# Compositional Structure of Partial Evaluations

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MIT Categories Seminar 9/10/20

# Partial Evaluations

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$1+2+3$

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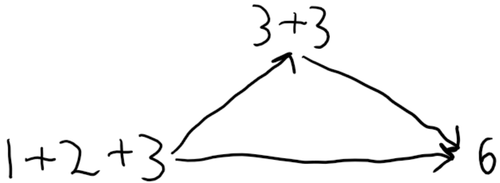

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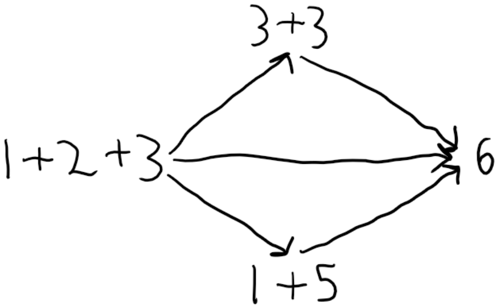

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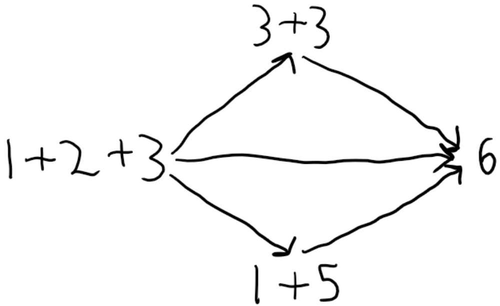

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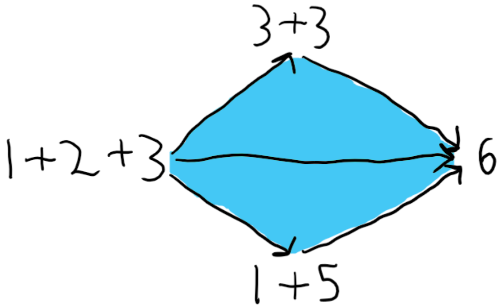

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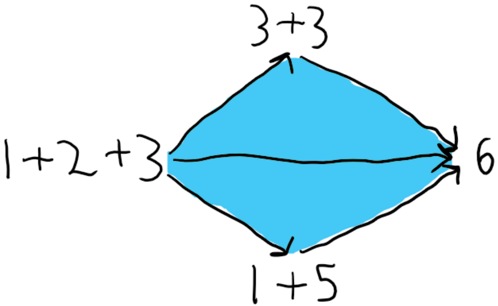
- How does this space relate to algebra? Computation?



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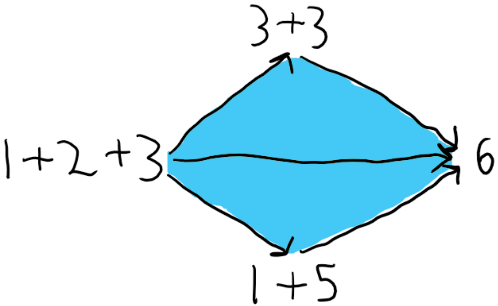

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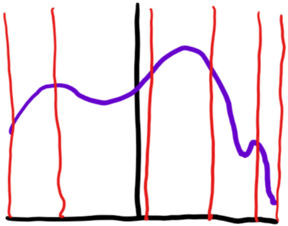

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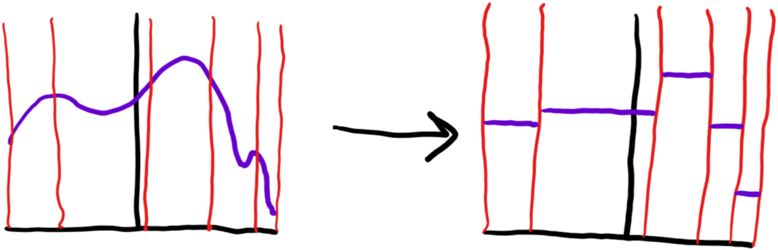

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$X$

$\{a, b, c\}$

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$X$

$\{a, b, c\}$

$TX$

$\boxed{a}$

$\boxed{b} + \boxed{b}$

$\boxed{a} + \boxed{c} + \boxed{b}$

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- A natural "unit" map $\eta : X \to TX$

Example: "Free (commutative) monoid" monad

| $X$ | $TX$ |
|---|---|
| $\{a, b, c\}$ | $\boxed{a}$ |
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$X$ $\qquad\qquad$ $TX$

$a$

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$$\begin{array}{ccc} X & & TX \\ a & \overset{\eta}{\longmapsto} & \boxed{a} \end{array}$$

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$TTX$ $\qquad\qquad$ $TX$

$\boxed{\boxed{a}+\boxed{b}}+\boxed{\boxed{c}}$

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$$\boxed{a} + \boxed{b} \xmapsto{\eta T} \boxed{\boxed{a} + \boxed{b}}$$

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$$\boxed{\boxed{a}} + \boxed{\boxed{b}} \xleftarrow{T\eta} \boxed{a} + \boxed{b}$$

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Example: "Free (commutative) monoid" monad

$$\boxed{\boxed{\boxed{a}+\boxed{b}}}+\boxed{\boxed{\boxed{c}+\boxed{a}}}$$

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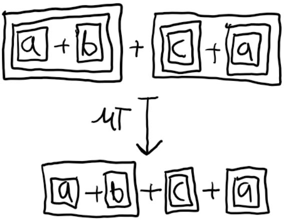

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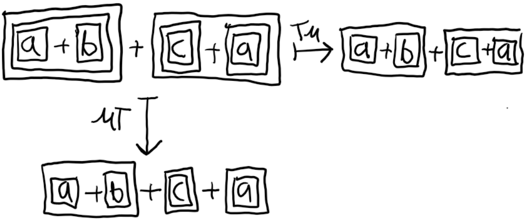

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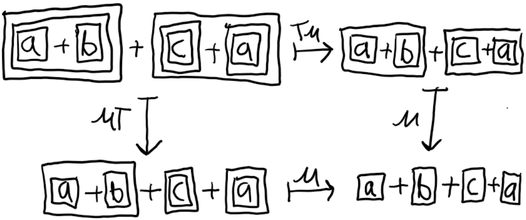

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Example: Distribution monad

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Example: Distribution monad

$X$ $\qquad$ $TX$

$\{a, b, c\}$ $\qquad$ $\boxed{a}$

$\frac{1}{3}\boxed{a} + \frac{2}{3}\boxed{b}$

$\frac{3}{7}\boxed{a} + \frac{1}{7}\boxed{b} + \frac{2}{7}\boxed{c}$

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Example: Distribution monad

$X$ $\qquad\qquad$ $TX$

$\{a, b, c\}$ $\qquad$ $\boxed{a}$

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$a \xmapsto{\eta} 1\boxed{a}$

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Example: Distribution monad

$X \qquad\qquad TX$

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A *monad* is a functor $T : \mathcal{C} \to \mathcal{C}$ where $TX$ describes formal expressions on $X$, where we have:

- A natural "unit" map $\eta : X \to TX$
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- Unit and associativity equations:

$$\begin{tikzcd} TX \arrow[r, "\eta T"] \arrow[dr, equal] & TTX \arrow[d, "\mu"] & TX \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & TX & \end{tikzcd} \qquad \begin{tikzcd} TTTX \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & TTX \arrow[d, "\mu"] \\ TTX \arrow[r, "\mu"] & TX \end{tikzcd}$$

Example: Distribution monad

$TTX$

$$\frac{1}{2}\boxed{\frac{1}{3}\boxed{a} + \frac{2}{3}\boxed{b}} + \frac{1}{2}\boxed{\frac{2}{3}\boxed{a} + \frac{1}{3}\boxed{c}}$$

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Example: Free $S$-module monad ($S$ a semiring)

$$\underbrace{\tfrac{1}{2} \boxed{\tfrac{1}{3} \boxed{a} + \tfrac{2}{3} \boxed{b}} + \tfrac{1}{2} \boxed{\tfrac{2}{3} \boxed{a} + \tfrac{1}{3} \boxed{c}}}_{TTX} \overset{\mu}{\longmapsto} \underbrace{\tfrac{1}{2} \boxed{a} + \tfrac{1}{3} \boxed{b} + \tfrac{1}{6} \boxed{c}}_{TX}$$

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$\boxed{1}+\boxed{2}+\boxed{3}$

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$$\begin{tikzcd}
T\mathbb{N} & & \mathbb{N} \\
\boxed{2} \arrow[rr, shift left, mapsto, "e"] & & 2 \arrow[ll, shift left, mapsto, "\eta"]
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Example: (Commutative) monoid $\mathbb{N}$

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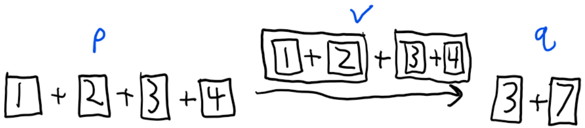

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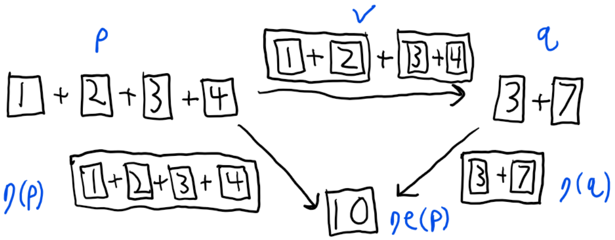

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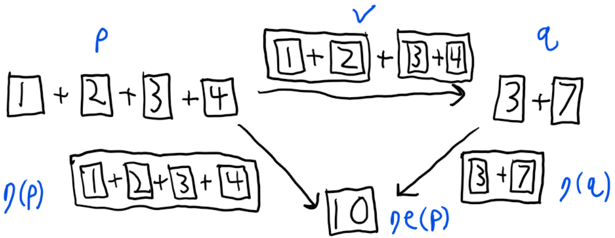


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$T\{*\}$ $\qquad$ $TT\{*\}$ $\qquad$ $T\{*\}$

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$$\begin{tikzcd}
 & 2\boxed{*} \arrow[dr, "\sqrt{2}\boxed{\sqrt{2}\boxed{*}}"] & \\
1\boxed{*} \arrow[ur, "1\boxed{0\boxed{*}} + 1\boxed{1\boxed{*}}"] \arrow[rr, dashed] & & \sqrt{2}\boxed{*}
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1\boxed{*} \arrow[ur, "1\boxed{0\boxed{*}}+1\boxed{1\boxed{*}}"] \arrow[rr, dashed, "s_1\boxed{r_1\boxed{*}}+\cdots+s_n\boxed{r_n\boxed{*}}"'] & & \sqrt{2}\boxed{*} \arrow[d, phantom, "\parallel"] \\
 & & s_1+\cdots+s_n
\end{tikzcd}$$

# Do Partial Evaluations Compose?

- A *partial evaluation* from $p$ to $q$ is a doubly nested expression $v \in TTA$ with $\mu(v) = p$ and $Te(v) = q$
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$$(s_1 r_1 + \cdots + s_n r_n)\boxed{*} \xrightarrow{s_1 \boxed{r_1 \boxed{*}} + \cdots + s_n \boxed{r_n \boxed{*}}} (s_1 + \cdots + s_n)\boxed{*}$$

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$$1 = \sqrt{2}\, r_1$$

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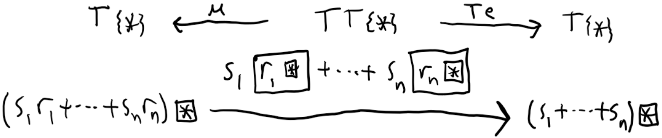


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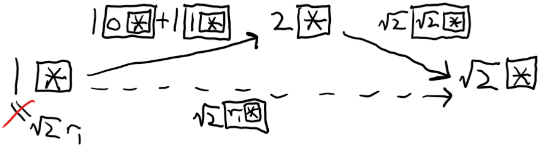

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$$1\boxed{*} \dashrightarrow^{\cancel{\sqrt{2}\boxed{r_1\boxed{*}}}} \sqrt{2}\boxed{*}$$

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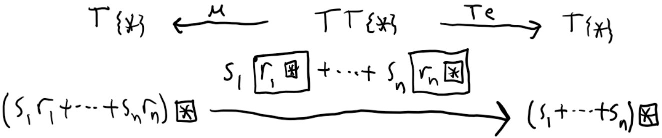


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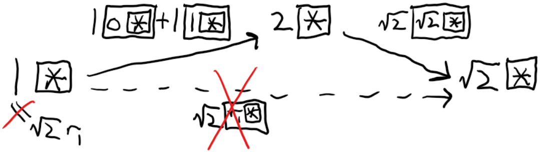


- (CFPS) Partial evaluations don't always compose

# Bar Construction

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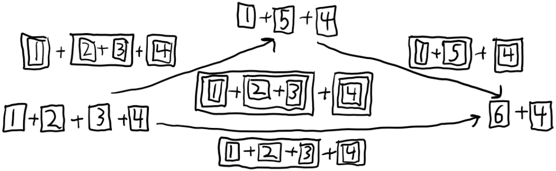

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...are given by the *simplicial identities*.

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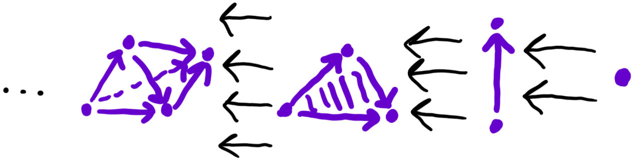

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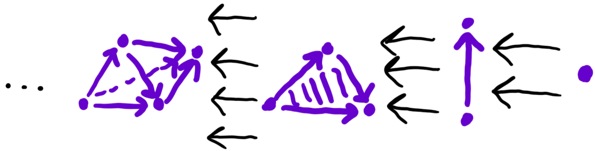

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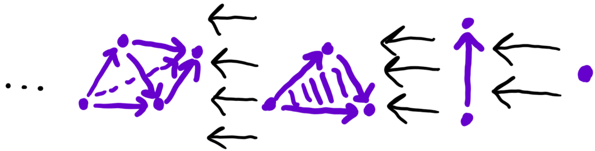
# Simplicial Sets



$$\cdots\ T^4A \ \substack{\xrightarrow{T^3e} \\ \xrightarrow{T^2\mu} \\ \xrightarrow{T\mu} \\ \xrightarrow{\mu}}\ T^3A \ \substack{\xrightarrow{T^2e} \\ \xrightarrow{T\mu} \\ \xrightarrow{\mu}}\ T^2A \ \substack{\xrightarrow{Te} \\ \xrightarrow{\mu}}\ TA$$

$$\cdots\ \{\text{3-simplices}\} \ \substack{\xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \\ \xrightarrow{d_3}}\ \{\text{2-simplices}\} \ \substack{\xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2}}\ \{\text{1-simplices}\} \ \substack{\xrightarrow{d_0} \\ \xrightarrow{d_1}}\ \{\text{0-simplices}\}$$

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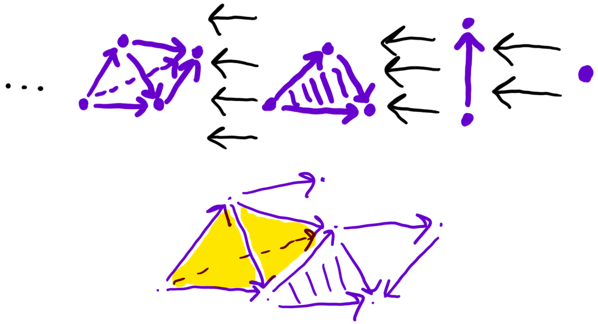

$$
\begin{tikzcd}[column sep=large]
\cdots \{\text{3-simplices}\} \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1"] \arrow[r, shift right=1.5, "d_2"] \arrow[r, shift right=4.5, "d_3"] & \{\text{2-simplices}\} \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"] & \{\text{1-simplices}\} \arrow[r, shift left=1.5, "d_0"] \arrow[r, shift right=1.5, "d_1"] & \{\text{0-simplices}\}
\end{tikzcd}
$$

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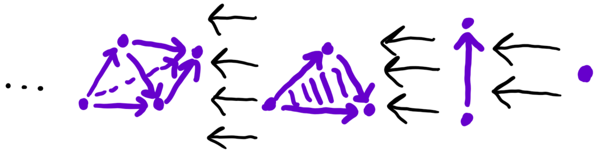

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$$

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\cdots\, \{\text{3-simplices}\} \arrow[r, shift left=4.5, "d_0"] \arrow[r, shift left=1.5, "d_1"] \arrow[r, shift right=1.5, "d_2"] \arrow[r, shift right=4.5, "d_3"] & \{\text{2-simplices}\} \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] \arrow[r, shift right=3, "d_2"] & \{\text{1-simplices}\} \arrow[r, shift left=1.5, "d_0"] \arrow[r, shift right=1.5, "d_1"] & \{\text{0-simplices}\}
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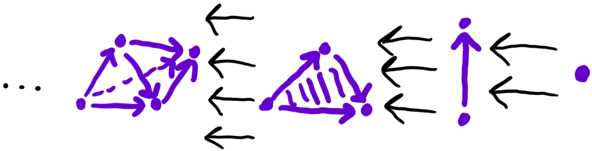

$$\cdots T^4A \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; T^3A \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; T^2A \;\substack{\longrightarrow\\ \longrightarrow}\; TA$$

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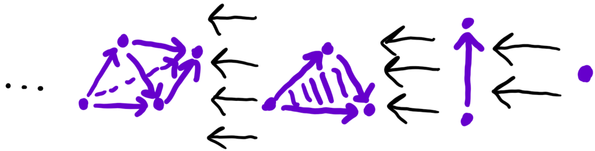

$$\begin{tikzcd}[column sep=large]
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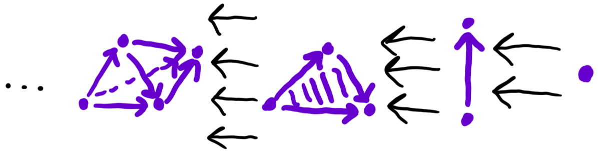

$$\cdots\ \{3\text{-simplices}\}\ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \xleftarrow{s_0}\\ \xleftarrow{s_1}\\ \xleftarrow{s_2}}\ \{2\text{-simplices}\}\ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \xleftarrow{s_0}\\ \xleftarrow{s_1}}\ \{1\text{-simplices}\}\ \substack{\longrightarrow\\ \longrightarrow\\ \xleftarrow{s_0}}\ \{0\text{-simplices}\}$$

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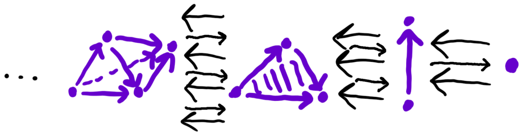

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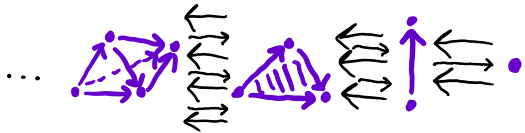

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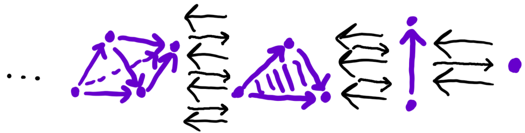

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$$\begin{tikzcd}
\cdots X_3 \arrow[r, shift left=9, "d_0"] \arrow[r, shift left=6, "d_1"] \arrow[r, shift left=3, "d_2"] \arrow[r, "d_3"] & X_2 \arrow[l, shift left=3, "s_0"] \arrow[l, shift left=6, "s_1"] \arrow[l, shift left=9, "s_2"] \arrow[r, shift left=6, "d_0"] \arrow[r, shift left=3, "d_1"] \arrow[r, "d_2"] & X_1 \arrow[l, shift left=3, "s_0"] \arrow[l, shift left=6, "s_1"] \arrow[r, shift left=3, "d_0"] \arrow[r, "d_1"] & X_0 \arrow[l, shift left=3, "s_0"]
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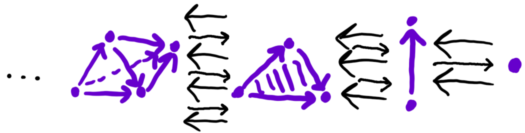

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$$
\begin{tikzcd}
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\arrow[r, shift left=6, "T^2\mu"]
\arrow[r, shift left=3, "T\mu"]
\arrow[r, "\mu"]
& T^3A
\arrow[l, shift left=3, "T^3\eta"]
\arrow[l, shift left=6, "T^2\eta"]
\arrow[l, shift left=9, "T\eta"]
\arrow[r, shift left=6, "T^2e"]
\arrow[r, shift left=3, "T\mu"]
\arrow[r, "\mu"]
& T^2A
\arrow[l, shift left=3, "T^2\eta"]
\arrow[l, shift left=6, "T\eta"]
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\begin{tikzcd}[column sep=large]
\cdots T^4A
\arrow[r, shift left=15, "T^3e"]
\arrow[r, shift left=10, "T^2\mu"]
\arrow[r, shift left=5, "T\mu"]
\arrow[r, "\mu"]
& T^3A
\arrow[l, shift left=5, "T^3\eta"]
\arrow[l, shift left=10, "T^2\eta"]
\arrow[l, shift left=15, "T\eta"]
\arrow[r, shift left=10, "T^2e"]
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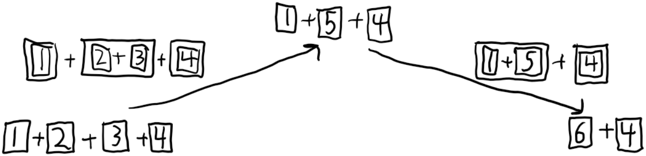
\arrow[l, shift left=6, "T^2\eta"]
\arrow[l, shift left=9, "T\eta"]
\arrow[r, shift left=6, "T^2e"]
\arrow[r, shift left=3, "T\mu"]
\arrow[r, "\mu"]
& T^2A
\arrow[l, shift left=3, "T^2\eta"]
\arrow[l, shift left=6, "T\eta"]
\arrow[r, shift left=3, "Te"]
\arrow[r, "\mu"]
& TA
\arrow[l, shift left=3, "T\eta"]
\end{tikzcd}
$$

- 1-simplices in this simplicial set are partial evaluations:



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\cdots T^4A \arrow[r, shift left=9, "T^3e"] \arrow[r, shift left=6, "T^2\mu"] \arrow[r, shift left=3, "T\mu"] \arrow[r, "\mu"] & T^3A \arrow[l, shift left=3, "T^3\eta"] \arrow[l, shift left=6, "T^2\eta"] \arrow[l, shift left=9, "T\eta"] \arrow[r, shift left=6, "T^2e"] \arrow[r, shift left=3, "T\mu"] \arrow[r, "\mu"] & T^2A \arrow[l, shift left=3, "T^2\eta"] \arrow[l, shift left=6, "T\eta"] \arrow[r, shift left=3, "Te"] \arrow[r, "\mu"] & TA \arrow[l, shift left=3, "T\eta"]
\end{tikzcd}
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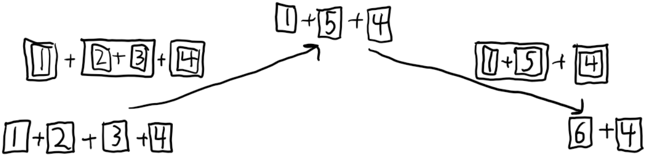
- 2-simplices in this simplicial set are "composition strategies":



# Simplicial Sets

- A *simplicial object* $X$ in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. Like the bar construction $Bar_T(A)$

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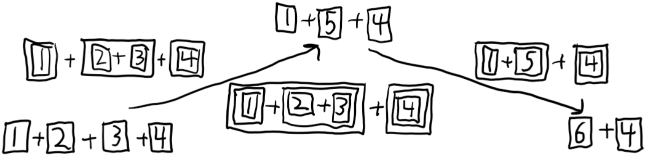

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$$\begin{tikzcd}[column sep=large]
\cdots T^4A \arrow[r, shift left=12, "T^3e"] \arrow[r, shift left=8, "T^2\mu"] \arrow[r, shift left=4, "T\mu"] \arrow[r, "\mu"] & T^3A \arrow[l, shift left=4, "T^3\eta"] \arrow[l, shift left=8, "T^2\eta"] \arrow[l, shift left=12, "T\eta"] \arrow[r, shift left=8, "T^2e"] \arrow[r, shift left=4, "T\mu"] \arrow[r, "\mu"] & T^2A \arrow[l, shift left=4, "T^2\eta"] \arrow[l, shift left=8, "T\eta"] \arrow[r, shift left=4, "Te"] \arrow[r, "\mu"] & TA \arrow[l, shift left=4, "T\eta"]
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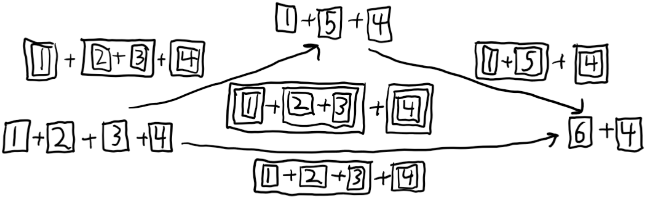

# Compositions

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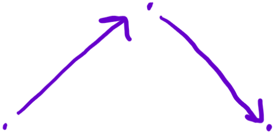

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& \cdot \arrow[dr] & \\
\cdot \arrow[ur] & & \cdot
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\begin{tikzcd}
\{*\} \arrow[drr] \arrow[drrrr] & & & & & \\
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& & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & T^2A \arrow[dll, "\mu"] \arrow[drr, "Te"'] & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & \\
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# Compositions

- If the square is a *weak pullback* (aka *weakly cartesian*), the dashed map always exists but not necessarily uniquely

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$$\begin{tikzcd}
C \arrow[drr, dashed, red, "\exists"] \arrow[drrr, red, bend left, "\forall"] \arrow[ddrr, red, "\forall"'] & & & \\
& & T^3A \arrow[r, "T^2e"] \arrow[d, "\mu"'] \arrow[dr, phantom, "wpb"] & T^2A \arrow[d, "\mu"] \\
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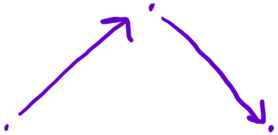

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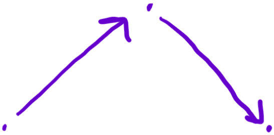

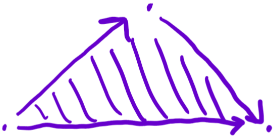

$$
\begin{tikzcd}
C \arrow[dr, dashed, red, "\exists"] \arrow[drr, red, "\forall"] \arrow[ddr, red, "\forall"'] & & \\
& T^3A \arrow[r, "T^2e"] \arrow[d, "\mu"'] \arrow[dr, phantom, "wpb"] & T^2A \arrow[d, "\mu"'] \\
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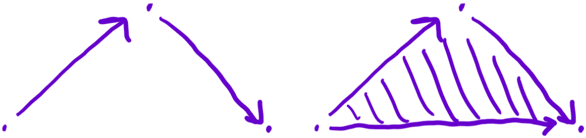

$$
\begin{tikzcd}
\Lambda \arrow[r, "\forall"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
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- When is $Bar_T(A)$ the nerve of a category? A quasicategory?

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$$
\begin{tikzcd}[column sep=small]
 & & & & & T^{n+1}A \arrow[dl, "\mu"'] \arrow[dr, "T^n e"] & & & & & \\
 & & & & T^nA \arrow[dl, "\mu"'] \arrow[dr, "T^{n-1}e"] & & T^nA \arrow[dl, "\mu"'] \arrow[dr, "T^{n-1}e"] & & & & \\
 & & & \cdots T^{n-1}A \cdots \arrow[dl] \arrow[dr] & & \cdots T^{n-1}A \cdots \arrow[dl] \arrow[dr] & & \cdots T^{n-1}A \cdots \arrow[dl] \arrow[dr] & & & \\
 & & T^3A \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] & & \cdots \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] & & \cdots \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] & & T^3A \arrow[dl, "\mu"'] \arrow[dr, "T^2e"] & & \\
 & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & \cdots \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & \\
TA & & TA & & TA & \cdots & TA & & TA & & TA
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- For $X = Bar_T(A)$, this means $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$

$$
\begin{tikzcd}[column sep=tiny]
 & & & & & T^{n+1}A \arrow[dl, "\mu"'] \arrow[dr, "T^n e"] & & & & & \\
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 & & & T^{n-1}A \arrow[dl] \arrow[dr] & & T^{n-1}A \arrow[dl] \arrow[dr] & & T^{n-1}A \arrow[dl] \arrow[dr] & & & \\
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 & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & \cdots \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & & T^2A \arrow[dl, "\mu"'] \arrow[dr, "Te"] & \\
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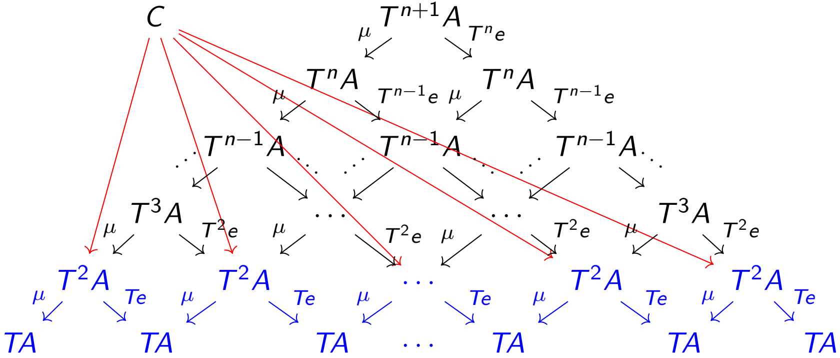

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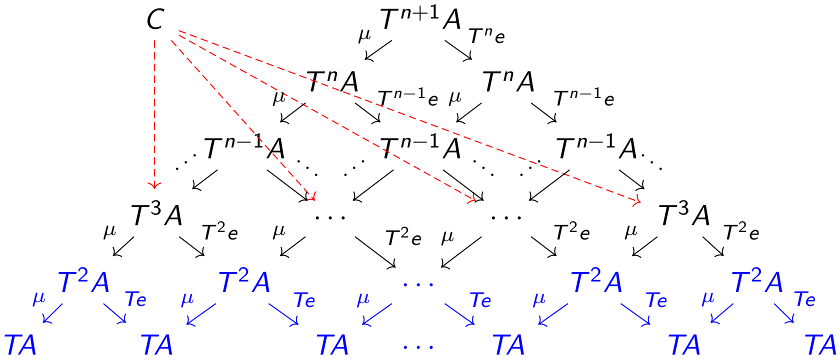

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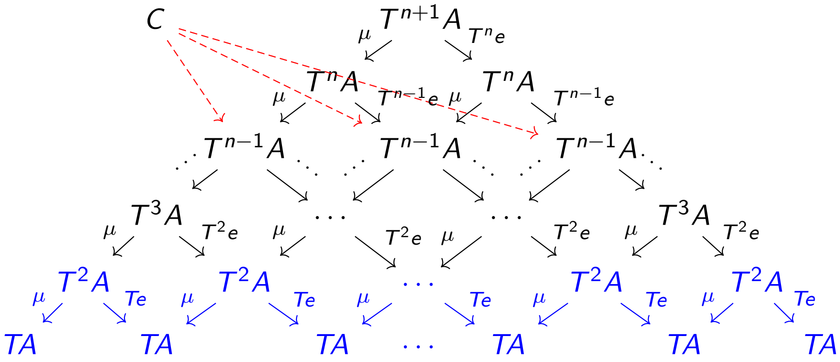

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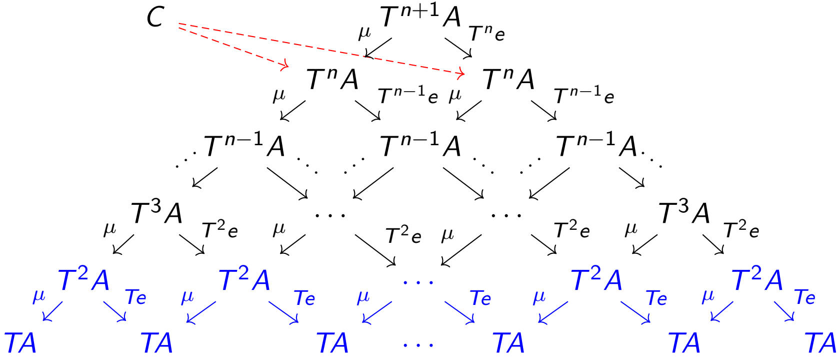

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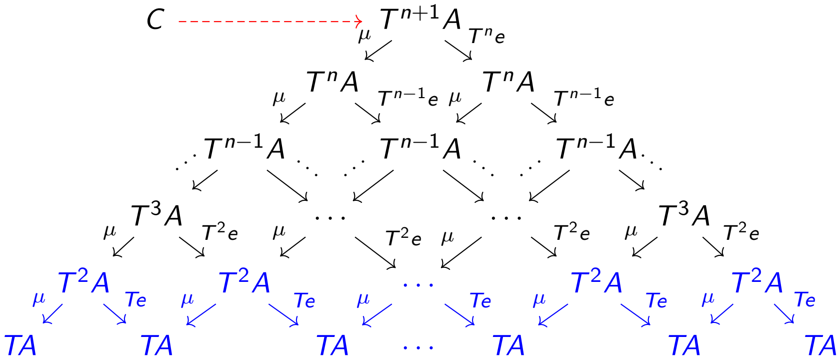
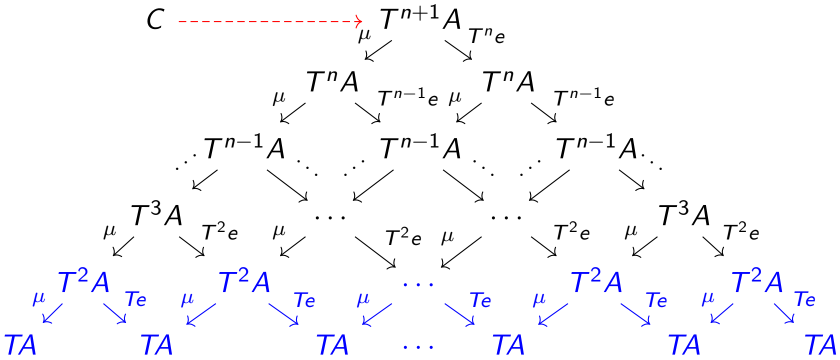


# Compositions

- When do partial evaluations form a category?
- If the naturality squares of $\mu$ are cartesian
- For $X = Bar_T(A)$, this means $X_n \cong X_1 \times_{X_0} \overset{n}{\cdots} \times_{X_0} X_1$
- This makes $Bar_T(A)$ the nerve of a category with formal expressions as objects and partial evaluations as morphisms



# BC Monads

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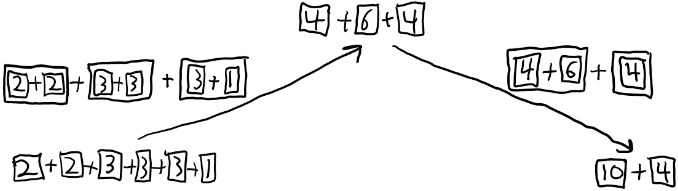


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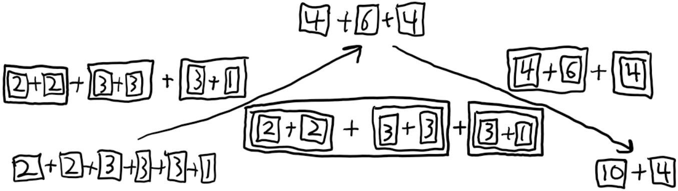


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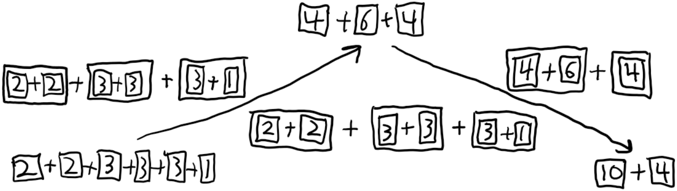


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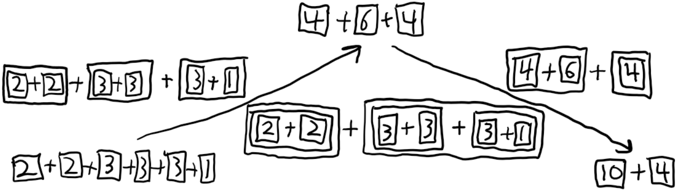


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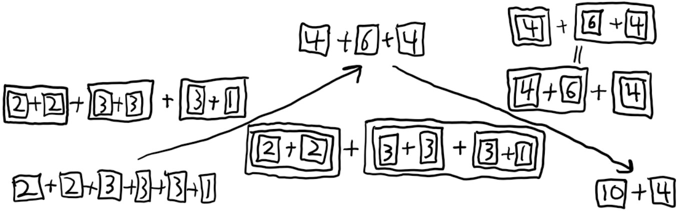


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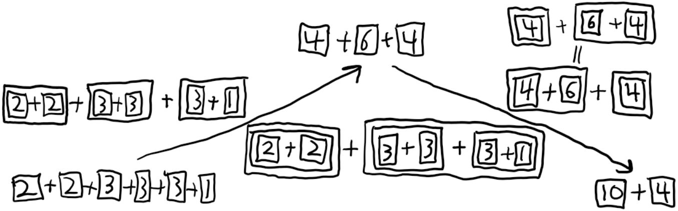


$$\begin{tikzcd}
T^{m+3}A \arrow[r, "T^{m+2}e"] \arrow[d, "\mu"'] \arrow[dr, phantom, "wpb"] & T^{m+2}A \arrow[d, "\mu"'] \\
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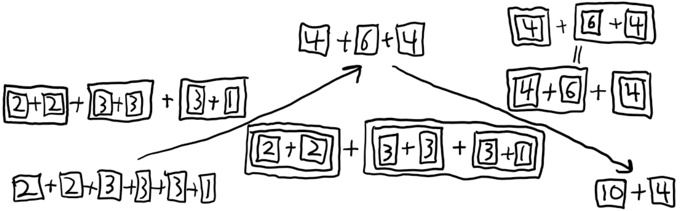


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\begin{array}{ccc}
T^{m+3}A & \xrightarrow{T^{m+2}e} & T^{m+2}A \\
\mu\downarrow & wpb & \mu\downarrow \\
T^{m+2}A & \xrightarrow{T^{m+1}e} & T^{m+1}A
\end{array}
\qquad
\begin{array}{ccc}
T^{m+4}A & \xrightarrow{T^{m+2}\mu} & T^{m+3}A \\
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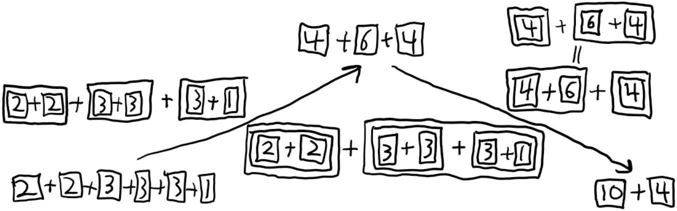


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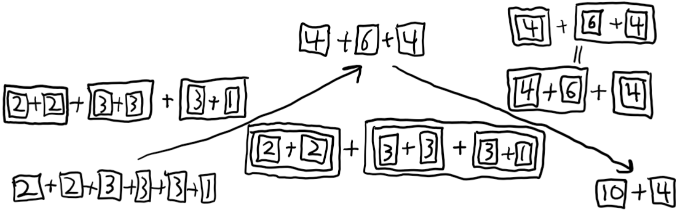


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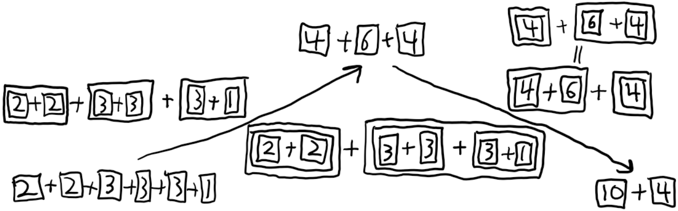


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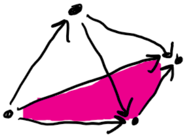

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T^n A \arrow[r, "T^{n-1}e"'] & T^{n-1}A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^{n+1}A \arrow[r, "T^{n-i}\mu"] \arrow[d, "T^{n-j}\mu"'] \arrow[dr, phantom, "wpb"] & T^n A \arrow[d, "T^{n-j}\mu"] \\
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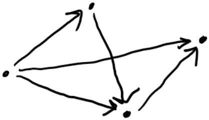

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$$\begin{tikzcd} \Delta^{n-1} \sqcup_{\Delta^{n-2}} \Delta^{n-1} \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

# Filler Conditions

- What properties does $Bar_T(A)$ have when $T$ is BC?
- Let $n \geq 2$, $j - i > 1$
- A simplicial set $X$ with this property is *inner span complete*

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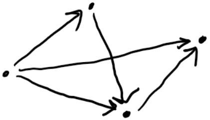

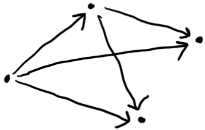

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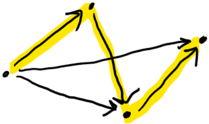
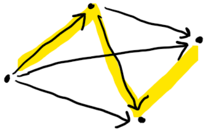
inner



outer



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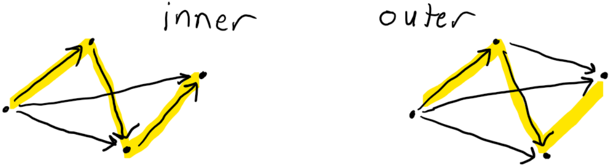


$$\begin{tikzcd} X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "wpb?"] & X_p \arrow[d] \\ X_q \arrow[r] & X_r \end{tikzcd}$$

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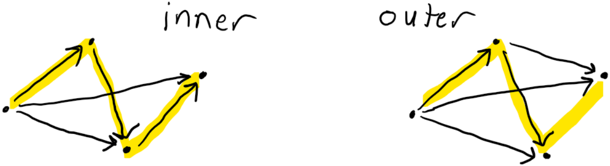


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- What properties does $Bar_T(A)$ have when $T$ is BC?
- Let $n \geq 2$, $j - i > 1$
- A simplicial set $X$ with this property is *inner span complete*
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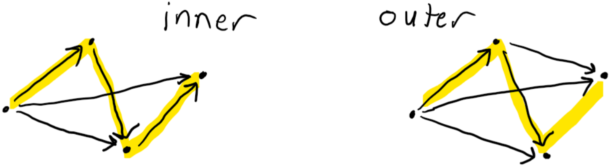


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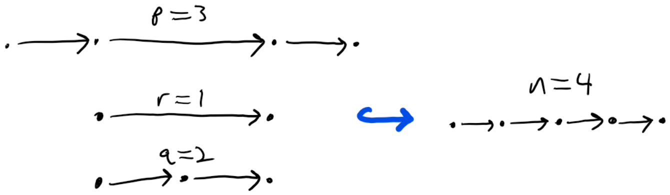


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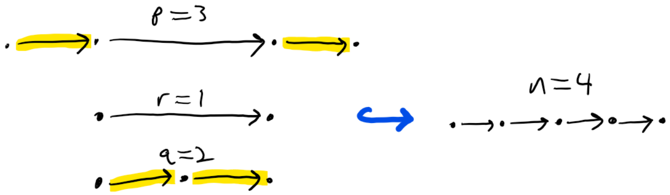


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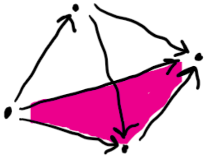

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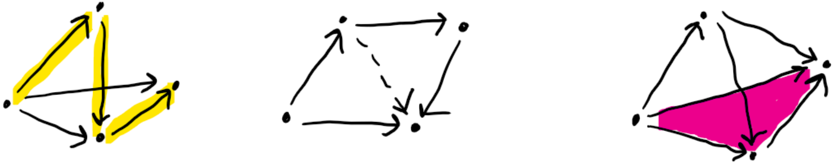

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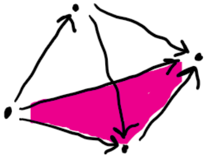

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Thank you!

# References


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