Orthogonal Spectra

Last time: sequential spectra

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\Sigma^\infty_+"] \arrow[r, phantom, "\perp"] & \text{Spectra} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

Quillen pair

play nicely with homotopy theory

$$\text{Spectra} \xleftarrow{H} \text{Ab}$$

$$HA \longleftarrow A$$

with n^{th} space $K(A,n)$

H faithfully embeds Ab into Spectra

$$\begin{tikzcd} \text{Spectra} & \text{Ab} \arrow[l, "H"'] \\ & \text{CRings} \arrow[u] \end{tikzcd}$$

A ring is a monoid in $(\text{Ab}, \otimes, \mathbb{Z})$

Need to define a symmetric monoidal product on Sp

Problem: there is a finite list of desireable properties for a category $\mathcal{C}$ with $ho(\mathcal{C})$ the stable homotopy category

such as a monoidal product

Lewis proves you can only ever have 4 of 5 properties. Impossible to have all of them.

Remark: Actually, you can with ∞-cats

Models of Spectra:

a category $\mathcal{C}$ that has 4 of 5 properties

- symmetric spectra
- orthogonal spectra
- EKMM spectra
- Γ-spaces

a spectrum X is connective if $\pi_n X = 0$ for $n < 0$

Γ-spaces only give connective spectra

Def: An orthogonal G spectrum X consists of the following data:

- pointed spaces X_n with $O(n)$ $G\times O(n)$ -action — orthogonal gp of $n\times n$ matrices
- maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$

such that the composite $\sigma_{n+m-1}\circ\cdots\circ\sigma_n$

$G\times O(n)$, $O(n)$ $G\times O(m)$ $G\times O(n+m)$

$$X_n \wedge S^m \longrightarrow X_{n+m}$$

is $O(n)\times O(m) \subseteq O(n+m)$ -equivariant

$G\times O(n)\times O(m)$

$O(m) \circlearrowright S^m$ if we think of S^m as the 1-pt compactification of $\mathbb{R}^m$.

$$O(n)\times O(m) \hookrightarrow O(n+m)$$

block diagonal $\begin{bmatrix} A_n & 0 \\ 0 & A_m \end{bmatrix}$

Def: A morphism of orthogonal spectra $f: X \to Y$ is a sequence of $G\times O(n)$ - equivariant maps $X_n \xrightarrow{f_n} Y_n$ such that the following diagram commutes:

$$\begin{tikzcd}
X_n \wedge S^m \arrow[r, "f_n \wedge \mathrm{id}"] \arrow[d, "\sigma"] & Y_n \wedge S^m \arrow[d, "\sigma"] \\
X_{n+m} \arrow[r, "f_{n+m}"] & Y_{n+m}
\end{tikzcd}$$

Examples: $\mathbb{S}$ n^{th} space S^n = 1-pt compactification of $\mathbb{R}^n$

ring spectrum

let $O(n) \circlearrowright \mathbb{R}^n \subseteq S^n = \mathbb{R}^n \cup \{\infty\}$

basepoint

$S^n \times S^m \longrightarrow S^{n+m}$

$(v,u) \longmapsto \begin{bmatrix} v \\ u \end{bmatrix}$

initial ring spectrum

Eilenberg-Maclane Spectra HA

If A is a ring, HA is a ring spectrum

$$HA_n = A[S^n] = \left\{ \sum a_i \cdot s_i \;\middle|\; \begin{matrix} a_i \in A \\ s_i \in S^n \end{matrix} \right\}$$

think $A[\mathbb{R}^n] \cup A \cdot \{\infty\}$

Suspension Spectra: $\Sigma^\infty_+ X$

n^{th} space $S^n \wedge X$

$\circlearrowright O(n)$ trivial action

Def: An orthogonal ring spectrum R is an orthogonal spectrum R together with

"graded multiplication" $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ $O(n) \times O(m)$-equivariant

$i_n : S^n \longrightarrow R_n$ for $n \geq 0$ $O(n)$-equivariant

has associativity, unit properties

Associative

$$\begin{tikzcd}
R_n \wedge R_m \wedge R_p \arrow[r, "1\wedge\mu"] \arrow[d, "\mu\wedge 1"'] & R_n \wedge R_{m+p} \arrow[d, "\mu"] \\
R_{n+m} \wedge R_p \arrow[r, "\mu"] & R_{n+m+p}
\end{tikzcd}$$

Unit

$$R_n \cong R_n \wedge S^0 \xrightarrow{id \wedge i_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

(and the symmetric thing)

multiplicativity $\quad \mu_{n,m} \circ i_n \wedge i_m = i_{n+m}$

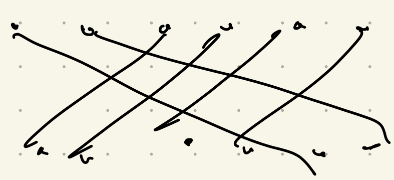
$$S^n \wedge S^m \xrightarrow{i_n \wedge i_m} R_n \wedge R_m \xrightarrow{\mu} R_{n+m}$$

Centrality

$$\begin{tikzcd}
R_m \wedge S^n \arrow[r, "id\wedge i_n"] \arrow[d] & R_m \wedge R_n \arrow[r, "\mu"] & R_{m+n} \arrow[d, "\tau_{n,m}"] \\
S^n \wedge R_m \arrow[r, "i_n\wedge id"] & R_n \wedge R_m \arrow[r, "\mu"] & R_{n+m}
\end{tikzcd}$$

permutation matrix

$$\tau_{n,m} = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}$$



Properties of Stable Homotopy Category:

$Ho(Sp)$ is the stable homotopy category

- it is abelion
  finite products = finite coproducts
  has all kernels (= fibers)
  all cokernels (= cofibers)
  fiber seq:s $A \to B \to C$ are cofiber seqs

- it is triangulated ← associated w/ chain complexes
  has a shift functor Σ
  shift in the other direction Ω
  has a notion of "distinguished triangle"
  which yields long exact sequences

- it is closed symmetric monoidal
  $\wedge = \otimes_{\mathbb{S}}$
  has a monoidal product $\wedge$ ← smash product
  w/ unit $\mathbb{S}$
  has an internal hom $F(X,Y)$ ← function spectrum
  $map(X,Y)$

$$F(X \wedge Y, Z) \simeq F(X, F(Y,Z))$$

$$\begin{tikzcd} \text{Spaces} \arrow[r, bend left, "\Sigma^\infty_+"] \arrow[r, phantom, "\perp"] & \text{Spectra} \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

$$\Omega^\infty X = \operatorname{colim} \Omega^n X_n$$

$$\Sigma^\infty_+(X \longrightarrow \Sigma^2 Y)$$

$$\mathrm{Hom}_{Ch_{\geq 0}(\mathbb{Z})}(A_\bullet, B_\bullet) = \bigoplus_{n \in \mathbb{Z}} \text{degree } n \text{ homs}$$

$\mathrm{Hom}_{ho(Sp)}(X, Y)$ includes maps of degree $\neq 0$

$$X \longrightarrow \Sigma^2 Y$$

$$X_0 \wedge S^1 \xrightarrow{\sigma} X_1 \quad \underset{\text{adjunction}}{\overset{\Sigma \dashv \Omega}{\rightsquigarrow}} \quad X_0 \xrightarrow{\tilde{\sigma}} \mathrm{Map}(S^1, X_1) \cong \Omega X_1$$

$$\Omega^\infty X = \operatorname{colim}\left(X_0 \xrightarrow{\tilde{\sigma}} \Omega X_1 \xrightarrow{\tilde{\sigma}} \Omega^2 X_2 \xrightarrow{\tilde{\sigma}} \cdots\cdots \right)$$

$$\pi_n \Omega^\infty X = \pi_n^s(X) = \pi_n(\Sigma^\infty X)$$

Cary Malkiewicz has a good overview on his webpage

Stefen Schwede - Lectures on Equivariant Stable homotopy theory

Birgit Richter - Commutative Ring Spectra

Handbook of Htpy Theory (on nlab)

G-equivariant spectra should be indexed on G-reps (real, orthogonal)

let V be a G-rep, X an orthogonal spectrum

$$X(V) = \mathrm{lin}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

htpy "groups" of G-spectra are Mackey functors