1} \Sigma^{j+1} X$ is iso. ∎

Def: the ith stable homotopy group of X is this eventual constant.

$$\pi_i^s(X) := \operatorname{colim}_j \pi_{i+j} \Sigma^j X$$

Note that this makes sense for $i < 0$ too

Fact: stable homotopy groups form a homology theory

$$\pi_{-2}^s X := \operatorname{colim}\left(\pi_0 \Sigma^2 X \longrightarrow \pi_1 \Sigma^3 X \longrightarrow \pi_2 \Sigma^4 X \longrightarrow \cdots \right)$$

use the convention that $\pi_{-i} X = 0$ for $i \geq 0$.

Instead of studying π_n, study π_n^s

Effectively, instead of studying a space X, study $\{\Sigma^i X\}_{i\geq 0}$

<u>Def</u>: A sequential spectrum X is a sequence $X_0, X_1, X_2, \_\_\_\_$ of based spaces together with maps $\Sigma X_i \xrightarrow{\sigma_i} X_{i+1}$.

The homotopy groups of a spectrum are

$$\pi_i X = \operatorname{colim}\Big(\pi_n X_0 \xrightarrow{\Sigma} \pi_{n+1} \Sigma X_0 \xrightarrow{(\sigma_0)_*} \pi_{n+1} X_1 \xrightarrow{\Sigma} \pi_{n+2} \Sigma X_1 \xrightarrow{(\sigma_1)_*} \pi_{n+2} X_2 \xrightarrow{\Sigma} \cdots \Big)$$

"stable"

spectrum

Note that this makes sense for $i < 0$ as well

$\pi_i X$ is <u>always</u> an abelian group

Examples:

Sphere spectrum $\mathbb{S}$

$S^0, S^1, S^2, S^3,$ ____

$\sigma_i : \Sigma S^i \longrightarrow S^{i+1}$ is identity

Suspension spectrum $\Sigma^\infty X$ ← some space X

$X, \Sigma X, \Sigma^2 X, \Sigma^3 X,$ ____

$\sigma_i = id$

K^* Complex topological K-theory KU

U is infinite unitary group

$\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U,$ ____

Bott periodicity: $\pi_{i+2} U = \pi_i U$

always: $\pi_i U = \pi_{i+1}(\mathbb{Z} \times BU)$

cobordism Thom Spectrum MO w/ n^{th} space $Th(\gamma^n \to Gr_n(\mathbb{R}^\infty))$

$\pi_n MO$ is the nth cobordism group of real manifold

Eilenberg-MacLane spectra HA $\qquad H^n(-;A)$

if A is an abelian group, then HA is spectrum w/ n^{th} space $K(A,n)$

$$\pi_m K(A,n) = \begin{cases} A & m=n \\ 0 & \text{else} \end{cases}$$

$$\Sigma K(A,n) \simeq K(A,n+1)$$

$\sigma_i \simeq \mathrm{id}$

$$\pi_i HA = \begin{cases} A & i=0 \\ 0 & \text{else} \end{cases}$$ (exercise)

Exercise: prove that $\pi_0 \mathbb{S} \cong \mathbb{Z}$

$\pi_n S^n \cong \mathbb{Z}$

Where do the examples come from?

Theorem: (Brown Representability)

If a sequence of functors $h^n : \text{Spaces} \longrightarrow Ab$ is a generalized cohomology theory, then there exists a spectrum E such that
Ω-spectrum

$$h^n(X) \cong [X, E_n] = \mathrm{Map}_*(X, E_n)$$

Def: A spectrum is an Ω-spectrum if the adjoints (under $\Sigma \dashv \Omega$) to σ_i are htpy equivalences

$$\Sigma X_i \xrightarrow{\sigma_i} X_{i+1} \rightsquigarrow X_i \xrightarrow{\tilde{\sigma}_i} \Omega X_{i+1}$$

$$\Sigma S^i \xrightarrow{id} S^{i+1} \qquad S^i \longrightarrow \Omega S^{i+1}$$

htpy equivalence

Any Ω-spectrum E yields a generalized cohomology theory

$$E^n X = [X, E_n]$$

(exercise)

Next time:

- stable htpy category
- algebra w/ spectra: rings, modules, smash product, etc
- equivariant spectra (?)

$\pi_j \Omega S^{i+1} = \pi_{j+1} S^{i+1} \simeq \pi_j S^i$

only equal in the stable range

$$
\begin{tikzcd}
\mathrm{Spaces} \arrow[r, bend left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & \mathrm{Spectra} \arrow[l, bend left, "\Omega^\infty"] & \mathrm{Ab} \arrow[l, bend right, "H"']
\end{tikzcd}
$$

$$H^*(X) = [\Sigma^\infty X, HA]_*$$

$$
\begin{array}{ccc}
H^n(X) & \cong & H^{n+1}(\Sigma X) \\
\wr\wr & & \wr\wr \\
[X, K(A,n)] & & [\Sigma X, K(A,n+1)] \\
\wr\wr & & \\
[\Sigma^\infty X, \Sigma^n HA] & &
\end{array}
$$