Outline.

① Monad / Comonad
② (Co) Algebra over a (Co)monad

$\Big\}$ motivation : adjunction.

③ Eilenberg-Moore Cat $\mathcal{C}^T$
⑤ Simplicial Objects   augmentation / Moore complex
⑥ Bar Construction   Group (co)homology / Hochschild (co)homology
⑦ Two-sided Bar construction.

§1 Monad.

Def. A monad on cat $\mathcal{C}$ consists

- endofunctor $T: \mathcal{C} \to \mathcal{C}$
- unit $\eta: 1 \Rightarrow T$
- multiplication $\mu: T^2 \Rightarrow T$

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] \arrow[dr, phantom, "\circlearrowleft"] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Comonad

Def

- endofuncto $K: \mathcal{D} \to \mathcal{D}$
- counit $\varepsilon: K \Rightarrow 1$
- comultiplication $\delta: K \Rightarrow K^2$

Rmk. Monads are monoids in the monoidal cat $\underline{\mathrm{End}(\mathcal{C}, \mathcal{C})}$

Motivation example : adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd} \qquad \eta: 1_{\mathcal{C}} \to UF \qquad \varepsilon: FU \Rightarrow 1_{\mathcal{D}}$$

monad $T = UF: \mathcal{C} \to \mathcal{C}$ comonad. $K = FU: \mathcal{D} \to \mathcal{D}$

unit η counit ε

multiplication $\mu = U\varepsilon F: UFUF \Rightarrow UF$ comult $F\eta U: FUFU \Rightarrow FU$

Eg. $\mathrm{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Mon}$ free monoid $\dashv$ forgetful adjunction.

$\eta : 1 \Rightarrow UF \qquad x \longmapsto [x]$ list $[x_1 \cdots x_n]$

$\mu = U\varepsilon F : UFUF \Rightarrow UF$ concatenation of list.

§2 (Co)algebra over (co)monad

Def. Give (T, η, μ)

- $A \in \mathrm{ob}(\mathcal{C})$
- "augmentation" $a : TA \to A$

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{Id}"'] & TA \arrow[d, "a"] \arrow[dl, phantom, "\circlearrowright"] \\ & A \end{tikzcd}$$

Def. (K, ε, δ)

- $B \in \mathrm{ob}(\mathcal{D})$
- $b : B \to TB$ s.t.

$$\begin{tikzcd} T^2B & TB \arrow[l] \\ TB \arrow[u] & B \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} B & TB \arrow[l, "\varepsilon_B"'] \\ & B \arrow[ul, equal] \arrow[u, "b"'] \end{tikzcd}$$

Eg. adjunction. $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \qquad \eta : 1_{\mathcal{C}} \Rightarrow UF \qquad \varepsilon : FU \Rightarrow 1_{\mathcal{D}}$

$\boxed{T = UF} \qquad \eta : 1 \Rightarrow UF \qquad \mu = U\varepsilon F : UFUF \Rightarrow UF$

$\boxed{A = UX}$, $\quad X \in \mathrm{ob}(\mathcal{D}) \qquad a = U\varepsilon_A : UFUA \to UA$

$K = FU$

$B = FY \qquad Y \in \mathrm{ob}(\mathcal{C})$

§3 Eilenberg-Moore Cat $\mathcal{C}^T$

Def. Given a monad (T, η, μ) on cat $\mathcal{C}$, then $\mathcal{C}^T$ consists of

- Objects $(A,\ a : TA \to A)$
- Morphisms $A \xrightarrow{f} B$ s.t.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\circlearrowright"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Prop. There is an adjunction $C \underset{U^T}{\overset{F^T}{\rightleftarrows}} C^T$

$F^T(A) = (TA,\ a = \mu_A : T^2A \to TA)$

$$\begin{tikzcd} \Delta^{op} \arrow[r] \arrow[dr] & \mathcal{G} \arrow[d] \\ & \mathrm{Set} \end{tikzcd}$$

finite set

U^T forgetful functor

unit $\eta : \mathbb{1} \to U^T F^T = T \quad A \mapsto TA$

counit $\varepsilon : F^T U^T \mapsto \mathbb{1} \qquad (TA, \mu_A) \xrightarrow{a} (A, a)$

Adjunction.

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\circlearrowright"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

$B \longrightarrow TB$

$$\begin{array}{ccc} N & \longrightarrow & C \otimes N \\ x & \longmapsto & \sum c_i \otimes x_i \end{array} \xrightarrow{\text{Hopf.}} \quad G \times \overset{\mathrm{Spec}\,R.}{\overset{\shortparallel}{S}} \longrightarrow S$$

§4 Simplicial Objects

- Augmented simplicial objects $X : \Delta_a^{op} \to C$
  - The augmented simplicial cat Δ_a

    Obj : $\mathrm{Ob}(\Delta) \cup \{\underline{[-1]}\} \qquad [n]$

    Mor : $[-1] \longrightarrow [n]$

Explicitly,

$$X = \quad X_{-1} \xleftarrow{\varepsilon} X_0 \overset{d_i}{\leftleftarrows} X_1 \overset{d_i}{\leftleftarrows} X_2 \ \cdots$$

- Moore complex. If C is abelian. $X.$ Moore complex

$X_{n-1} \xleftarrow{\partial_n} \underline{X_n}$ with $\partial = \sum_{i=0}^{n} (-1)^i d_i \qquad \partial^2 = 0.$

- Homology $H_\cdot(X_\cdot)$
- Example Singular (co)homology

$$\mathrm{Sing}: \mathrm{Top} \longrightarrow \mathrm{sSet} \xrightarrow{\mathbb{Z}[-]} \mathrm{sAb}$$
$$X \longmapsto S(X) = \{\mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)\}_{n\geq 0} \longmapsto \mathbb{Z}[S(X)].$$
$$H_\cdot(\mathrm{Sing}(X)) = H_\cdot(X).$$

§5 Bar construction.

$$C \underset{U^T}{\overset{F^T}{\rightleftarrows}} C^T \qquad \eta: 1\!\!1 \Rightarrow U^T F^T = T \qquad \varepsilon: F^T U^T \Rightarrow 1\!\!1$$

comonad $(F^T U^T, \varepsilon, \delta = F^T \eta U^T)$ $\qquad F^T U^T: C^T \to C^T$

$$\mathrm{Bar}_T : \Delta_a^{op} \longrightarrow \mathrm{End}(C^T)$$
$$[n] \longmapsto (\underline{F^T U^T})^{n+1}$$

$$\mathcal{U}\mathrm{Bar}_T(A) : \Delta_a^{op} \longrightarrow C$$
$$[n] \longmapsto \underline{T^{n+1}A}$$

Explicitly

$$\mathcal{U}\mathrm{Bar}_T(A) = \left\{ \boxed{A \xleftarrow{a}} \; TA \underset{Ta}{\overset{\mu_T}{\leftleftarrows}} T^2A \cdots\cdots T^nA \leftleftarrows T^{n+1}A \right\}$$

$$d_i = T^i \mu_{T^{n-i-1}A} : T^nA \longrightarrow T^{n-1}A$$

$$s_i = T^i \delta_{T^{n-i-2}A} : T^nA \longrightarrow T^{n+1}A$$

$$T \xrightarrow{\delta} T^2$$

Example 1. Group (Co) homology

$$F = \mathbb{Z}[G] \underset{\mathbb{Z}}{\otimes} (-) : Ab \rightleftarrows \underset{\mathrm{Mod}(\mathbb{Z}[G])}{\overset{\shortparallel}{\mathrm{Mod}(G)}} : U$$

$$T = UF = \mathbb{Z}[G] \underset{\mathbb{Z}}{\otimes} [-]$$

$$\eta : \mathbb{1} \Rightarrow UF \qquad \eta_A : A \longmapsto \mathbb{Z}[G]\otimes A \qquad a \mapsto 1\otimes a$$

$$\varepsilon : FU \Rightarrow \mathbb{1}_D \qquad \mu_A : \mathbb{Z}[G]\otimes\mathbb{Z}[G]\otimes A \to \mathbb{Z}[G]\otimes A$$
$$x\otimes y\otimes a \longmapsto xy\otimes a$$

take $A = \mathbb{Z}$ as trivial module $\quad a : \mathbb{Z}[G] \to \mathbb{Z}$

$$\boxed{\mathrm{Bar}_T(\mathbb{Z})} = \Big\{ \overset{-1}{\mathbb{Z}} \leftarrow \overset{0}{\mathbb{Z}[G]\underset{\mathbb{Z}}{\otimes}\mathbb{Z}} \underset{T^n a}{\leftleftarrows} \overset{X_2}{\mathbb{Z}[G]^{\otimes 2}\underset{\mathbb{Z}}{\otimes}[\mathbb{Z}]} \leftleftarrows \cdots \Big\}$$

with

$$d_i(x[x_1|\cdots|x_n]) = \begin{cases} xx_1[x_2|\cdots|x_n] & i=0 \\ x[x_1|\cdots|x_ix_{i+1}|\cdots|x_n] & 1\le i\le n-1 \\ x[x_1|\cdots|x_{n-1}] & i=n. \end{cases}$$

$$s_i(x[x_1|\cdots|x_n]) = x[x_1|\cdots|x_{i-1}|1|x_i|\cdots|x_n] \quad 0\le i\le n.$$

Rmk: This is the free resolution of $\mathbb{Z}$ in $\mathrm{Mod}(G)$.

Homology w/ coeff M:

Cohomology w/ coff A:

$$\underset{X_0}{\overset{}{\mathrm{Hom}(\mathbb{Z}[G], A])}} \to \underset{X_1}{\mathrm{Hom}(\mathbb{Z}[G]^{\otimes 2}, A)} \to \cdots$$

Example 2 Hochschild (co)homology

A k-algebra $\qquad A^e = A \underset{k}{\otimes} A^{op}$

$T = A \underset{k}{\otimes} (-) : \mathrm{Mod}(A^e) \longrightarrow \mathrm{Mod}(A^e)$ is a monad.

unit $\quad \eta : 1 \to T \qquad M \longrightarrow A \underset{k}{\otimes} M, \; m \mapsto 1 \otimes m$

multiplication $\quad \mu : T^2 \Rightarrow T^2 \quad A \underset{k}{\otimes} A \underset{k}{\otimes} M \longrightarrow A \underset{k}{\otimes} M$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \otimes y \otimes m \mapsto xy \otimes m.$

$$\mathrm{Bar}_T(A) = \Big\{ \underset{-1}{M} \leftarrow \underset{0}{A \underset{k}{\otimes} A} \leftarrow \cdots \leftarrow \underset{n}{A \overset{\otimes n+1}{\underset{k}{\otimes}} A} \leftarrow \Big\}$$

w/ $\quad d_i(x_0 \otimes \cdots \otimes x_{n+1}) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}$

$\quad s_j = \ldots \; x_i \otimes 1 \otimes x_{i+1} \ldots$

$$H_\bullet(\underset{\sim}{\mathrm{Bar}_T(A)}) = HH_\bullet(A)$$

§6. Resolution.

- Deformation retract. $\quad \varepsilon : X_\bullet \longrightarrow X_{-1}$

  if $\exists\, s : X_{-1} \to X$ section of ε s.t. $s\varepsilon \simeq \mathrm{Id}_X$.
  simplicial

- Sufficient condition. $\qquad A \quad T \quad \eta_A : A \to TA$

  $s_{n+1} : X_n \longrightarrow X_{n+1}$ s.t.

  ① $s_0 = s$

  ② $\begin{cases} d_i s_{n+1} = s_n d_i \\ d_{n+1} s_{n+1} = \mathrm{Id} \\ s_i s_n = s_{n+1} s_i \end{cases}$

$$\begin{array}{ccccccc} A & & TA & & T^2A & & \\ \| & & \| & & \| & & \\ X_{-1} & \underset{s_0}{\overset{\varepsilon}{\leftrightarrows}} & X_0 & \underset{s_1}{\leftleftarrows} & X_1 & \underset{s_2}{\Lleftarrow} & X_2 \end{array}$$

$X_{-1} \overset{\varepsilon}{\underset{\sim}{\leftarrow}} X_\bullet$ htpy equivalence.

$$|\overline{W}(G(X))| \simeq \Omega X$$

§* Décalge / shift functor

Def. $\sigma : \Delta_a \times \Delta_a \to \Delta_a$

$([m],[n]) \longmapsto [m+n+1]$

$$\begin{tikzcd} PX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^I \arrow[d] \\ |X| \arrow[r] & X \end{tikzcd}$$

$\sigma(-,[0])^* : s\mathcal{C} \to as\mathcal{C}$

$X. \longmapsto \mathrm{Dec}_0(X) = (\sigma(-,[0]))^* X = PX$

Explicity.

$$\boxed{\begin{aligned} &\bullet\ \mathrm{Dec}_0(X)_n = X_{n+1} \\ &\bullet\ d_i^{n,\mathrm{Dec}_0(X)} = d_i^{n+1,X} \\ &\bullet\ s_j^{n,\mathrm{Dec}_0(X)} = s_j^{n+1,X} \end{aligned}}$$

shift dim -1
forgetting the last face/deg maps of X

Prop. $\mathrm{Dec}_0(X) \xrightarrow{\sim} X_0$ w.h.e. (deformation retract)

§* Two-sided bar construction

$\mathcal{B}$ bicat.

A, B C
0-cell

1 cell

$$\begin{tikzcd} A \arrow[r, "X"] & B \arrow[d, "M"] \\ C & B \arrow[l, "Y"'] \end{tikzcd}$$

left module

2 cell

$\boxed{\alpha : MX \to X}$

$\boxed{\beta : YM \to Y}$

right module.

$\mathrm{Bar}(X,M,Y)_n = YM^nX$

Σ.Top

$\mathrm{Bar}(\odot, G, \cdot) = BG.$

$\mathrm{Bar}(G,G,\cdot) = EG$

$\underline{\mathrm{Bar}(T,T,A)}$

$\boxed{\text{twist cochain}}$

Remaining question

1. Difference between algebra and coalgebra.
2. Décalage / shift funtor as path object.
3. Bar construction and classifying space.

Fact 1. Dual of a coassociative coalgebra is an algebra

C R-coalgebra. A R-algebra

$\Rightarrow B = \mathrm{Hom}_R(C, A)$ is R-algebra

$\forall f, g : C \to A$, define $fg(x) = \sum_x f(x_{(1)}) \otimes g(x_{(2)})$

if we use Sweedler notation $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$

$$(fg)h(x) = \sum \big(f(x_1) \otimes g(x_2)\big) \otimes h(x_3)$$
$$= \sum f(x_1) \otimes \big(g(x_2) \otimes h(x_3)\big) = f(gh)(x)$$

Fact 2. Dual of an algebra is not always a coalgebra.

A R-algebra $\qquad B = \mathrm{Hom}_R(A, R) = A^*$

$$\Delta : A^* \to A^* \otimes A^* \subseteq (A \otimes A)^* \xleftarrow{m^*} A^*$$

find smaller $A^\circ \subset A^*$ s.t $m^*(A^\circ) \subset A^* \otimes A^*$

Hopf dual $A^\circ = \{ f \in A^* \mid \ker f \supset I,\ \dim A/I < \infty \}$

$\Delta_\circ(f) = \sum f_1 \otimes f_2$ iff $\sum f_1(a) \otimes f_2(b) = f(ab)$, $\forall a, b \in A$

$\varepsilon_\circ(f) = f(1_A)$

Fact 3. Hopf dual of Hopf dual is not identity $(H^\circ)^\circ \neq H$

Eg 1. $H = k[G]$ $\quad \Delta(g) = g \otimes g \quad \varepsilon(g) = 1$

G binate group $\rightarrow$ no nontrivial finite dim reps / k. field

$f: H \to k$ $\quad I = \ker(f)$ $\quad \mathrm{codim}\, I < \infty$ $\quad \dim H/I < \infty$

H/I trivial $\therefore f = c \cdot \varepsilon$ $\quad \therefore H^\circ = k \cdot \varepsilon \cong k$ $\quad (H^\circ)^\circ \cong k^\circ \cong k$.

Eg2. $k \subset \mathbb{C}$ subfield.

Higman group $G = \langle a, b, c, d \mid ab = ba^2, bc = cb^2, cd = dc^2, da = ad^2 \rangle$

G has no nontrivial finite dim reps.

Eg3. $H = \mathbb{C}[x]$ $\quad \Delta(x) = 1 \otimes x + x \otimes 1$, $\quad \varepsilon(x) = 1$

$\dim(H) = \aleph_0$ countable

$\dim \mathbb{C}((H^\circ)^\circ)$ uncountable.

2. Décalage / shift functor and path object.

I. Décalge.

$(\Delta_a, \sigma = +, 0 = [-1])$ $\qquad$ initial $[-1] = 0$

$\sigma: \Delta_a \times \Delta_a \longrightarrow \Delta_a$ $\qquad$ terminal $[0] = 1$

$([i], [j]) \longmapsto [i+j+1]$

monoid $\quad 1 = [0]$ $\quad (2 = [1] \xrightarrow{\mu} 1 = [0], \quad 0 = [-1] \xrightarrow{1} 1 = [0])$

$\rightsquigarrow$

comonoid $\quad 1 \in \Delta_a^{op}$

$\rightsquigarrow$

comonad $\quad D_0 = \sigma(-, [0]) = (-) + [0] : \Delta_a^{op} \to \Delta_a^{op}$

$\rightsquigarrow$

comonad $\quad \mathrm{Dec}_0 = D_0^* = \sigma(-, [0])^* : \mathrm{asSet} \to \mathrm{asSet}$.

- counit $\quad d_{n+1}^* : \mathrm{Dec}_0 \Rightarrow \mathrm{Id}$ $\qquad \{ \mathrm{Dec}_0(X)_n = X_{n+1} \xrightarrow{d_{n+1}} X_n \}$

II. path object.

$P: \mathrm{Top} \to \mathrm{Top}$ $\qquad \mathrm{Dec}_0 : \mathrm{sSet} \to \mathrm{sSet}$

$$\begin{tikzcd} PX \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^I \arrow[d, "ev_0"] \\ |X| \arrow[r, hook, "i"] & X \end{tikzcd}$$

discrete

$|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$

$\mathrm{Dec}_0 \circ \mathrm{Sing} = \mathrm{Sing} \circ P$

II. Cone.

$\sigma: \Delta \times \Delta \to \Delta$ ordinal sum, $[i],[j] \mapsto [i+j+1]$

$\sigma^*: sSet \xrightarrow[\xleftarrow{\sigma_!}]{} ssSet$ w/ left adjoint $\sigma_!: ssSet \to sSet$

$\square: sSet \times sSet \to ssSet$ $\boxed{X * Y}$ join

$(X_\bullet, Y_\bullet) \longmapsto \{X_k \times Y_\ell\}_{(k,\ell)}$

$*: sSet \times sSet \xrightarrow{\square} ssSet \xrightarrow{\sigma_!} sSet$

$(X_\bullet, Y_\bullet) \longmapsto X \square Y \longmapsto \sigma_!(X \square Y) =: X * Y$

$$(X*Y)_n = X_n \cup Y_n \cup \Big(\bigcup_{i+j=n-1} X_i \times Y_j \Big)$$

$$d_i(x_j, y_k) = \begin{cases} y_{n-1} & j=0 \\ (d_i x_j, y_k) & i \le j,\ j \ne 0 \\ (x_j, d_{i-j-1} y_k) & i > j,\ k \ne 0 \\ x_{n-1} & k=0 \end{cases}$$

$\Delta[n] = \underbrace{\Delta[0] * \cdots * \Delta[0]}_{n+1}$

$\Delta[n] * \Delta[0] = \Delta[n+1]$ $\quad\quad \Delta^n * \Delta^0 = \Delta^{n+1}$

$C := \sigma_!((-) \square \Delta[0]) : sSet \rightleftarrows sSet : Dec_0$

cone $\quad X \longmapsto X * \Delta[0] = C(X)$ shift / Décalage

$Dec_0(X)_n = Hom(\Delta[n], Dec_0(X))$ $\quad \Delta^{op} \to Set.$

$= \underline{Hom}(\Delta[n] * \Delta[0], X)$ $\quad$ Cone in X

$\Delta[n] * \Delta[0] = \Delta[n+1]$ $\quad X \in sSet$

Hom

$= \underline{X_{n+1}}$

3. Classifying space of simplicial groups

$T : \mathrm{ssSet} \longrightarrow \mathrm{sSet} : \mathrm{Dec}$ $\qquad \mathrm{Dec}_0 = \sigma^*(-,[0])$

total simplicial set functor / Artin-Mazur codiagonal

$$(T X_{\bullet,\bullet})_n = \mathrm{eq}\left\{ \prod_{i=0}^{n} X_{i,n-i} \rightrightarrows \prod_{i=0}^{n-1} X_{i,n-i-1} \right\}$$

$\sigma : \Delta\times\Delta \to \Delta$

$[i],[j] \mapsto [i+j+1]$

$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_i} X_{i,n-i} \xrightarrow{d_0^v} X_{i,n-i-1}$$

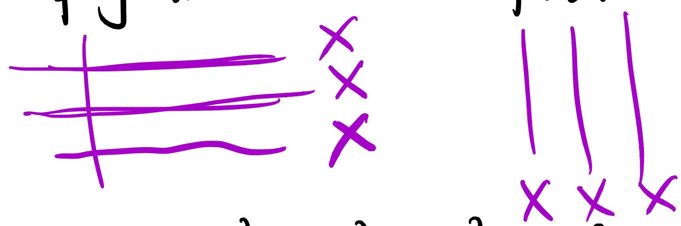
$$\prod_{i=0}^{n} X_{i,n-i} \xrightarrow{p_{i+1}} X_{i+1,n-i-1} \xrightarrow{d_{i+1}^h} X_{i,n-i-1}$$

$i=1,2$

Prop. $T p_i^* X = X$ $\qquad p_i =$ projection on the ith factor

$$dX \overset{\sim}{\rightleftarrows} TX$$

diagonal $\qquad$ total.



Def. $NG_\bullet = \{ (NG_n)_\bullet \} \in \mathrm{ssSet}.$

$$\begin{tikzcd} G_0^2 \arrow[r] & G_1^2 \arrow[r] \arrow[d] & G_2^2 \arrow[r] \arrow[d] & G_3^2 \arrow[r] \arrow[d] & \cdots \\ G_0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G_3 \arrow[r] \arrow[d] & \cdots \\ * \arrow[r] & * \arrow[r] & * \arrow[r] & * \arrow[r] & \cdots \end{tikzcd}$$

G constant. simple. group

Claim $\overline{W}G = TNG \rightsquigarrow \overline{W}G = BG$

$$(g_{n-1},\cdots,g_0) \longmapsto \left(d_0^{i-1}(g_{n-1}), d_0^{i-2}(g_{n-2}), \cdots, d_0(g_{n-i+1}), g_{n-i}\right) \in (NG_{n-i})_i$$

$T(NG)$

$G^2 \quad G^2$

$G \quad G$

Bar construction. to G.

Def. (Twisting function) $G_\bullet \in sGr$. $X_\bullet \in sSet$

Def. A twisting function $\tau : X_\bullet \to G_{\bullet-1}$ is a collection of maps

$\{\tau_n : X_n \to G_{n-1}\}$ s.t.

$$\begin{cases} d_0(\tau(x)) = \tau\left((d_0 x)^{-1}\right)\tau(d_0 x) & \\ d_i(\tau(x)) = \tau(d_{i+1} x) & i \geq 1 \\ s_j(\tau(x)) = \tau(s_{j+1} x) & j \geq 0 \\ \tau(s_0 x) = 1_{G_n} & x \in G_n. \end{cases}$$

① algebra analogue.

dg coalgebra C,

dg algebra A.

$\boxed{\text{"twisted cochain"}}$

$\tau : C_\bullet \to A_{\bullet-1}$

Cartan-

$\boxed{d\cdot\tau + \tau\cdot d = -}$

$$Tw : sSet \times sGr \to Set.$$

$$(X_\bullet, G_\bullet) \mapsto Tw(X, G)$$

②.

Def. A twisted Cartesian product w/ fibre $G_\bullet$, base $X_\bullet$ is a sSet $E_\bullet = G_\bullet \times_\tau X$

$$E_n = \{G_n \times X_n\}$$

$$d_i(g,x) = \begin{cases} (\tau(x)\cdot d_0 g,\ d_0 x) \\ (d_i g,\ d_i x) \end{cases}$$

$$s_j(g,x) = (s_j g,\ s_j x)$$

$$E_\bullet \underset{pr_2}{\overset{\boxed{6}}{\rightrightarrows}} X$$

$$(g, x) \mapsto g\cdot x$$

?

$$E_\bullet = G_\bullet \times_\tau X$$

$$E_\bullet \times G_\bullet \underset{pr_1}{\overset{\rho}{\rightrightarrows}} E$$

Prop. principal fibration / bundle

$$E_\bullet \underset{s_2}{\overset{s_1}{\rightrightarrows}} X_\bullet$$

Def. $\rho : E_\cdot \times G_\cdot \longrightarrow E_\cdot$ (simplicial principal bundles)

① degreewise principal, $\rho(x,g) = x \Rightarrow g = 1_G$.

② $eq\{E_\cdot \times G_\cdot \overset{\rho}{\underset{pr}{\rightrightarrows}} E\} = E_\cdot / G_\cdot = X_\cdot$

$Tw(X_\cdot, G_\cdot) \longleftrightarrow \mathrm{Principal}_G(X)$

Prop. Any principal bundle $E_\cdot \to X_\cdot$ w/ right $G_\cdot$-action + local section $X \xrightarrow{6} E_\cdot$, $\longleftrightarrow E_\cdot = G_\cdot \underset{\tau}{\times} X \longrightarrow X$

top. principal bundle $E \underset{6}{\overset{p}{\rightleftarrows}} X$, $E \cong G \times X$

$$\boxed{\begin{aligned} Tw_G : sSet &\longrightarrow Set \\ X &\longmapsto Tw(X_\cdot, G_\cdot) \end{aligned}} \qquad \boxed{\begin{aligned} Tw_X : sGr &\longrightarrow Set \\ G &\longmapsto Tw(X, G) \end{aligned}}$$

both representable

$\boxed{\text{classifying space of sGr.}}$ $\qquad$ $\boxed{\text{Kan-loop group construction}}$

$$\boxed{\overline{W}(G_\cdot)_n} = \begin{cases} *, & n=0 \\ G_{n-1} \times G_{n-2} \times \cdots \times G_0, & n>0 \end{cases}$$

$$d_0(g_{n-1}, \cdots, g_0) = (g_{n-2}, \cdots, g_0)$$

$$d_{i+1}(g_{n-1}, \cdots, g_0) = (d_i g_{n-1}, \cdots, d_1 g_{n-i}, g_{n-i-2} \cdot d_0 g_{n-i-1}, g_{n-i-3}, \cdots, g_0)$$

$$s_0(\text{———}) = (1, \text{———})$$

$s_{j+1}(\longrightarrow) = (s_j g_{n-2}, \cdots, s_0 g_{n-j-2}, 1, g_{n-j-3}, \cdots, g_0)$

$\widehat{\tau_n}(G) : \overline{W}_n(G) \to G_{n-1}$

$(g_{n-1}, \cdots, g_0) \longmapsto g_{n-1}$

$$\begin{tikzcd} X_\bullet \arrow[r, "\tau"] \arrow[d, dashed, "\exists!"'] & G_{\bullet-1} \\ \overline{W}(G) \arrow[ur, dashed, "\hat{\tau}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

Goal : Bar construction ?

① . $\boxed{\text{categorical def.}}$ (alg / top example) standard non simplicial bar cnst.

HH. / Group BG

② Déclage / shift functor. $\boxed{\mathrm{Dec}_0}$

simplicial level

$T : C \to C$. monad.

$$\boxed{\begin{aligned} \mathrm{Bar}_T(A) : \Delta^{op} &\to C^T \\ n &\mapsto T^n A. \end{aligned}}$$

$\mathrm{Bar}_T : \Delta^{op} \to \mathrm{End}(C^T)$

resolution.

$\mathrm{Dec}_0 : sC \to sC$.

$U\mathrm{Bar}_T(A) \xrightarrow{\text{acyclic.}} A$

$\boxed{T \circ U\mathrm{Bar}_T(A) = U\mathrm{Bar}_T(A) \circ \mathrm{Dec}_0}$ $\rightsquigarrow$ $U\mathrm{Bar}_T(A)$ resolutio

$\mathrm{Tot} : \mathrm{ssSet} \rightleftarrows \mathrm{sSet} : \mathrm{Dec}$

$C : \mathrm{sSet}. \rightleftarrows \mathrm{sSet} : \mathrm{Dec}_0$

$$\begin{tikzcd} \mathrm{sGr} \arrow[r, "\overline{W}"] \arrow[d, "N"'] & \mathrm{sSet}. \\ \mathrm{ssSet}. \arrow[ur, "\mathrm{Tot}"'] & \end{tikzcd}$$

Stand

$$\begin{tikzcd} \mathrm{Gr} \arrow[d, "N"'] \\ \mathrm{sSet}. \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gr} \arrow[d, "N"'] & G \\ \mathrm{sSet} & BG = N(G). \end{tikzcd}$$

Cat.

Bar construction $\overset{G.}{\underset{\text{cat}}{\rightleftarrows}}$ classifying space of group

$\mathrm{Bar}_T(-)$

Dec.

simplicial classifying space

"simplicial bar construction"

∞-cat. bar construction.

{ E_∞ - bar construction }