Bar Construction

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CHAPTER 1

Categorical Construction

The bar construction is a very useful tool to compute cohomology for general algebraic theories (groups, algebras, Lie algebras, etc), as well as for other things like constructing classifying bundles. Given any monad M and an M-algebra X, the bar construction is an efficient machine that produces a 'free resolution' of X, to which one can then apply the machinery of derived functors to compute the cohomology of X.[1]

1. Monad

Definition 1.1. A **monad** on a category $\mathcal{C}$ consists of

- an endofunctor $T : \mathcal{C} \to \mathcal{C}$,
- a **unit** natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow T$, and
- a **multiplication** natural transformation $\mu : T^2 \Rightarrow T$,

so that the following diagrams commute in $\mathcal{C}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

$$\begin{array}{ccc} T^3 & \xRightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \xRightarrow{\mu} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xRightarrow{\eta T} & T^2 & \xLeftarrow{T\eta} & T \\ & {\scriptstyle 1_T}\searrow & \Downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

REMARK. A monad on $\mathcal{C}$ is precisely a monoid in the monoidal category $\mathcal{C}^{\mathcal{C}}$ of endofunctors on $\mathcal{C}$, where the binary functor $\circ : \mathcal{C}^{\mathcal{C}} \times \mathcal{C}^{\mathcal{C}} \to \mathcal{C}^{\mathcal{C}}$ is composition, and the unit object is the identity endofunctor $1_{\mathcal{C}} \in \mathcal{C}^{\mathcal{C}}$.

1.1. Motivation Example: Adjunction. One source of intuition is that a monad is the "shadow" cast by an adjunction on the category appearing as the codomain of the right adjoint. Consider the adjunction

$$\mathbf{Set} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{Ab}, \quad \eta : 1_{\mathbf{Set}} \Rightarrow UF, \quad \epsilon : FU \Rightarrow 1_{\mathbf{Ab}}$$

and suppose we have forgotten entirely about the category of abelian groups. What structure remains visible on the category of sets? First, there is an endofunctor, UF, which sends a set to the set of finite formal sums of elements with integer coefficients. There is also the natural transformation η, the "unit map" whose component $\eta_S : S \to UFS$ sends an element of the set S to the corresponding singleton sum. However, the "evaluation map" ϵ, whose component $\epsilon_A : FUA \to A$ is the group homomorphism that evaluates a finite formal sum of elements of the abelian group A to its actual sum in A, is not directly visible. There is, however, a related natural transformation $(U\epsilon F)_S : UFUFS \to UFS$ between endofunctors of **Set**, which is the evaluation map regarded as a function of sets of finite formal sum, not a group homomorphism, and considered only in the special case of free abelian groups.

[1] A wonderful notes on what is bar construction and how to understand it is here.

For any adjunction, this triple of data defines a monad:

Lemma 1.2. *Any adjunction*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}, \quad \eta : 1_{\mathcal{C}} \Rightarrow UF, \quad \epsilon : FU \Rightarrow 1_{\mathcal{D}}$$

gives rise to a monad on the category $\mathcal{C}$ *serving as the domain of the left adjoint, with*

- *the endofunctor* T *defined to be* UF,
- *the unit* $\eta : 1_{\mathcal{C}} \Rightarrow UF$ *serving as the unit* $\eta : 1_{\mathcal{C}} \Rightarrow T$ *or the monad, and*
- *the whiskered counit* $U\epsilon F : UFUF \Rightarrow UF$ *serving as the multiplication* $\mu : T^2 \Rightarrow T$ *for the monad.*

Example 1.3. (1) The **free monoid monad** is induced by the free $\dashv$ forgetful adjunction between monoids and sets. The endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ is defined by

$$TA = \coprod_{n \geq 0} A^n$$

that is, TA is the set of finite lists of elements in A; in computer science contexts, this monad is often called the **list monad**. The components of the unit $\eta_A : A \to TA$ are defined by the coproduct inclusions, sending each element of A to the corresponding singleton list. The components of the multiplication $\mu_A : T^2 A \to TA$ are the concatenation functions, sending a list of lists to the composite list. In general, the free monoid monad can also be defined in any monoidal category with coproducts that distribute over the monoidal product.

(2) The free $\dashv$ forgetful adjunction between sets and the category of R-modules induces the **free R-module monad** $R[-] : \mathbf{Set} \to \mathbf{Set}$. Define $R[A]$ to be the set of finite formal R-linear combinations of elements of A. Formally, a finite R-linear combination is a finitely supported function $\chi : A \to R$, meaning a function for which only finitely many elements of its domain take non-zero values. Such a function might be written as $\sum_{a \in A} \chi(a) \cdot a$. The components $\eta_A : A \to R[A]$ of the unit send an element $a \in A$ to the singleton formal R-linear combination corresponding to the function $\chi_a : A \to R$ that sends a to $1 \in R$ and every other element to zero. The components $\mu_A : R[R[A]] \to R[A]$ of the multiplication are defined by distributing the coefficients in a formal sum of formal sums. Special cases of interest include the **free abelian group monad** and the **free vector space monad**.

2. Comonad

Definition 2.1. A **comonad** on $\mathcal{C}$ is a monad on $\mathcal{C}^{op}$: explicitly, a comonad consists of an endofunctor $K : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\epsilon : K \Rightarrow 1_{\mathcal{C}}$ and $\delta : K \Rightarrow K^2$ so that the diagrams dual to Definition 1.1 commute in $\mathcal{C}^{\mathcal{C}}$.

A **comonad** on a category $\mathcal{D}$ consists of

- an endofunctor $K : \mathcal{D} \to \mathcal{D}$,
- a **counit** natural transformation $\epsilon : K \Rightarrow 1_{\mathcal{D}}$, and
- a **comultiplication** natural transformation $\delta : K \Rightarrow K^2$,

so that the following diagrams commute in $\mathcal{D}^{\mathcal{D}} = \mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

$$\begin{array}{ccc} K^3 & \xLeftarrow{K\delta} & K^2 \\ {\scriptstyle \delta K}\Uparrow & & \Uparrow{\scriptstyle \delta} \\ K^2 & \xLeftarrow[\delta]{} & K \end{array} \qquad\qquad \begin{array}{ccccc} K & \xLeftarrow{K\epsilon} & K^2 & \xRightarrow{\epsilon K} & K \\ & {\scriptstyle 1_K}\nwarrow\!\!\!\!\nwarrow & \Uparrow{\scriptstyle \delta} & \nearrow\!\!\!\!\nearrow{\scriptstyle 1_K} & \\ & & K & & \end{array}$$

Similarly we have

Lemma 2.2. *Any adjunction*

$$\mathcal{C} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}, \quad \eta : 1_{\mathcal{C}} \Rightarrow UF, \quad \epsilon : FU \Rightarrow 1_{\mathcal{D}}$$

gives rise to a comonad on the category $\mathcal{D}$ serving as the domain of the right adjoint, with

- *the endofunctor T defined to be UF,*
- *the unit $\eta : 1_{\mathcal{C}} \Rightarrow UF$ serving as the unit $\eta : 1_{\mathcal{C}} \Rightarrow T$ or the monad, and*
- *the whiskered counit $U\epsilon F : UFUF \Rightarrow UF$ serving as the multiplication $\mu : T^2 \Rightarrow T$ for the monad.*

3. Algebra Over A Monad

Definition 3.1. Let $\mathcal{C}$ be a category with a monad (T, η, μ). The **Eilenberg-Moore category** for T or the **category of T-algebras** is the category $\mathcal{C}^T$ whose:

- objects are pairs $(A \in Ob(\mathcal{C}), a : TA \to A)$, called T-algebras, so that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} T^2A & \xrightarrow{\mu_A} & TA \\ {\scriptstyle Ta}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$

  commutes in $\mathcal{C}$, and
- morphisms $f : (A, a) \to (A, a')$ are T-algebra homomorphisms: maps $f : A \to A$ in $\mathcal{C}$ so that the square

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TA' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

  commutes, with composition and identities as in $\mathcal{C}$.

Lemma 3.2. *For any monad (T, η, μ) acting on a category $\mathcal{C}$, there is an adjunction*

$$\mathcal{C} \underset{U^T}{\overset{F^T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{C}^T,$$

between $\mathcal{C}$ and the Eilenberg-Moore category whose induced monad is (T, η, μ).

Proof. The functor $U^T : \mathcal{C}^T \to \mathcal{C}$ is the forgetful functor. The functor $F^T : \mathcal{C} \to \mathcal{C}^T$ carries an object A in $\mathcal{C}$ to the free T-algebra

$$(TA, \mu_A : T^2A \to TA)$$

and carries a morphism $f : A \to A'$ to the free T-algebra morphism

$$F^T f : TA \xrightarrow{Tf} TB.$$

Note that $U^T F^T = T$.

The unit of the adjunction $F^T \dashv U^T$ is given by the natural transformation $\eta : 1_{\mathcal{C}} \Rightarrow T$. The counit $\epsilon : F^T U^T \Rightarrow 1_{\mathcal{C}^T}$ is defined as follows:

$$\epsilon_{(A,a)} : (TA, \mu_A) \xrightarrow{a} (A, a), \quad \begin{array}{ccc} T^2A & \xrightarrow{Ta} & TA \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ TA & \xrightarrow[a]{} & A \end{array}$$

Note, in particular, that $U^T \epsilon F^T_A = \mu_A$, so that the monad underlying the adjunction $F^T \dashv U^T$ is (T, η, μ). □

4. Coalgebra Over A Comonad

Definition 4.1. Let $\mathcal{D}$ be a category with a comonad (K, ϵ, δ). The **co-Eilenberg-Moore category** for K or the **category of K-coalgebras** is the category $\mathcal{D}^K$ whose:

- objects are pairs $(B \in Ob(\mathcal{D}), b : B \to KB)$, called K-coalgebras, so that the diagrams

$$\begin{array}{ccc} B & \xleftarrow{\epsilon_B} & KB \\ & {\scriptstyle 1_B}\nwarrow & \uparrow{\scriptstyle b} \\ & & B \end{array} \qquad \begin{array}{ccc} K^2B & \xleftarrow{\delta_B} & KB \\ {\scriptstyle Kb}\uparrow & & \uparrow{\scriptstyle b} \\ KB & \xleftarrow[b]{} & B \end{array}$$

  commutes in $\mathcal{D}$, and
- morphisms $g : (B, b) \to (B', b')$ are K-algebra homomorphisms: maps $g : B \to B'$ in $\mathcal{D}$ so that the square

$$\begin{array}{ccc} KB & \xrightarrow{Kg} & KB' \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle b'} \\ B & \xrightarrow[g]{} & B' \end{array}$$

  commutes, with composition and identities as in $\mathcal{D}$.

Remark. An (c)algebra over a (c)monad is a special case of a (c0)module over a (co)monad in a bicategory.

5. Bar Construction

5.1. (Augmented) Simplicial Objects. A **simplicial object** in a category $\mathcal{C}$ is a functor $X : \Delta^{op} \to \mathcal{C}$. An **augmented simplicial object** is a simplicial object $X : \Delta^{op} \to \mathcal{C}$ together an object $X_{-1} \in Ob(\mathcal{C})$ and an arrow $\varepsilon : X_0 \to X_{-1}$ such that $\varepsilon d_0 = \varepsilon d_1 : X_1 \to X_{-1}$.

5.1.1. *Moore Complex.* When $\mathcal{A}$ is an abelian category, a simplicial object S in $\mathcal{A}$ gives homology via a suitable "boundary" operation. We have a chain complex

$$S_0 \xleftarrow{\partial} S_1 \xleftarrow{\partial} S_2 \xleftarrow{\partial} \cdots$$

called Moore complex, where the boundary homomorphism $\partial : S_n \to S_{n-1}$ is defined as the alternating sum $\partial = \sum_{i=0}^{n}(-1)^i d_i$. Then $H_n(S)$ is the n-th homology of S.

Example 5.1 (Singular Homology)**.** Consider the singular chain complex functor

$$S : \mathbf{Top} \longrightarrow \mathbf{sSet}$$

with left adjoint geometric realization

$$\mathbf{sSet} \underset{S}{\overset{|-|}{\rightleftarrows}} \mathbf{Top}$$

We can compose it with the levelwise free abelian group functor

$$S : \mathbf{Top} \longrightarrow \mathbf{sSet} \xrightarrow{\mathbb{Z}} \mathbf{sAb}$$

which assigns to each topoligical space X an (augmented) simplicial object $S = S(X)$, where each S_n is the free abelian group generated by all the n-simplices in X. The associated chain complex is the **singular homology chain complex** of X, with its homology the **singular homology**.

5.2. Bar Construction. Let $\mathcal{C}$ be a category, (T, η, μ) is a monad on $\mathcal{C}$. we have the adjunction

$$\mathcal{C} \underset{U^T}{\overset{F^T}{\rightleftarrows}} \mathcal{C}^T,$$

This in turn gives a comonad FU acting on T-algebras, that is to say, a comonoid in a monoidal category of endofunctors. By the dual of Lemma **??**, there is a unique monoidal functor

$$\mathrm{Bar}_T : \Delta^{op} \to \mathrm{Fun}(\mathcal{C}^T, \mathcal{C}^T)$$

which sends the comonoid [1] in Δ^{op} to FU. This is the **bar construction**.

Applying the functor which evaluates at a T-algebra (A, a), we have an augmented simplicial object

$$\mathrm{Bar}_T(A) = \left[\cdots \longrightarrow T^3A \;\underset{T^2a}{\overset{\mu_{TA}}{\underset{-T\mu_A}{\rightrightarrows}}}\; T^2A \;\underset{Ta}{\overset{\mu_A}{\rightrightarrows}}\; TA \xrightarrow{a} A \right].$$

When $\mathcal{C}$ is an abelian category, there is a chain complex QA associated with $\mathrm{Bar}_T(A)$, with augmentation $a : QA \to A$.

This complex is a standard resolution of A in the sense of homological algebra, and so may be used to construct derived functors; in particular, various cohomology functors.

Example 5.2 (Cohomology of groups)**.** Let $U : \mathbf{Rng} \to \mathrm{Mon}$ be the forgetful functor which forgets the addition. It has a left adjoint $\mathbb{Z} : \mathrm{Mon} \to Rng$ which sends each monoid M to the monoid ring $\mathbb{Z}[M]$. In particular, when $M = G$ is a group, $\mathbb{Z}[G]$ is the group ring.

Let $\mathrm{Mod}(\mathbb{Z}[G])$ be the category of left $\mathbb{Z}[G]$-modules. The forgetful functor $U : \mathrm{Mod}(\mathbb{Z}[G]) \to \mathbf{Ab}$ has a left adjoint

$$F = \mathbb{Z}[G] \otimes - : \mathbf{Ab} \to \mathrm{Mod}(\mathbb{Z}[G])$$

$$B \longmapsto \mathbb{Z}[G] \otimes_{\mathbb{Z}} B$$

with unit

$$\eta = \{\eta_B : B \to \mathbb{Z}[G] \otimes_{\mathbb{Z}} B, b \mapsto 1 \otimes b\}$$

and counit

$$\varepsilon = \{\varepsilon_A : \mathbb{Z}[G] \otimes_{\mathbb{Z}} UA \to A, x \otimes a \mapsto xa\}.$$

The composite $\mathrm{Mod}(\mathbb{Z}[G]) \xrightarrow{U} \mathbf{Ab} \xrightarrow{F} \mathrm{Mod}(\mathbb{Z}[G])$ determines a comonad $\langle L = UF, \varepsilon, \delta = U\eta F\rangle$ in the category $\mathrm{Mod}(\mathbb{Z}[G])$, where

$$\delta_A : LA = \mathbb{Z}[G] \otimes A \to L^2A = \mathbb{Z}[G] \otimes \mathbb{Z}[G] \otimes A$$

$$x \otimes a \longmapsto x \otimes 1 \otimes a$$

Take $A = \mathbb{Z}$ viewed as a trivial $\mathbb{Z}[G]$-module, the simplicial object is

$$\mathrm{Bar}_L(\mathbb{Z}) = \Big[\ \mathbb{Z}[G] \leftleftarrows \mathbb{Z}[G]^{\otimes 2} \leftleftarrows \mathbb{Z}[G]^{\otimes 3} \cdots\ \Big]$$

with face maps $d_i : \mathbb{Z}[G]^{\otimes(n+1)} \to \mathbb{Z}[G]^{\otimes n}$ given by

$$d_i(x[x_1|\cdots|x_n]) = \begin{cases} xx_1[x_2|\cdots|x_n], & i = 0, \\ x[x_1|\cdots|x_ix_{i+1}|\cdots|x_n], & 0 < i < n, \\ x[x_1|\cdots|x_{n-1}], & i = n. \end{cases}$$

and degeneracy maps $s_i : \mathbb{Z}[G]^{\otimes n} \to \mathbb{Z}[G]^{\otimes(n+1)}$ given by

$$s_i(x[x_1|\cdots|x_n]) = x[x_1|\cdots|x_{i-1}|1|x_i|\cdots|x_n].$$

This (augmented) simplicial object determines an augmented chain complex in $\mathrm{Mod}(\mathbb{Z}[G])$

$$\mathbb{Z} \longleftarrow \mathbb{Z}[G] \longleftarrow \mathbb{Z}[G]^{\otimes 2} \longleftarrow \cdots \longleftarrow \mathbb{Z}[G]^{\otimes n} \longleftarrow \cdots$$

which is a free resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$.

The cohomology of G is obtained from the resolution as follows. By applying the functor $\mathrm{Hom}_G(-, A) : \mathrm{Mod}(\mathbb{Z}[G])^{op} \to \mathbf{Ab}$ to the chain complex (dropping the augmentation) we get a cochain complex

$$\mathrm{Hom}_G(\mathbb{Z}[G], A) \longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G]^{\otimes 2}, A) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_G(\mathbb{Z}[G]^{\otimes n}, A) \longrightarrow \cdots$$

The cohomology groups of this complex are the cohomology groups $H^n(G, A)$ of the group G with coefficients in A.

For example,

$$H^0(G, A) = \{a \in A | xa = a, \forall x \in G\} = A^G,$$

and $H^1(G, A)$ is the group of crossed homomorphisms $G \to A$ modulo the principal crossed homomorphisms, and $H^2(G, A)$ is the group of all group extensions of the additive group A by the multiplicative group G, with operation (conjugation) given by the $\mathbb{Z}[G]$-module structure of A.

The homology of G with coefficients in a right $\mathbb{Z}[G]$-module C can be constructed in a similar way. Apply the contravariant functor $C \otimes_{\mathbb{Z}[G]} (-) : \mathrm{Mod}(\mathbb{Z}[G]) \to \mathbf{Ab}$ to the chain complex, we get a chain complex in $\mathbf{Ab}$, whose homology groups are the homology groups of G with coefficients in C.

In order to see the resolution property of bar construction, we will review some simplicial methods.

6. Décalage and Resolutions

To explain why the bar construction $\mathrm{Bar}_T(A)$ is an acyclic resolution of (the constant simplicial object) A, we recall the fundamental décalage construction.

6.1. The Décalage (or Shift) Functor.

6.1.1. *Definition.* If you take a simplicial set and 'throw away' the last face and degeneracy, and relabel, shifting everything down one 'notch', you get a new simplicial set. This is what is called the **dëcalage** (shift in French) of a simplicial set.

Let Δ_a be the autgmented simplicial category, i.e. the simplex category Δ together with the additional object $[-1]$, the empty set (the initial object of Δ_a). We will write $\mathbf{as}\mathcal{C} = \mathrm{Fun}(\Delta_a, \mathcal{C})$ for the category of augmented simplicial objects in a category $\mathcal{C}$, which we will assume to be complete and cocomplete.

Δ_a is a monoidal category with unit $[-1]$ under the operation of ordinal sum, which operation we will denote by σ, if $[m], [n] \in Ob(\Delta_a)$, then $\sigma([m], [n]) = [m + n + 1]$, and the operation σ gives rise to a bifunctor

$$\sigma : \Delta_a \times \Delta_a \to \Delta_a$$

which sends a morphism $(\alpha, \beta) : ([m], [n]) \to [m'], [n'])$ in $\Delta_a \times \Delta_a$ to the morphism $\alpha(\alpha, \beta)$ defined by

$$\alpha(\alpha, \beta)(i) = \begin{cases} \alpha(i), & 0 \le i \le n, \\ \beta(i - m - 1) + m' + 1, & n < i \le m + n + 1. \end{cases}$$

Note (Δ_a, σ) is not a symmetric monoidal category.

The monoidal structure on Δ_a allows us to define a functor $\sigma(-, [0]) : \Delta_a \to \Delta$ which sends $[n] \in Ob(\Delta_a)$ to $\sigma([n], [0]) = [n + 1] \in Ob(\Delta)$.

Definition 6.1 ([I]). Define $\mathrm{Dec}_0 : \mathbf{s}\mathcal{C} \to \mathbf{as}\mathcal{C}$ to be the functor given by restriction along $\sigma_0 = \sigma(-, [0]) : \Delta_a \to \Delta$, so that if X is a simplicial object in $\mathcal{C}$ then $\mathrm{Dec}_0 X$ is the augmented simplicial object obtained by shifting every dimension down by one, 'forgetting' the last face and degeneracy of X in each dimension:

- $\mathrm{Dec}_0 X_n \coloneqq X_{n+1}$,
- $d_k^{n, \mathrm{Dec}_0 X} \coloneqq d_k^{n+1}$,
- $s_k^{n, \mathrm{Dec}_0 X} \coloneqq s_k^{n+1}$.

Thus the augmented simplicial object $\mathrm{Dec}_0 X$ can be pictured as

$$X_0 \xleftarrow{d_0} X_1 \; \underset{d_0}{\overset{d_1}{\leftleftarrows}}\!\!\xrightarrow{s_0} \; X_2 \; \underset{d_0}{\overset{d_2}{\leftarrow}} \; \xleftarrow{d_1} \; \rightrightarrows \; X_3 \cdots$$

Note that the simplicial identity $d_0 d_1 = d_0 d_0$ shows that $d_0 : X_1 \to X_0$ is an augmentation.

There is an analogous functor $\mathrm{Dec}^0 : \mathbf{s}\mathcal{C} \to \mathbf{as}\mathcal{C}$ given by restriction along the functor $\sigma([0], -) : \Delta_a \to \Delta$ thus Dec^0 is the functor which forgets the bottom face and degeneracy map at each level.

The functors Dec_0 and Dec^0 are usually called the **décalage** or **shifting** functors. More generally we can define functors $\mathrm{Dec}^n : \mathbf{s}\mathcal{C} \to \mathbf{as}\mathcal{C}$ and $\mathrm{Dec}^n : \mathbf{s}\mathcal{C} \to \mathbf{as}\mathcal{C}$ induced by restriction along $\sigma(-, [n]) : \Delta_a \to \Delta$.

The relation between Dec_n and Dec^n can be easily understood through the notion of the opposite simplicial object.

$$(\mathrm{Dec}_n X)^{op} = \mathrm{Dec}^n(X^{op}).$$

There are canonical comonads underlying the functors Dec_0 and Dec^0, when these functors are thought of as endofunctors on $\mathbf{s}\mathcal{C}$ by forgetting augmentations. And we see that the functors Dec_n and Dec^n (also thought of as endofunctors on $\mathbf{s}\mathcal{C}$) are given by $\mathrm{Dec}_n = (Dec_0)^n$ and $\mathrm{Dec}^n = (\mathrm{Dec}^0)^n$ respectively.

6.2. Path Object. Décalage is essentially a kind of path space construction, i.e., in the case $\mathcal{C} = \mathbf{Set}$ it is a simplicial sets analogue of a topological pullback

$$\begin{array}{ccccc} PX & \longrightarrow & X^I & \xrightarrow{ev_1} & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle ev_0} & & \\ |X| & \xrightarrow{\mathrm{id}} & X & & \end{array}$$

where $\mathrm{id} : |X| \to X$ is the identity inclusion of the underlying set with the discrete topology. PX is essentially a sum of spaces of based paths $\alpha : (I, 0) \to (X, x_0)$ over all possible choices of basepoint x_0, fibered over X by taking α to $\alpha(1)$. Each space of based paths is contractible and therefore PX is acyclic.

Definition 6.2. An **acyclic structure** on a simplicial object X is a P-coalgebra structure $X \to \mathrm{Dec}_0(X)$.

A Dec_0-coalgebra structure on X is the same as a right σ_0-coalgebra (or σ_0-comodule) structure, given by a simplicial map $h : X \to X \circ \sigma_0$ satisfying certain equations. Explicitly, it consists of a series of maps $h_n : X([n]) \to X([n+1])$ satisfying suitable equations.

The map $h : X \to X \circ \sigma_0$ may be viewed as a homotopy. The coalgebra structure $h : X \to \mathrm{Dec}_0(X)$ has a retraction given by the counit $\epsilon :: \mathrm{Dec}_0 X \to X$, so X becomes a retract of an acyclic space, hence acyclic itself.

In our case, there is a homotopy

$$h : U\,\mathrm{Bar}_T \xrightarrow{\eta_{U\,\mathrm{Bar}_T}} TU\,\mathrm{Bar}_T = U\,\mathrm{Bar}_T\,D$$

which is an acyclic structure, i.e., a right σ_0-coalgebra structure. Thus $U\,\mathrm{Bar}_T$ is acyclic.

Next we will check directly that $U\,\mathrm{Bar}_T(A)$ is acyclic by direct computation. In order to do that, we will use a standard homological algebra trick. Explicitly, we will forget the augmentation of $U\,\mathrm{Bar}_T(A)$, and consider $A = U\,\mathrm{Bar}_T(A)[-1]$ as a constant simplicial object, and show the natural map $U\,\mathrm{Bar}_T(A) \to A$ is a homotopy equivalence.

6.3. Homotopy of Simplicial Maps.

Definition 6.3 ([**M1**],Definition 5.1)**.** Let $f.g : K \to L$ be simplicial maps between simplicial sets. Then f is homotopic to g, written $f \simeq g$, if there exists $h_i : K_n \to L_{n+1}$ for $0 \le i \le n$, satisfying

(1) $\partial_0 h_0 = f$, $\partial_{n+1} h_n = g$,
(2) $\partial_i h_j = h_{j-1}\partial_i$ for $i < j$, and $\partial_{i+1}h_{i+1} = \partial_{i+1}h_i$, and $\partial_i h_j = h_j \partial_{i-1}$ for $i > j+1$.
(3) $s_i h_j = h_{j+1} s_i$ for $i \le j$ and $s_i h_j = h_j s_{i-1}$ for $i > j$.

h is called a homotopy from f to g.

Proposition 6.4 ([M1], Proposition 6.2). *Let $f.g : K \to L$ be simplicial maps between simplicial sets. Then $f \simeq g$ if and only if there is a simplicial map $H : K \times \Delta[1] \to L$ such that*

- $H(x, 0) = g(x), \forall x \in X$, *and*
- $H(x, 1) = f(x), \forall x \in X$.

Proposition 6.5 ([M1], Corollary 6.11). *Homotopy is an equivalence relation on maps into Kan complexes.*

6.4. Contractibility of the Décalage Functor. It is an important fact that $\mathrm{Dec}_0 X$ and $\mathrm{Dec}^0 X$ are not just augmented simplicial objects, they are actually contractible augmented simplicial objects in the following sense.

Definition 6.6. Let $\epsilon : X \to X_{-1}$ be an augmented simplicial object in $\mathcal{C}$. The augmentation map σ is a **deformation retraction** if there exists a simplicial map $s : X_{-1} \to X$ (with X_{-1} is regarded as a constant simplicial object) which is a section of the projection ϵ and is such that $s\epsilon$ is simplicially homotopic to the identity map on X.

A sufficient condition for $s\epsilon$ to be simplicially homotopic to the identity map on X is that there exist for each $n \ge -1$, maps $s_{n+1} : X_n \to X_{n+1}$ with $s_0 = s$, which act as 'extra degeneracies on the right' in the sense that the following identities hold:

$$\begin{cases} d_i s_{n+1} = s_n d_i, & 0 \le i \le n, \\ d_{n+1} s_{n+1} = \mathrm{id}, \\ s_i s_n = s_{n+1} s_i, \end{cases}$$

Given the data of such a collection of maps s_{n+1} as above, we define maps $h_i : X_n \to X_{n+1}$ by the formula

$$h_i = s_0^{n-i} s_{n+1} d_0^{n-i}.$$

The h_i then piece together to define a simplicial homotopy $h : X \times \Delta[1] \to X$ from $s\epsilon$ to the identity on X.

Lemma 6.7. *Let $\sigma : X \to X_{-1}$ be a contractible augmented simplicial object in $\mathcal{C}$. Then there is a simplicial homotopy $h : X \otimes \Delta[1] \to X$ in $\mathbf{s}\mathcal{C}$ between $S\epsilon$ and 1_X.*

Lemma 6.8. *For any simplicial object X in C , the augmentation $d_0 : \mathrm{Dec}_0 X \to X_0$ is a deformation retract. An analogous statement is true for $\mathrm{Dec}^0 X$.*

A prime example where simplicial objects with extra degeneracies appear is in the construction of simplicial comonadic resolutions. Suppose that L is a comonad on a category $\mathcal{C}$, and X is an object of $\mathcal{C}$. Then L determines an augmented simplicial object $L_* X$ whose object of n-simplices is $L^n X$ and whose face and degeneracy maps are defined by

$$d_i = L^i \varepsilon L^{n-i}, s_j = L^i \delta L^{n-i-1}.$$

respectively, where $\varepsilon : L \to 1$ denotes the counit and $\delta : L \to L^2$ denotes the comultiplication of the comonad. Suppose that there exists a section $s : A \to LA$ of the counit $a : LA \to A$.

Then σ determines extra degeneracies $s_{n+1} : L^n X \to L^{n+1} X$ given by $s_{n+1} = L^n \sigma$. It follows from the discussion above that there is a simplicial homotopy $h : L_* X \times \Delta[1] \to L_* X$ in $\mathbf{s}\mathcal{C}$ between $s\varepsilon$ and the identity on $L_* X$. In the case of bar construction, $\eta_A : A \to TA$ gives a desired section, so the augmentation is in fact a homotopy equivalence.

7. Two-sided Bar Construction

7.1. Left and Right Modules. Let $\mathcal{B}$ be a 2-category, and let $M : B \to B$ be a monad with multiplication $m : M^2 \to M$ and unit $u : 1_B \to M$.

A **left module** over M consists of a 1-cell $X : A \to B$ and a 2-cell $\alpha : MX \to X$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{uX} & MX \\ & {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle \alpha} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} M^2X & \xrightarrow{mX} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array}$$

commute.

A **right module** over M consists of a 1-cell $Y : B \to C$ and a 2-cell $\beta : YM \to Y$ such that the diagrams

$$\begin{array}{ccc} Y & \xrightarrow{Yu} & YM \\ & {\scriptstyle 1_Y}\searrow & \downarrow{\scriptstyle \beta} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} YM^2 & \xrightarrow{Ym} & YM \\ {\scriptstyle \beta M}\downarrow & & \downarrow{\scriptstyle \beta} \\ MX & \xrightarrow[\beta]{} & X \end{array}$$

commute.

7.2. Two-sided Bar Construction. Suppose given a 2-category $\mathcal{B}$ together with a monad$M : B \to B$ in $\mathcal{B}$, together with a left module $X : A \to B$ and a right module $Y : B \to C$. There is a unique 2-functor $\mathcal{I} \to \mathcal{B}$ which preserves the monad and module structures, and this induces a functor $\Delta^{op} = \mathcal{I}(0, 2) \longrightarrow \mathcal{B}(A, C)$ This functor is the **two-sided bar construction**, denoted $B(Y, M, X)$.

The structure of the two-sided bar construction may be given more concretely as follows:

- The n-dimensional component of $B(Y, M, X)$ is

$$B(Y, M.X)_n = YM^nX$$

- The $n + 2$ face maps $d_i^n : YM^{n+1}X \to YM^nX$

$$d_i^n = \begin{cases} \beta M^n X, & i = 0, \\ YM^{i-1} m M^{n-i} X, & 1 \le i \le n, \\ YM^n \alpha, & i = n+1. \end{cases}$$

- The $n + 1$ degeneracy maps $s_i^n : YM^nX \to YM^{n+1}X$ are $YM^i u M^{n-i} X, 0 \le i \le n$.

7.3. Classifying bundle. Consider the cartesian monoidal category **Top** as a 1-object bicategory Σ**Top** (which we may strictify to a 2-category). A topological monoid M is the same as a monad in Σ**Top**, and the usual meaning of left and right M-modules is preserved by thinking of them as modules over the monad.

In particular, M may be regarded as a left or right M-module, and the 1-point space $*$ carries a unique structure of left or right M-module. As a result we may consider the simplicial space

$$BM = B(*, M, *)$$

as base space, and the simplicial space

$$EM = B(M, M, *)$$

as total space, of a simplicial fibration

$$B(\pi, M, *) : B(M, M, *) \to B(*, M, *)$$

induced by the unique left module map $\pi : M \to *$. This is the classifying bundle of the monoid M.

7.4. Cofibrant replacement. If T is a monad and $(A, a : TA \to A)$ is a (left-sided) T-algebra, then with T acting upon itself on the right, there is a simplicial object $B(T, T, A)$ which may be regarded as a cofibrant replacement of A, a simplicial T-algebra which as a simplicial object is homotopy-equivalent to the constant simplicial object at A.

7.5. Canonical two-sided bar construction of an adjunction. Suppose given any adjoint pair

$$A \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} B, \quad \eta : 1_{\mathcal{C}} \Rightarrow UF, \quad \epsilon : FU \Rightarrow 1_{\mathcal{D}}$$

in a 2-category $\mathcal{B}$. There is an associated monad $M = UF : B \to B$, and a canonical left M-action on U:

$$\alpha = U\epsilon : UFU \Rightarrow U$$

and a canonical right M-action on F:

$$\beta = \epsilon F : FUF \to F.$$

We may then form the canonical simplicial object $B(F, M, U)$. By general abstract nonsense, the tensor product $F \bigotimes_M U$ is 1_A, so if we regard 1_A as a constant simplicial object $\Delta^{op} \to \mathcal{B}(A, A)$, the cofibrant replacement result above specializes as follows.

Proposition 7.1. *The canonical simplicial map $B(F, M, U) \to 1_A$ is a simplicial homotopy equivalence.*

7.6. Homotopy colimits. Suppose that $\mathcal{C}$ is a small category and $F : \mathcal{C} \to$ **Top** is a functor. We may regard $\mathcal{C}$ as a monad $\mathcal{C} : C_0 \to C_0$ in the bicategory of spans in **Top**, where C_0 is the set of objects with the discrete category, and we may regard F as a left module over the monad $\mathcal{C}$.

As always, the terminal object 1 carries a unique right module structure. The usual colimit, $\operatorname{colim} F$, may be described as the tensor product

$$\operatorname{colim} F \cong 1 \circ_{\mathcal{C}} F$$

As a result, we have the cofibrant replacement $B(1, \mathcal{C}, F)$ of $\operatorname{colim} F$. The geometric realization of the simplicial space $B(1, C, F)$ is none other than the homotopy colimit of F.

8. Total Décalage Functor

The ordinal sum map

$$\sigma : \Delta \times \Delta \longrightarrow \Delta$$

induces a functor

$$\mathrm{Dec} : \mathbf{sSet} \longrightarrow \mathbf{ssSet}$$

with $\mathrm{Dec}\, X([m],[n]) = X_{m+n+1}$.

$\mathrm{Dec}\, X$ is both row and column augmented. The row augmentation $\epsilon_r : \mathrm{Dec}\, X \to p_1^* X$ is given by the map $d_{\text{last}} : \mathrm{Dec}_0\, X \to X$, while the column augmentation $\epsilon_c : \mathrm{Dec}\, X \to p_2^* X$ is given by the map $d_{\text{first}} : \mathrm{Dec}^0\, X \to X$.

Suppose that X is a simplicial set, and regard $\mathrm{Dec}\, X$ as a (vertical) simplicial space whose rows are the simplicial sets $Dec_n X$ for $n \geq 0$. Then the functor $p_1^* : \mathbf{sSet} \to \mathbf{ssSet}$ which sends a simplicial set K to the constant simplicial space whose rows are K, has a left adjoint $\pi_0 : \mathbf{ssSet} \to \mathbf{sSet}$.

Lemma 8.1. *For any simplicial set X, we have $\pi_0\, \mathrm{Dec}\, X = X$.*

Dec has both a left and right adjoint. The left adjoint of Dec is related to the notion of the join of simplicial sets. The right adjoint to Dec is denoted $T : \mathbf{ssSet} \to \mathbf{sSet}$, called the **total simplicial set functor**. It has the following explicit description: if X is a bisimplicial set then the set $(TX)_n$ of n-simplices of the simplicial set TX is given by the equalizer of some diagram.

Lemma 8.2. *Let X be a simplicial set. Then there are isomorphisms $Tp_1^* X = Tp_2^* X = X$, natural in X.*

8.1. Kan's Simplicial Loop Group Construction.

Definition 8.3. Let G be a simplicial group. Then $\overline{W}G$ is the simplicial set with a single vertex, and whose set of n-simplices, $n \geq 1$, is given by

$$\overline{W}G_n = G_{n-1} \times \cdots \times G_0$$

with face and degeneracy maps given by

$$d_i(g_{n-1}, \cdots, g_0) = \begin{cases} (g_{n-2}, \cdots, g_0), & i = 0, \\ (d_i g_{n-1}, \cdots, d_1 g_{n-i+1}, g_{n-i} d_0 g_{n-i}, g_{n-i-2} \cdots, g_0) & i > 0. \end{cases}$$

and

$$s(g_{n-1}, \cdots, g_0) = \begin{cases} (1, g_{n-1}, \cdots, g_0) & i = 0, \\ (s_{i-1} g_{n-1}, \cdots, s_0 g_{n-i}, 1, g_{n-i-1}, \cdots, g_0) & i > 0. \end{cases}$$

Let NG denotes the bisimplicial set which, when viewed as a (vertical) simplicial object in **sSet**, has as its object of n-simplices the (horizontal) simplicial set NG_n, i.e. the nerve of the group G_n.

Proposition 8.4. *The classifying complex functor $\overline{W}$ factors as*

$$\overline{W} = TN,$$

so that $\overline{W}G = TNG$ for any simplicial group G.

9. Simplicial Principal Bundle

9.1. Twisting Function. Let $X_\bullet$ be a simplicial set and $G_\bullet$ a simplicial group. Then a **twisting function** $\tau : X_\bullet \to G_\bullet$ is a family of maps $\varphi = \{\tau_n : X_n \to G_{n-1}, n \geq 1\}$ such that

$$\begin{aligned} d_0(\tau(x)) &= \tau(d_1(x))\tau(d_0 x)^{-1}, \\ d_i\tau(x) &= \tau(d_{i+1}x), i > 0, \\ s_i\tau(x) &= \tau(s_{i+1}x), i \geq 0, \\ \tau(s_0 x) &= 1_G. \end{aligned}$$

9.2. Twisted Cartesian Product. Given a simplicial set Y_{bullet} with left $G_\bullet$-action, one then defines a twisted Cartesian product, (TCP), $X_\bullet \times_\tau Y_\bullet$ with

$$(X_\bullet \times_\tau Y_\bullet)_n = X_n \times Y_n$$

and

$$\begin{aligned} d_i(x, f) &= (d_i x, d_i f), i > 0 \\ d_0(x, f) &= (d_0 x, \tau(x) d_0 f), \\ s_i(x, y) &= (s_i x, s_i y). \end{aligned}$$

By the adjunction between W-bar and the Dwyer-Kan loop groupoid functor, a twisting function $\tau : X_\bullet \to G_\bullet$ corresponds exactly to a simplicial map from X to $\overline{W}(G_\bullet)$ delooping of the simplicial group. It also corresponds to a morphism of simplicial groupoids $G(X_\bullet) \to G_\bullet$.

9.3. Simplicial Principal Bundle.

Definition 9.1. Let G be a simplicial group. For E a Kan complex, an action of G on E

$$\rho : E \times G \to E$$

is called **principal** if it is degreewise principal, i.e. if for all $n \in \mathbb{N}$ the only elements $g\ inG_n$ that have any fixed point $e \in E_n$ in that $\rho(e, g) = e$ are the neutral elements.

Definition 9.2. For G a simplicial group, a morphism $E \to X$ of Kan complexes equipped with a G-action on E is called a G-**simplicial principal bundle** if

- the action is principal;
- the base is isomorphic to the quotient

$$E/G := \mathrm{eq}\{\ E \times G \underset{pr_1}{\overset{\rho}{\rightrightarrows}} E\ \}$$

by the G-action, $E/G \simeq X$.

Proposition 9.3 ([M1], Lemma 18.2)**.** *A simplicial G-principal bundle $P \to X$ is necessarily a Kan fibration.*

Proposition 9.4 ([M1], Proposition 18.4)**.** *Let $E \to B$ be a twisted cartesian product of the simplicial set B with a simplicial group G. Then with respect to the canonical G-action this is a simplicial principal bundle.*

9.4. Universal Simplicial G-Principal Bundle.

Definition 9.5. For G a simplicial group, define the simplicial set WG to be the décalage of $\overline{W}G$

$$WG := \mathrm{Dec}_0 \overline{W}G.$$

For $X_\bullet$ any Kan complex, there is an ordinary pullback diagram

$$\begin{array}{ccc} P_\bullet & \longrightarrow & WG \\ \downarrow & & \downarrow \\ X_\bullet & \xrightarrow{g} & \overline{W}G \end{array}$$

We call $P_\bullet := X_\bullet \times_g WG$ the simplicial G-principal bundle corresponding to g.

Proposition 9.6. *Let τ be the twisting function corresponding to $g : X_{bullet} \to \overline{W}G$. Then the simplicial set $P_\bullet := X_\bullet \times_g WG$ is explicitly given by the twisted Cartesian product $X_\bullet \times_\tau G_\bullet$.*

CHAPTER 2

Simplicial Classifying Space

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