A GILLET–WALDHAUSEN THEOREM FOR CHAIN COMPLEXES OF SETS

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Abstract. The (A)CGW categories of Campbell and Zakharevich show how finite sets and varieties behave like the objects of an exact category for the purpose of algebraic K-theory. We further develop that program by defining chain complexes and quasi-isomorphisms for finite sets, which satisfy an analogue of the classical Gillet–Waldhausen Theorem: their K-theory agrees with the K-theory of finite sets. Along the way, we define new double categorical structures that modify those of Campbell and Zakharevich to include the data of weak equivalences. These *FCGWA categories* produce K-theory spectra which satisfy analogues of the Additivity and Fibration Theorems. The weak equivalences are determined by a subcategory of acyclic objects satisfying minimal conditions, resulting in a Localization Theorem that generalizes previous versions in the literature.

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Introduction

In recent work [CZ], Campbell and Zakharevich introduced a new type of structure, called ACGW-categories. These are double categories satisfying a list of additional axioms that seek to extract the properties of abelian categories which make them so particularly well-suited for algebraic K-theory.

Their key insight lies in the fact that the only morphisms that a K-theory machinery for abelian categories truly sees are the monomorphisms and epimorphisms, and moreover, that these are not required to interact with each other outside of the short exact sequences — or more generally, the bicartesian squares. This suggests that the monomorphisms and epimorphisms could form the horizontal and vertical morphisms in a double category, with squares the bicartesian squares, and that one should be able to axiomatize in the language of double categories any remaining crucial properties in order to obtain a K-theory machinery analogous to the Q-construction.

The main appeal of these double categories is that they generalize the structure of exact sequences in abelian categories to key non-additive settings such as finite sets and reduced schemes, where the notion of complements replaces that of kernels and cokernels. As well as versions of the $S_\bullet$- and Q-constructions, they recover analogues of classical results such as Quillen's Localization and Devissage Theorems, which were not previously available in any setting other than abelian (or exact) categories.

Much like Quillen's Q-construction, the K-theory of ACGW categories is not equipped to handle settings where there is a notion of "weak equivalences". We expand on the work of [CZ] to allow for the additional structure of this homotopical information. Naturally, this requires us to use an $S_\bullet$-construction of K-theory instead of a Q-construction, so we strengthen the axioms of [CZ] in order for $S_\bullet$ to have the expected behavior. We call our main structures FCGWA categories, which stands for Functorial CGW categories with Acyclics. As the name suggests, the weak equivalences in an FCGWA category are determined by a class of acyclic objects, much like the weak equivalences considered in [Sar20]. Then, an FCGWA category consists of a pair $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is an *FCGW category* (a modified version of an ACGW category) and $\mathcal{W}$ is a class of acyclic objects, called an *acyclicity structure*.

Our main motivating example, and the driving force behind this generalization, is that of chain complexes. Aside from being the building blocks of homological algebra, chain complexes on an exact category also play a crucial role in algebraic K-theory. When endowed with quasi-isomorphisms as the class of weak equivalences, they form a Waldhausen category, and the Gillet–Waldhausen Theorem tells us that the K-theory spectrum of an exact category $\mathcal{C}$ —with isomorphisms— is equivalent to the K-theory spectrum of bounded chain complexes on $\mathcal{C}$ — with quasi-isomorphisms. Chain complexes then provide an often more convenient model for the K-theory of exact categories.

Our aim is to construct a similar chain complex model for the K-theory of non-additive categories such as sets and varieties. In this spirit, we construct an FCGWA category of chain complexes of sets, while in future work with Inna Zakharevich we plan to do the same for varieties. The differentials in our chain complexes of sets are given by partial functions, which correspond to basepoint-preserving functions between pointed sets. Under this correspondence, our chain complexes agree with the "naive" notion of a chain complex of finite pointed sets: a sequence of basepoint-preserving functions such that any two that are adjacent compose to the constant function. The familiarity of these objects is an appealing part of our theory, though the morphisms and weak equivalences between them which determine their K-theory are more subtle, obtained through a different analogy with classical chain complexes more natural to the FCGWA formalism.

These chain complexes satisfy an analogue of the Gillet–Waldhausen Theorem, thus forming a new model for the K-theory of finite sets. It also provides further evidence that most classical results of algebraic K-theory can be adapted to the FCGWA setting, which we see as the theme of this work.

Theorem. (Theorem 12.3) *There exists a homotopy equivalence*

$$K(\mathsf{FinSet}) \simeq K(\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}, \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})^{\mathsf{b}})$$

between the K-theory of finite sets (with isomorphisms) and the K-theory of the FCGWA category of bounded chain complexes of finite sets (with weak equivalences determined by bounded exact complexes).

Just as [CZ] captures the essential features required to carry out Quillen's major foundational theorems, our FCGWA categories allow us to obtain many of Waldhausen's structural results. Chief among them are the Additivity Theorem, which any K-theory machinery is expected to satisfy, and the Fibration Theorem, which compares the K-theory of a category equipped with two classes of weak equivalences by constructing a homotopy fiber.

Theorem. (Theorem 8.1) *Let $\mathcal{V}$ and $\mathcal{W}$ be two acyclicity structures on an FCGW category $\mathcal{C}$, such that $\mathcal{V} \subseteq \mathcal{W}$. Then, there exists a homotopy fiber sequence*

$$K(\mathcal{W}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{W})$$

As a consequence of this result in the case where $\mathcal{V}$ is trivial, we obtain a Localization Theorem that generalizes those existing in the literature; this includes Quillen's original theorem for abelian categories [Qui73], Schlichting's [Sch04] and Cardenas' [Car98] Localization Theorems for exact categories, the first author's Localization Theorem obtained from cotorsion pairs [Sar20], and the Localization Theorem for ACGW categories of [CZ]. In the setting of FCGW categories arising from exact categories, it reads as follows:

Theorem. (Section 9.1) *Let $\mathcal{B}$ be an exact category and $\mathcal{A} \subseteq \mathcal{B}$ a full subcategory such that if any two terms in an exact sequence in $\mathcal{B}$ are in $\mathcal{A}$, then the third term is as well. Then there exists an FCGWA category $(\mathcal{B}, \mathcal{A})$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}, \mathcal{A})$$

is a homotopy fiber sequence.

This version of the Localization Theorem has fewer requirements than any of the aforementioned Localization Theorems, though this comes at the expense that the model constructed for the cofiber is not generally an exact or even Waldhausen category, but instead an FCGWA category. However, we consider this not a shortcoming of the theorem but an advertisement for the relevance of FCGWA categories.

Outline. The first part of this work introduces the main protagonists. After a brief tour through the world of double categories in Section 1, we define pre-FCGW categories in Section 2 as double categories with some additional structure and properties. In Section 3 we introduce FCGW categories, which satisfy stronger axioms that allow us to prove our foundational K-theory theorems in Section 4. Finally, Section 4 contains the definition of our principal structures of interest: FCGWA categories. These are FCGW categories that allow for a notion of weak equivalences, defined from a class of acyclic objects.

The second part contains our main results regarding the K-theory of FCGWA categories. First, Section 5 introduces an $S_\bullet$ construction for FCGWA categories. We support this definition by showing that K_0 admits the expected explicit description as a Grothendieck group, and that this K-theory agrees with that of the known examples of exact categories with weak equivalences which form FCGWA categories, and with the K-theory of their underlying CGW categories as defined in [CZ] when the weak equivalences are simply isomorphisms.

The next sections are dedicated to several foundational results. Section 6 shows that our K-theory machinery satisfies the Additivity Theorem, and in Section 7 we show how our $S_\bullet$ construction, which lends itself to iteration, produces a spectrum. Section 8 proves our version of Waldhausen's Fibration Theorem, which relates the K-theory spectra of an FCGW category equipped with two comparable classes of weak equivalences by constructing a homotopy fiber. In a similar vein, we obtain a Localization Theorem in Section 9 that allows us to relate the K-theories of an inclusion of FCGW categories by constructing a homotopy cofiber; we then compare this to previous Localization Theorems in the literature.

In the third part, we construct our main novel example of FCGWA categories: chain complexes of sets, with a notion of quasi-isomorphisms. Section 10 is devoted to proving that chain complexes of finite sets form an FCGW category; Section 11 further gives an FCGWA structure by considering exact chain complexes as acyclics. In turn, Section 12 contains a Gillet–Waldhausen Theorem that establishes these chain complexes as an alternate model for the K-theory of sets.

Finally, the appendix deals with a collection of technical results building up to the proofs that each level of the $S_\bullet$ construction and the grids used to prove the Fibration Theorem form FCGWA categories.

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Part 1. FCGWA categories

1. Double categorical preliminaries

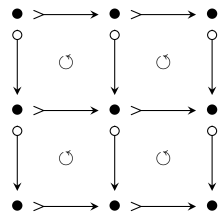
Double categories, originally defined as categories internal to categories, describe categorical settings with two different types of morphisms, related by higher cells called squares. In this section, we recall the well-known notions of double categories, double functors, and the natural transformations between them, as well as the space associated to a double category. We also introduce a notion of double categories with shared isomorphisms and discuss a natural notion of equivalence between them that will be useful in later sections.

Definition 1.1. A double category $\mathcal{C}$ consists of:

- a set of objects $\mathrm{Ob}(\mathcal{C})$
- two categories $\mathcal{M}$ and $\mathcal{E}$ with the same objects as $\mathcal{C}$. We call their maps **m-morphisms** ($\rightarrowtail$) and **e-morphisms** ($\circ\!\!\rightarrow$), respectively
- a set of squares of the form

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ {\scriptstyle g}\,\downarrow\!\circ & \circlearrowleft & \circ\!\downarrow\,{\scriptstyle g'} \\ C & \underset{f'}{\rightarrowtail} & D \end{array}$$

- categories $\mathrm{Ar}_\circlearrowleft \mathcal{M}$, $\mathrm{Ar}_\circlearrowleft \mathcal{E}$ with objects the m-morphisms (resp. e-morphisms) and maps from f to f' (resp. g to g') given by the squares above, such that
- composite and identity squares respect those of the e-morphisms (resp. m-morphisms) along their sides, and satisfy the interchange law: in a grid



applying the composition operations in either order yields the same result.

Remark 1.2. In the definition above, we use the symbol $\circlearrowleft$ to denote that there exists a square having the depicted boundary; this should not be interpreted as the square being a commutative diagram, especially since m- and e-morphisms need not compose among each other.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be double categories. A **double functor** $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ consists of an assignment on objects, m-morphisms, e-morphisms, and squares, which are compatible with domains and codomains and preserve all double categorical compositions and identities.

Definition 1.4. A double functor is **full** (resp. **faithful**) if it is surjective (resp. injective) on each set of m-morphisms and e-morphisms with fixed source and target, and on each set of squares with fixed boundary.

We say a double subcategory $\mathcal{C} \subseteq \mathcal{D}$ is full if the inclusion is a full double functor.

The category of double categories is cartesian closed, and thus there exists a double category whose objects are the double functors. We briefly describe the horizontal morphisms, vertical morphisms, and squares of this double category; the reader unfamiliar with double categories is encouraged to see [Gra20, §3.2.7] for more explicit definitions.

Definition 1.5. Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be double functors. A horizontal natural transformation $\mu\colon F \Rightarrow G$, which we henceforth call **m-natural transformation**, consists of

- an m-morphism $\mu_A\colon FA \rightarrowtail GA$ in $\mathcal{D}$ for each object $A \in \mathcal{C}$, and
- a square

$$\begin{array}{ccc} FA & \overset{\mu_A}{\rightarrowtail} & GA \\ {\scriptstyle Ff}\,\big\downarrow & \circlearrowleft & \big\downarrow\,{\scriptstyle Gf} \\ FB & \underset{\mu_B}{\rightarrowtail} & GB \end{array}$$

in $\mathcal{D}$ for each e-morphism $f\colon A \multimap B$ in $\mathcal{C}$,

such that the assignment of squares is functorial with respect to the composition of e-morphisms, and that these data satisfy a naturality condition with respect to m-morphisms and squares.

Dually, one defines a vertical natural transformation, which we call **e-natural transformation**.

Definition 1.6. Given m-natural transformations $\mu\colon F \Rightarrow G$, $\mu'\colon F' \Rightarrow G'$ and e-natural transformations $\eta\colon F \Rightarrow F'$, $\eta'\colon G \Rightarrow G'$ between double functors $\mathcal{C} \longrightarrow \mathcal{D}$, a **modification** α shown below left

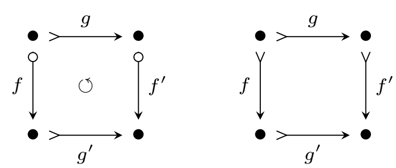
$$\begin{array}{ccc} F & \overset{\mu}{\longrightarrow} & G \\ {\scriptstyle \eta}\big\downarrow & \alpha & \big\downarrow{\scriptstyle \eta'} \\ F' & \underset{\mu'}{\longrightarrow} & G' \end{array} \qquad \begin{array}{ccc} FA & \overset{\mu_A}{\longrightarrow} & GA \\ {\scriptstyle \eta_A}\big\downarrow & \circlearrowleft\,\alpha_A & \big\downarrow{\scriptstyle \eta'_A} \\ F'A & \underset{\mu'_A}{\longrightarrow} & G'A \end{array}$$

consists of a square in $\mathcal{D}$ as above right for each object $A \in \mathcal{C}$, satisfying horizontal and vertical coherence conditions with respect to the squares of the transformations μ, μ', η, and η'.

The double categories of interest to this paper arise from taking m- and e-morphisms to be certain classes of morphisms in some category, and squares from certain commuting squares in the ambient category. For these, it will be convenient for the two classes of maps in the double category to have a common class of isomorphisms. To that purpose, we introduce the following notion.

Definition 1.7. A double category $\mathcal{C}$ has **shared isomorphisms** if:

- there is a groupoid I with identity-on-objects functors $\mathcal{M} \leftarrow I \rightarrow \mathcal{E}$ which create isomorphisms. For a morphism f in I, we write f for both the corresponding m-isomorphism and e-isomorphism, which we distinguish in diagrams by the different arrow shapes
- for isomorphisms f, f' and m-morphisms g, g' there is a (unique) square as below left if and only if the square below right commutes in $\mathcal{M}$



- the analogous correspondence holds between squares in $\mathcal{C}$ and commuting squares in $\mathcal{E}$ for isomorphisms f, f' and e-morphisms g, g'

In our double categories of interest, squares between fixed m- and e-morphisms will be unique when they exist, so the uniqueness of the squares above will be inconsequential.

The unification of m- and e-isomorphisms extends to natural isomorphisms between double functors as well, which allows us to define a canonical notion of equivalence of double categories with shared isomorphisms.

Definition 1.8. Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be double functors, where $\mathcal{D}$ has shared isomorphisms. A **natural isomorphism** $\alpha\colon F \cong G$ consists of an isomorphism $\alpha_A\colon FA \cong GA$ for each object A in $\mathcal{C}$, such that when we regard all α_A as m-morphisms (resp. e-morphisms), α is an m- (resp. e-) natural transformation.

Remark 1.9. Note that any natural isomorphism will be such that the component squares of the m- and e-natural transformation α are invertible (horizontally or vertically, as it corresponds), by the uniqueness of the squares in Definition 1.7. Definition 1.7 also shows that the naturality condition can be reduced to checking that the components of α form a natural transformation in the 1-categorical sense between the underlying functors on m-morphisms and e-morphisms, so it is not necessary here to provide naturality squares in the data of α.

We can use these natural isomorphisms to define a notion of equivalence between double categories with shared isomorphisms. A careful study of these equivalences is beyond the scope of this paper; our goal is simply to show that they induce homotopy equivalences of spaces after realization.

Definition 1.10. Let $\mathcal{C}, \mathcal{D}$ be double categories with shared isomorphisms. An **equivalence** between $\mathcal{C}$ and $\mathcal{D}$ is a pair of double functors $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ equipped with natural isomorphisms $1_{\mathcal{C}} \cong GF$ and $FG \cong 1_{\mathcal{D}}$.

A definition of this form is not possible for general double categories without making arbitrary choices for whether the natural isomorphisms are m- or e-transformations.

This is appropriate for the double categories we consider which arise from categories, and has the following convenient characterization.

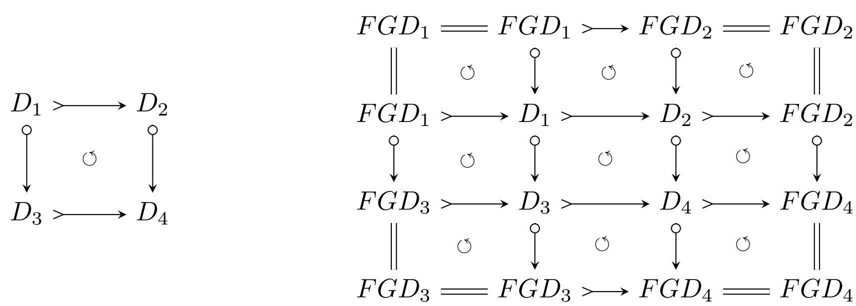
Proposition 1.11. *Let $F: \mathcal{C} \longrightarrow \mathcal{D}$ be a double functor between double categories with shared isomorphisms. Then, F belongs to an equivalence if and only if it is fully faithful and essentially surjective.*

Here essentially surjective means that every object in $\mathcal{D}$ is isomorphic to FC for some object C in $\mathcal{C}$, just as for ordinary categories.

Proof. Given an equivalence $F: \mathcal{C} \longrightarrow \mathcal{D}$, F is essentially surjective and fully faithful on m- and e-morphisms as the restrictions $F_{\mathcal{M}}: \mathcal{M}_{\mathcal{C}} \longrightarrow \mathcal{M}_{\mathcal{D}}$ and to $F_{\mathcal{E}}: \mathcal{E}_{\mathcal{C}} \longrightarrow \mathcal{E}_{\mathcal{D}}$ form equivalences of categories. Lastly, F is fully faithful on squares relative to their boundaries, since F is a double biequivalence (see [MSV20, Definition 3.7]) by [MSV20, Proposition 5.14], and thus in particular fully faithful on squares.

Given a fully faithful and essentially surjective double functor $F: \mathcal{C} \longrightarrow \mathcal{D}$, by the classical characterization of equivalences of categories, both $F_{\mathcal{M}}$ and $F_{\mathcal{E}}$ form equivalences of categories. Furthermore, as the objects and isomorphisms of $\mathcal{M}$ and $\mathcal{E}$ are the same for both $\mathcal{C}$ and $\mathcal{D}$ (in the sense of shared isomorphisms), the quasi-inverses $G_{\mathcal{M}}: \mathcal{M}_{\mathcal{C}} \longrightarrow \mathcal{M}_{\mathcal{D}}$ and $G_{\mathcal{E}}: \mathcal{E}_{\mathcal{C}} \longrightarrow \mathcal{E}_{\mathcal{D}}$ can be chosen to agree on objects and such that the isomorphisms $F_{\mathcal{M}}G_{\mathcal{M}}D \cong D$ and $F_{\mathcal{E}}G_{\mathcal{E}}D \cong D$ also agree, as in the classical construction of these quasi-inverses those choices are made arbitrarily (see, for example, [Rie16, Theorem 1.5.9]). It follows immediately from the proof in loc. cit. that under these choices, the isomorphisms $C \cong G_{\mathcal{M}}F_{\mathcal{M}}C$ and $C \cong G_{\mathcal{E}}F_{\mathcal{E}}C$ agree as well, by observing that any double functor between double categories with shared isomorphisms preserves the correspondence between m- and e-isomorphisms.

We can now define a double functor $G: \mathcal{D} \longrightarrow \mathcal{C}$ which restricts to $G_{\mathcal{M}}$ on $\mathcal{M}_{\mathcal{D}}$ and $G_{\mathcal{E}}$ on $\mathcal{E}_{\mathcal{D}}$. It remains only to define how G acts on squares; given a square α in $\mathcal{D}$ as below left, we construct the square below right in the image of F.



The outer squares on the right above are pseudo-commutative by Definition 1.7 and by naturality of the isomorphisms $F_{\mathcal{M}}G_{\mathcal{M}}D \cong D$ and $F_{\mathcal{E}}G_{\mathcal{E}}D \cong D$ in $\mathcal{M}_{\mathcal{D}}$, $\mathcal{E}_{\mathcal{D}}$. As F is fully faithful on squares, this composite square has a unique preimage in $\mathcal{C}$, which we define to be $G(\alpha)$.

It is then tedious but straightforward to check that G respects identities and composites of squares, and that the isomorphisms $C \cong GFC$ and $FGD \cong D$ for C in $\mathcal{C}$ and D in $\mathcal{D}$ are natural, making F, G into an equivalence of double categories. □

Finally, we recall that the process of constructing a space from a category by taking the geometric realization of its nerve has an analogue in double categories, as defined for example in [FP10, Definition 2.14]. This is an especially important construction for us, as it will be used to define the K-theory space of our double categories of interest.

Definition 1.12. The double nerve, or **bisimplicial nerve**, of a double category $\mathcal{C}$ is the bisimplicial set $N_{\square}\mathcal{C}$ whose (m, n)-simplices are the $m \times n$-matrices of composable squares in $\mathcal{C}$.

We let $|\mathcal{C}|$ denote the geometric realization of the bisimplicial set $N_\square\mathcal{C}$, or, equivalently, the geometric realization of its diagonal simplicial set $n \mapsto N_\square\mathcal{C}_{n,n}$. Going forward, we abuse notation and use these two spaces interchangeably.

Lemma 1.13. *Let $\mathcal{C}, \mathcal{D}$ be double categories with shared isomorphisms. If there exists an equivalence between $\mathcal{C}$ and $\mathcal{D}$, then $|\mathcal{C}|$ and $|\mathcal{D}|$ are homotopy equivalent.*

Proof. This can be deduced from [FP10, Proposition 2.22], since a natural isomorphism α as in Definition 1.8 determines a "2-fold natural transformation" in the sense of [FP10, Definition 2.20] (which agrees with what we call a modification in Definition 1.6) given by α in one direction and identities in the other. This can also be proven directly by noting that an m-natural transformation can be equivalently described as a double functor $\mathcal{C} \times \Delta^1_m \longrightarrow \mathcal{D}$, where Δ^1_m is the double category with a single non-identity m-morphism and whose geometric realization is the interval. □

2. pre-FCGW categories

In this section, we introduce pre-FCGW categories and establish the necessary categorical yoga. Pre-FCGW categories are almost identical to the pre-ACGW categories of [CZ], as their name suggests. The differences are that we begin with pseudo-commutative squares and define distinguished squares among them by a property, replace pullback squares of m- and e-morphisms with a more flexible notion of "good" squares, and don't require axioms (S) or (A) involving pushouts and sums. Pushouts (and consequently sums) will be axiomatized in the following section on FCGW categories.

All names aside, the purpose of these double categories is to capture the essential features of exact categories that make them so suitable for K-theory, while allowing for a non-additive setting. First of all, they have two classes of maps that mimic the role of admissible monomorphisms and admissible epimorphisms (reversing the direction of the latter): these will be the m- and e-morphisms in the double category. They also contain associated notions of (co)kernels and short exact sequences, but instead of defining these as certain (co)limits that would require an additive setting, their relevant features are axiomatized. This allows one to expand the classical intuition from exact categories to other settings such as sets and varieties, as done in [CZ].

Notation 2.1. Following the ACGW categories of [CZ], from now on the squares in a double category will be called "mixed" or "pseudo-commutative" squares. This last nomenclature was inspired by the fact that, when working with abelian categories, the role of the pseudo-commutative squares is played by the commutative squares between monomorphisms and epimorphisms in the category.

Throughout this paper, we work with several different categories with objects the m- or e-morphisms of $\mathcal{C}$, such as $\operatorname{Ar}_\circlearrowleft \mathcal{M}$, $\operatorname{Ar}_\circlearrowleft \mathcal{E}$ introduced in Definition 1.1. We also recall the following notation from [CZ, Definition 2.4].

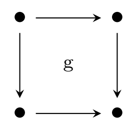
Definition 2.2. Given a double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$, let $\operatorname{Ar}_\triangle \mathcal{M}$ denote the category whose objects are morphisms $A \rightarrowtail B \in \mathcal{M}$, and where

$$\operatorname{Hom}_{\operatorname{Ar}_\triangle \mathcal{M}}(A \overset{f}{\rightarrowtail} B, A' \overset{f'}{\rightarrowtail} B') = \left\{ \begin{array}{cc} \text{commutative squares} & \begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ \cong\downarrow & & \downarrow \\ A' & \underset{f'}{\rightarrowtail} & B' \end{array} \end{array} \right\}.$$

Similarly, we have a category $\mathrm{Ar}_\triangle \mathcal{E}$ defined analogously.

We can imitate this definition for more general types of squares.

Definition 2.3. Given a category $\mathcal{A}$, a **class of good squares** is a subcategory $\mathrm{Ar}_{\mathrm{g}} \mathcal{A}$ of the category $\mathrm{Ar}\,\mathcal{A}$ with objects arrows in $\mathcal{A}$ and morphisms commuting squares between them. Good squares in $\mathrm{Ar}_{\mathrm{g}} \mathcal{A}$ are denoted by



Examples of classes of good squares include the *weak triangles* of $\mathrm{Ar}_\triangle \mathcal{A}$ and the pullback squares denoted $\mathrm{Ar}_\times \mathcal{A}$.

We now define pre-FCGW categories. The reader unfamiliar with (A)CGW categories is strongly encouraged to read each axiom together with its counterpart in exact categories, explained below in Example 2.6.

Definition 2.4. A **pre-FCGW category** is a double category $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ with shared isomorphisms, equipped with

- classes of good squares $\mathrm{Ar}_{\mathrm{g}} \mathcal{M}$ and $\mathrm{Ar}_{\mathrm{g}} \mathcal{E}$
- equivalences of categories $k\colon \mathrm{Ar}_{\circlearrowleft} \mathcal{E} \longrightarrow \mathrm{Ar}_{\mathrm{g}} \mathcal{M}$ and $c\colon \mathrm{Ar}_{\circlearrowleft} \mathcal{M} \longrightarrow \mathrm{Ar}_{\mathrm{g}} \mathcal{E}$

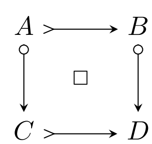
such that

(Z) $\mathcal{M}, \mathcal{E}$ each have initial objects which agree

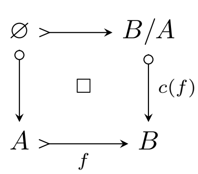
(M) All morphisms in $\mathcal{M}, \mathcal{E}$ are monic

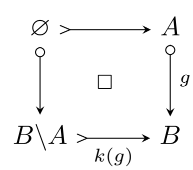
(G) $\mathrm{Ar}_\triangle \mathcal{M} \subseteq \mathrm{Ar}_{\mathrm{g}} \mathcal{M} \subseteq \mathrm{Ar}_\times \mathcal{M}$ and $\mathrm{Ar}_\triangle \mathcal{E} \subseteq \mathrm{Ar}_{\mathrm{g}} \mathcal{E} \subseteq \mathrm{Ar}_\times \mathcal{E}$

(D) k sends a pseudo-commutative square to $Ar_\triangle\mathcal{M} \subset Ar_{\mathrm{g}}\mathcal{M}$ if and only if c sends the square to $Ar_\triangle\mathcal{E} \subset Ar_{\mathrm{g}}\mathcal{E}$. In this case the square is called **distinguished** and is denoted as follows:



(K) For any m-morphism $f\colon A \rightarrowtail B$ there is a distinguished square as below left, and for any e-morphism $g\colon A \circ\!\!\rightarrow B$ there is a distinguished square as below right.





The notation $B/A, B\backslash A$ will only be used when the defining maps f and g are clear from context. Otherwise the cokernel and kernel objects will be denoted $\operatorname{coker} f, \ker g$ respectively.

Remark 2.5. The double subcategory of distinguished squares of any pre-FCGW category forms a CGW category[1] by restricting the functors k and c to this subcategory, where axiom (I) of

[1] A careful reader might observe that axiom (A) of CGW categories is missing in our formulation. However, this will hold in all examples of interest (FCGW-categories) as we discuss in Remark 3.3.

CGW categories follows from the properties of shared isomorphisms in Definition 1.7. Conversely, any CGW category satisfying these stronger isomorphism conditions gives a pre-FCGW category where the only squares are the distinguished ones, and the good squares are given by $\mathrm{Ar}_\triangle \mathcal{M}$ and $\mathrm{Ar}_\triangle \mathcal{E}$.

Therefore, it is not surprising that all of the basic examples of interest agree with those of [CZ, Section 3]. We include them here as well, since they illustrate the ideas behind the axioms; in particular, the first example illustrates the motivation behind good squares, which are new to our formulation.

Example 2.6. Let $\mathcal{A}$ be an exact category, and let $\mathcal{C} = (\mathcal{M}, \mathcal{E})$ be the double category with the same objects as $\mathcal{A}$, and where

$$\mathcal{M} = \{\text{admissible monomorphisms}\}$$

We want the functors k and c to be the usual kernel and cokernel functors, and the cokernel of an admissible monomorphism $i\colon A \rightarrowtail B$ is an admissible epimorphism $B \twoheadrightarrow \operatorname{coker} i$. Keeping axioms (M) and (K) in mind, this suggests we should let $\mathcal{E}$ be the admissible epimorphisms pointing in the opposite direction; i.e.,

$$\mathcal{E} = \{\text{admissible epimorphisms}\}^{\mathrm{op}}$$

We must now define the good squares and the pseudo-commutative squares in the double category accordingly. Given a pullback square of admissible monomorphisms as below, the induced map on cokernels is always a monomorphism.

$$\begin{array}{ccccc}
A & \overset{i}{\rightarrowtail} & B & \longleftarrow\!\circ & \operatorname{coker} i \\
\downarrow & & \downarrow & & \vdots \\
A' & \overset{i'}{\rightarrowtail} & B' & \longleftarrow\!\circ & \operatorname{coker} i'
\end{array}$$

We claim that this monomorphism will be admissible precisely when the induced morphism out of the pushout $B \cup_A A' \longrightarrow B'$ is an admissible monomorphism. Indeed, one can factor the diagram above as follows, where all rows are exact

$$\begin{array}{ccccc}
A & \overset{i}{\rightarrowtail} & B & \longleftarrow\!\circ & \operatorname{coker} i \\
\downarrow & & \downarrow & & \| \\
A' & \rightarrowtail & B \cup_A A' & \longleftarrow\!\circ & \operatorname{coker} i \\
\| & & \downarrow & & \vdots \\
A' & \overset{i'}{\rightarrowtail} & B' & \longleftarrow\!\circ & \operatorname{coker} i'
\end{array}$$

Applying the Snake Lemma to the bottom part of the diagram, we see that

$$\operatorname{coker}(B \cup_A A' \longrightarrow B') \cong \operatorname{coker}(\operatorname{coker} i \longrightarrow \operatorname{coker} i');$$

thus, one of these monomorphisms is admissible if and only if the other one is.

This leads us to define the good squares in $\mathcal{M}$ as the pullback squares of maps in $\mathcal{M}$ with this pushout property, which include weak triangles as pushouts preserve isomorphisms. The pseudo-commutative squares are then the squares who commute in $\mathcal{A}$, and such that the morphism induced on kernels (which is always a monomorphism) is admissible. One can show that the dual notion of good squares in $\mathcal{E}$ is also compatible with this class of pseudo-commutative squares.

Once the structure has been determined, the axioms are not hard to check. Axiom (Z) holds since 0 is both initial and terminal. Axiom (M) is immediate, since monomorphisms are monics, epimorphisms are epics, and epics become monic in the opposite category. Axiom (D) is also satisfied, and one finds that distinguished squares are the bicartesian squares. Axiom (K) is the familiar statement that any admissible monomorphism (resp. epimorphism) determines a short exact sequence by taking its cokernel (resp. kernel), which is constructed as the pushout (resp. pullback) along the unique map to (resp. from) the 0 object.

That this double category has shared isomorphisms follows immediately, as a map in an exact category is an isomorphism if and only if it is both an admissible monomorphism and an admissible epimorphisms, and pseudo-commutative squares are defined to agree with commuting squares, where a square with parallel isomorphisms always induces an isomorphism on kernels (resp. cokernels).

Remark 2.7. If the exact category $\mathcal{A}$ in the previous example is abelian, then all monomorphisms and epimorphisms are admissible and the pre-FCGW structure is somewhat simplified. In this case, the good squares are precisely the pullbacks of monomorphisms or pushouts of epimorphisms, and the pseudo-commutative squares are simply the commuting squares.

Example 2.8. We can define a double category of finite sets $\mathsf{FinSet} = (\mathcal{M}, \mathcal{E})$ by setting

$$\mathcal{M} = \mathcal{E} = \{\text{injective functions}\}$$

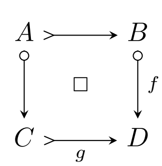
and letting both pseudo-commutative squares be the pullback squares. Both of the functors k and c take an injection $A \longrightarrow B$ to the inclusion of the complement of its image $B \backslash A \longrightarrow B$. With $\emptyset$ as the initial object and good squares also the pullbacks, this gives a pre-FCGW category. The distinguished squares are then the pushout squares of injections, so this agrees with [CZ, Example 3.3].

Example 2.9. We can define a double category Var whose objects are varieties, with m- and e-morphisms given by

$$\mathcal{M} = \{\text{closed immersions}\} \qquad \text{and} \qquad \mathcal{E} = \{\text{open immersions}\}$$

Like the example above, pseudo-commutative and good squares are given by (all) pullback squares (as varieties are closed under pullbacks), and the functors k and c take a morphism to the inclusion of its complement. This example is identical to [CZ, Example 3.4], except we swap open and closed immersions when defining m- and e-morphisms. The reason for this is explained in Example 3.7.

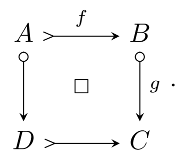
Axioms (Z), (M), and (G) are easily checked, and this is clearly a double category with shared isomorphisms. For axiom (D), one can verify that the distinguished squares



are the pullback squares in which $\mathrm{im} f \cup \mathrm{im} g = D$. Then, axiom (K) holds directly as well.

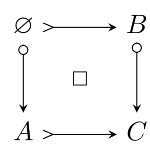
We conclude this section with a collection of useful technical results. For the sake of completeness, we first recall three lemmas from [CZ] which only rely on the underlying CGW category, and whose proofs apply verbatim in our setting.

Lemma 2.10. [CZ, Lemma 2.9] *For any diagram* $A \overset{f}{\rightarrowtail} B \overset{g}{\circ\!\!\rightarrow} C$ *there is a unique (up to unique isomorphism) distinguished square*

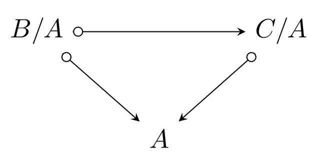


The analogous statement holds for any diagram $A \overset{f}{\circ\!\!\rightarrow} B \overset{g}{\rightarrowtail} C$.

Remark 2.11. As a corollary, we obtain a key consequence of axiom (K): the functors k and c are inverses on objects. It also invites us to consider distinguished squares of the form below as extensions of A by B, which is exactly what they are in Example 2.6.

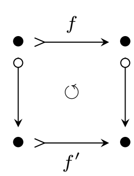


Lemma 2.12. [CZ, Lemma 2.10] *Given any composition* $C \rightarrowtail B \rightarrowtail A$*, there is an induced map* $B/A \circ\!\!\rightarrow C/A$ *such that the triangle below commutes.*



The same holds when the roles of m- and e-morphisms are reversed.

Lemma 2.13. [CZ, Lemma 5.12] *In a pseudo-commutative square as below, if* f' *is an isomorphism then so is* f.

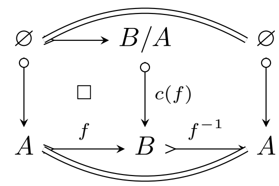


The same holds when the roles of m- and e-morphisms are reversed.

Lemma 2.14. *An m-morphism (resp. e-morphism) in an FCGW category is an isomorphism if and only if its cokernel (resp. kernel) has initial domain.*

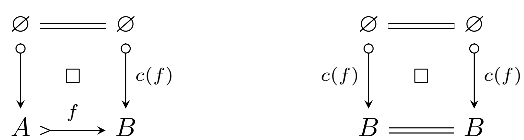
This generalizes [CZ, Lemma 2.8].

Proof. Given an isomorphism $f\colon A \cong B$, we can use axiom (K) to construct the following diagram:



By Lemma 2.10, the data $B/A \overset{c(f)}{\rightarrowtail} B \overset{f^{-1}}{\circ\!\!\longrightarrow} A$ completes to a distinguished square, whose composite with the left square above must (again by Lemma 2.10) agree with the outer identity square on $\varnothing \circ\!\!\longrightarrow A$ up to unique isomorphism. Therefore, we have a monic $B/A \rightarrowtail \varnothing$, which implies that B/A is initial.

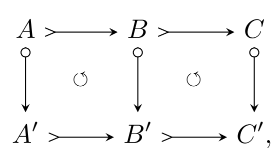
For the converse, note that the data $\varnothing \overset{\mathrm{id}}{\rightarrowtail} \varnothing \overset{c(f)}{\circ\!\!\longrightarrow} B$ can be completed to both of the distinguished squares



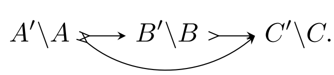
Then, by Lemma 2.10, these squares must be isomorphic; in particular, $f\colon A \rightarrowtail B$ is an isomorphism. □

Lemma 2.15. *Given a composite square of two pseudo-commutative squares, if two of the three squares are distinguished, then so is the third.*

Proof. Given a pasting



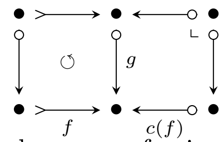
we take kernels of the vertical e-morphisms and obtain m-morphisms



If any two of these are isomorphisms then so is the third, so by definition the same is true of the squares. The same reasoning applies for vertical composites. □

Lemma 2.16. *In a pre-FCGW category, if there exists a square as below right completing the mixed cospan below left, then it is unique up to unique isomorphism.*

Proof. Given any such span, a square can be constructed by applying the inverse equivalence c^{-1} to the pullback of g and $c(f)$, as seen in the following diagram



Since pullbacks and the kernel-cokernel squares of axiom (K) are unique up to unique isomorphism, the same must be true of this pseudo-commutative square. □

In particular, the above lemma implies that a pseudo-commutative square (if it exists) is unique relative to its boundary. Then, for a given square of m- and e-morphisms, the existence of a pseudo-commutative square filler can be treated as a property rather than data. When such a pseudo-commutative filler exists, we say that the square *is* pseudo-commutative.

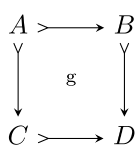
3. FCGW categories

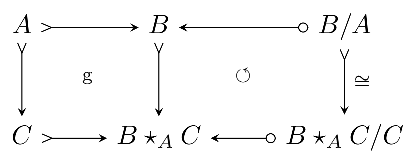
Thanks to Remark 2.5, pre-FCGW categories admit the same Q-construction introduced in [CZ] for CGW categories. However, we are interested in a model similar to Waldhausen's $S_\bullet$ construction, which naturally lends itself to iteration, as well as eventually allowing us to incorporate weak equivalences into our structures.

In this section we introduce FCGW categories, together with several technical results that will allow us to prove the necessary functoriality to iterate the $S_\bullet$ construction and prove the Additivity and Fibration theorems. Key among these is a way to define an FCGW structure on certain double categories of diagrams over an FCGW category. This proof is quite long, and will be deferred to Appendix B.

Definition 3.1. An **FCGW category** is a pre-FCGW category satisfying the following additional axioms:

($\star$) For every diagram $C \leftarrowtail A \rightarrowtail B$, if the category of good squares as below left (with morphisms maps $D \rightarrowtail D'$ commuting under B and C) is non-empty, then it has an initial object which we write $D = B \star_A C$.

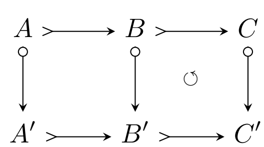




Furthermore, the induced maps $B/A \rightarrowtail B \star_A C/C$ and $C/A \rightarrowtail B \star_A C/B$ are isomorphisms (above right). The dual statement holds for spans of e-morphisms as well.

(PO) For every diagram $C \leftarrowtail A \rightarrowtail B$, the category of good squares as in axiom (PO) is non-empty. The dual statement need not hold for spans of e-morphisms.

(PBL) Pseudo-commutative squares satisfy the "pullback lemma": if the outer composite below is a pseudo-commutative square, then so is the square on the left.



The analogous statement holds for composites in the e-direction.

(POL) If the outer square in the commutative diagram below is good, then the right square is good.

$$\begin{array}{ccccc} A & \rightarrowtail & B & \rightarrowtail & D \\ \downarrow & \text{g} & \downarrow & & \downarrow \\ C & \rightarrowtail & B \star_A C & \rightarrowtail & E \end{array}$$

The same property holds for e-morphisms when the $\star$-pushout exists.

(GS) A square in $\mathcal{M}$ as below is a good square from f to k if and only if it is a good square from g to h.

$$\begin{array}{ccc} \bullet & \overset{f}{\rightarrowtail} & \bullet \\ {\scriptstyle g}\downarrow & \text{g} & \downarrow{\scriptstyle h} \\ \bullet & \underset{k}{\rightarrowtail} & \bullet \end{array}$$

In particular this means that good squares are closed under composition in both directions.

This definition warrants some explanation. Axiom (GS) is a categorical technicality that allows us to treat good squares in a symmetrical way. Axioms (PBL) and (POL) are in a way dual to each other, and they mean to capture the "pullback lemma" and "pushout lemma" which are known to hold in a category with pullbacks and pushouts. Axioms (PO) and ($\star$) deal with the existence of certain initial objects among good squares, which are intended to behave as pushouts do in an exact category. From this perspective, axiom (PO) then says that any span of morphisms in $\mathcal{M}$ admits a "pushout". This is not required of the maps in $\mathcal{E}$, where instead we only expect a "pushout" if the given span is already known to be part of a good square. While this is not necessary in an exact category where we have all pullbacks of admissible epimorphisms, the reader curious about this asymmetry is directed to Example 3.7 and Section 10 for examples of where this asymmetry may arise. This is the only asymmetry between m- and e-morphisms in our definition.

The need for these pushouts arises when studying the classical proofs of the Additivity Theorem (see, for example, [McC93], [Wal85, Section 1.4], [Wei13, Chapter V, Theorem 1.3]). We will see that $\star$-pushouts are adequately functorial and allow for a construction of $\star$ in categories of diagrams; in particular, this will allow us to define an $S_\bullet$ construction that can be iterated. Indeed, the "F" in FCGW stands for Functorial. A more detailed study of the properties of the $\star$-pushout can be found in Appendix A.

Remark 3.2. The good squares are meant to behave like the cofibrations in Waldhausen's category $F_1\mathcal{C}$. Recall that, given a Waldhausen category $\mathcal{C}$, $F_1\mathcal{C}$ is the subcategory of $\operatorname{Ar}\mathcal{C}$ whose objects are the cofibrations. Here, a morphism

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & D \end{array}$$

is a cofibration if the maps $A \longrightarrow C$, $B \longrightarrow D$ and $B \cup_A C \longrightarrow D$ are cofibrations.

In our setting, the pushout is replaced by the $\star$-pushout, and by axiom ($\star$) all good squares are such that there is an induced m-morphism $B \star_A C \rightarrowtail D$. Moreover, the converse also holds, and so this property characterizes good squares. Indeed, given a commutative square as below left

$$
\begin{array}{ccc}
A & \rightarrowtail & B \\
\downarrow & & \downarrow \\
C & \rightarrowtail & D
\end{array}
\qquad\qquad
\begin{array}{ccccc}
A & \rightarrowtail & B & = & B \\
\downarrow & & \downarrow & & \downarrow \\
C & \rightarrowtail & B \star_A C & \rightarrowtail & D
\end{array}
$$

together with an m-morphism $B \star_A C \rightarrowtail D$ over D, we can rewrite it as the composite above right, which implies the square is good.

Remark 3.3. Our FCGW categories are very similar in nature to the ACGW categories of [CZ]. The key distinctions are that we do not require all pullback squares to participate in the equivalences k and c as in [CZ, Definition 5.3], but rather consider the class of good squares which specialize pullbacks; and our requirements of $\star$-pushouts are more relaxed than axioms (S) and (PP) of [CZ], reducing the necessary $\star$-pushouts — we then prove the extra functoriality properties asserted in those axioms as consequences of ours (Lemma A.6, Proposition A.4). These distinctions turn out to be crucial both when iterating the process of the $S_\bullet$ construction, and for including new examples such as exact categories and varieties which are not ACGW categories.

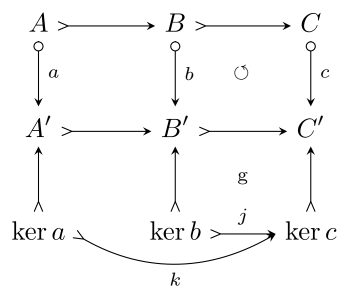
The reader might also notice that we do not require an analogue to axiom (A) in [CZ]. This is due to the fact that a stronger, functorial version of this notion (which is intended to axiomatize the existence of a trivial extension) can be recovered from our axioms by taking the star pushout of the span below left.

$$
\begin{array}{ccc}
\varnothing & \rightarrowtail & A \\
\downarrow & & \\
B & &
\end{array}
$$

For our first example, recall that an exact category is called weakly idempotent complete when every monomorphism that admits a retraction is admissible, or equivalently, every epimorphism that admits a section is admissible.

Example 3.4. Given an exact category $\mathcal{A}$ which is weakly idempotent complete, the pre-FCGW structure described in Example 2.6 can be upgraded to an FCGW structure by defining $\star$-pushouts as the pushouts. This is well-defined and satisfies axiom (PO), as admissible monomorphisms (resp. epimorphisms) are stable under pushout (resp. pullback). Axiom ($\star$) is easily checked, as pushouts of admissible monomorphisms preserve cokernels, and dually for pullbacks of epimorphisms. Axiom (POL) is satisfied as pushouts in an exact category (unlike $\star$-pushouts in the full generality of an FCGW category) have a universal property with respect to commutative (and not necessarily good) squares.

Weak idempotent completeness plays a role when verifying axiom (PBL). Given a pasting as in axiom (PBL), we can take kernels to obtain the following diagram



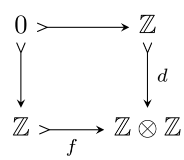
where the bottom outer diagram is a good square.

Since good squares are pullbacks, there exists an induced morphism $i\colon \ker a \longrightarrow \ker b$ such that $k = ji$. Thus i is a monomorphism, but in a general exact category, there is no way to ensure that it is admissible. This property is guaranteed by the fact that $\mathcal{A}$ is weakly idempotent complete, as proven in [BÏ0, Proposition 7.6]. Similarly, the vertical pasting in axiom (PBL) uses the fact that, given a composite $r = qp$ where p, r are admissible epimorphisms, the weak idempotent completeness implies that q is also an admissible epimorphism.

In fact, using this same property, one can easily observe that in weakly idempotent complete categories, all commutative squares of mixed type are pseudo-commutative.

Remark 3.5. (Weakly idempotent complete) exact categories do not in general have all pullbacks, and so they are not examples of ACGW (or pre-ACGW) categories in [CZ].

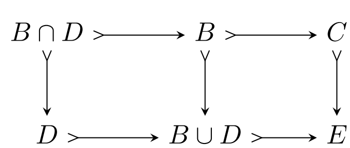
Even when pullbacks exist, our restriction from pullback squares to good squares is not vacuous, as we now illustrate. Let $\mathcal{C}$ denote the exact category of finitely generated projective (i.e., free) abelian groups. This category is idempotent complete, and thus it is in particular weakly idempotent complete. If we consider the square below



where d is the diagonal map $d(x) = (x, x)$ and f is given by $f(x) = (x, -x)$, we see that this is a pullback square in $\mathcal{C}$ which is not good. Indeed, the map induced on cokernels is the monomorphism $i\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $i(x) = 2x$, which is not admissible since its cokernel $\operatorname{coker} i = \mathbb{Z}/2\mathbb{Z}$ is not free.

Example 3.6. The pre-FCGW structure on finite sets described in Example 2.8 can be upgraded to an FCGW structure by defining $\star$-pushouts as pushouts of sets; this is the same as its structure as an ACGW category. Here axiom (PBL) holds as pseudo-commutative squares are pullbacks, which satisfy the pullback lemma. Axiom (GS) is immediate, axiom (PO) follows from the existence of pushouts of injections, and axiom ($\star$) follows from the universal property of the pushout and the observation that a square of injections induces an injection from the pushout precisely when the original square is a pullback.

Axiom (POL) can be deduced from the distributivity of intersections over unions among subsets. In this setting, the diagram in the axiom can be written as



where the union and intersection are taken with respect to E. If the outer square is good (a pullback), we have $C \cap D = B \cap D$. It follows that

$$C \cap (B \cup D) = (C \cap B) \cup (C \cap D) = B \cup (B \cap D) = B$$

so the right square is also a pullback. It is worth noting that axiom (POL) could also be verified here by a similar argument to that for exact categories, but this would require going "outside" the pre-FCGW structure by mentioning functions induced from a pushout by arbitrary commuting squares of sets, which are not necessarily monic. Distributivity allows for a proof using only the information "seen" by the pre-FCGW structure.

Example 3.7. The pre-FCGW category of varieties of Example 2.9 can be upgraded to the structure of an FCGW category, by letting $\star$-pushouts be the pushouts of varieties.

Axiom (GS) is immediate in this setting, and axiom (PB) is satisfied as pseudo-commutative squares are pullbacks. Axiom (PO) holds, since pushouts of closed immersions exist, and the resulting square is a pullback. We note that this does not hold for e-morphisms, as the pushout of open immersions need not exist. However, it does when the span of open immersions is known to belong to a pullback square, and thus $\star$-pushouts of both m- and e-spans satisfy axiom ($\star$). Finally, axiom (POL) can be verified in a similar manner as either of the previous examples.

Remark 3.8. Just as Example 3.4, varieties give another example that fits our axioms, and not those of ACGW categories (although, unlike exact categories, varieties are pre-ACGW). In this case, this is due to the fact that our $\star$-pushouts need not exist in the case of e-morphisms, while $\star$-pushouts of both classes of morphisms are required in axiom (PP) of [CZ, Definition 5.4].

As usual, FCGW categories have natural notions of functors and subcategories.

Definition 3.9. An **FCGW functor** is a double functor that preserves all of the relevant structure up to natural isomorphism.

Definition 3.10. A double subcategory $\mathcal{D}$ of an FCGW category $\mathcal{C}$ is an **FCGW subcategory** if it inherits an FCGW structure from $\mathcal{C}$.

For full double subcategories of an FCGW category, many of the axioms are automatically preserved, so it is easy to check whether they are FCGW.

Lemma 3.11. *A full double subcategory of an FCGW category $\mathcal{C}$ is an FCGW subcategory if it is closed under $k, c, \star$, and contains $\varnothing$.*

The most common way for us to construct new FCGW categories from familiar ones will be through functor categories. Given an FCGW category $\mathcal{C}$ and any double category $\mathcal{D}$, we wish to describe an FCGW structure on a double subcategory of the double category $[\mathcal{D}, \mathcal{C}]$ of double functors described in Definition 1.5.

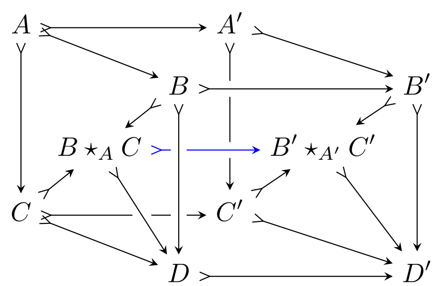
Definition 3.12. For $\mathcal{C}$ an FCGW category and $\mathcal{D}$ any double category, we define the double subcategory $\mathcal{C}^{\mathcal{D}} \subset [\mathcal{D}, \mathcal{C}]$ as follows:

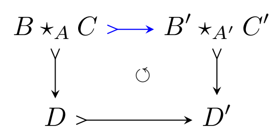
- objects are all double functors $\mathcal{D} \longrightarrow \mathcal{C}$
- $\mathcal{M}$ consists of the "good" m-natural transformations whose naturality squares of m-morphisms are good
- $\mathcal{E}$ is given by the "good" e-natural transformations whose naturality squares of e-morphisms are good
- mixed squares consist of all modifications between the m- and e-morphisms, which are pointwise pseudo-commutative in $\mathcal{C}$

Note that $\mathcal{M}$ and $\mathcal{E}$ here are in fact categories, as good squares are closed under identities and composition and there are no restrictions placed on the mixed naturality squares of these transformations.

As we saw in Example 2.6, it is not enough to consider squares whose sides are all in $\mathcal{M}$, and instead we need to work with a more well-behaved notion of good square. Similarly, when working with m-natural transformations, it will not suffice to ask that all the squares involved are good, but instead we need a stronger notion of "good cube". In order to do this, we present the following definition, which adapts the good cubes of [Zak18, Definition 2.3] to our setting.

Definition 3.13. Let $\mathcal{C}$ be an FCGW category. A commutative cube of morphisms in $\mathcal{M}$ is a **good cube** if each face is a good square, and if the induced m-morphism between $\star$-pushouts[2] is such that the square below right is good.





We call this the "southern square". Good cubes in $\mathcal{E}$ are defined in the same way.

Remark 3.14. A priori, it seems as if our definition of good cube is subject to a choice of direction. Indeed, we could have taken $\star$-pushouts of the back and front faces, instead of the left and right faces, and induced a different southern square. However, as we show in Remark A.8, if any of these induced squares are good, then all of them are. Moreover, it is possible to define a "southern arrow" as in [Zak18, Definition 2.3] and show that any of the southern squares of a cube is good if and only if there exists a southern arrow that is an m-morphism.

Theorem 3.15. *For $\mathcal{C}$ an FCGW category and $\mathcal{D}$ any double category, the functor double category $\mathcal{C}^{\mathcal{D}}$ admits the structure of an FCGW category as follows:*

- $\operatorname{Ar}_{\mathrm{g}} \mathcal{M}$ *are the commutative squares of m-natural transformations whose component cubes of naturality squares between m-morphisms are good cubes.* $\operatorname{Ar}_{\mathrm{g}} \mathcal{E}$ *is defined dually*
- *the functors k and c are defined pointwise from those of $\mathcal{C}$, as is $\star$ in the sense that the $\star$-pushout of a span of $\mathcal{D}$-shaped diagrams in $\mathcal{C}$ is the $\mathcal{D}$-shaped diagram of pointwise $\star$-pushouts*

Showing that this defines an FCGW structure is nontrivial, especially for $\star$-pushouts, but the axioms of FCGW categories were designed to enable this kind of construction. As the technical details of this proof are not needed to describe our main results, we defer it to Appendix B, along with several helpful corollaries providing FCGW structures on more specialized subcategories of $\mathcal{C}^{\mathcal{D}}$.

4. Adding weak equivalences

One of the benefits of Waldhausen's $S_\bullet$-construction over Quillen's Q-construction is that it allows us to incorporate homotopical data in the form of weak equivalences. In practice, when

[2]Such a morphism always exists; see Proposition A.3.

a Waldhausen category has additional algebraic structure (such as that of an exact or abelian category), the weak equivalences often interact nicely with that structure.

In particular, one often finds that the class of weak equivalences can be completely determined by the acyclic monomorphisms and epimorphisms, and that in turn, these can be characterized by having acyclic (co)kernels. Such is the case, for example, in the category of bounded chain complexes over an exact category, with quasi-isomorphisms as weak equivalences.

In this section, we borrow this intuition and define *m- and e-quivalences* on an FCGW category, constructed from a given class of acyclic objects.

Definition 4.1. An **acyclicity structure** on an FCGW category $\mathcal{C}$ is a class of objects of $\mathcal{C}$, which we call **acyclic objects**, such that:

(IA) any initial object is acyclic

(A23) for any kernel-cokernel pair $A \rightarrowtail B \leftarrow\!\circ\, C$, if any two of A, B, C are acyclic then so is the third

We refer to the pair $(\mathcal{C}, \mathcal{W})$ as an **FCGWA category**, where $\mathcal{W}$ is the full double subcategory of acyclic objects.

Definition 4.2. An **FCGWA functor** $(\mathcal{C}, \mathcal{W}) \longrightarrow (\mathcal{C}', \mathcal{W}')$ is an FCGW functor $\mathcal{C} \longrightarrow \mathcal{C}'$ that preserves acyclic objects.

Definition 4.3. An m-morphism (resp. e-morphism) in an FCGWA category $(\mathcal{C}, \mathcal{W})$ is a **weak equivalence** if its cokernel (resp. kernel) is acyclic.

Notation 4.4. We will refer to the m-morphisms (resp. e-morphisms) which are weak equivalences as m-equivalences (resp. e-equivalences), and denote them by $\overset{\sim}{\rightarrowtail}$ (resp. $\overset{\sim}{\circ\!\!\rightarrow}$). When it is not relevant whether the weak equivalence is horizontal or vertical, we denote them by $\overset{\sim}{\longrightarrow}$.

FCGWA categories can be equivalently defined in terms of the weak equivalences rather than their acyclic objects, but as we now show, the desired properties of weak equivalences are more easily expressed in terms of acyclic objects. This is reminiscent of the construction of Waldhausen structures on exact categories via cotorsion pairs of [Sar20].

Much of the theory we develop holds equally well in a more general setting in which weak equivalences are not determined by acyclic objects, but this complicates the proofs significantly and is not necessary for any of our examples.

Example 4.5. In any FCGW category $\mathcal{C}$, acyclic objects can be chosen to be the initial objects. By Lemma 2.14, they satisfy Definition 4.1 and weak equivalences are precisely the isomorphisms. The K-theory of this FCGWA category as defined in Section 5 is the same as that of the underlying CGW category of $\mathcal{C}$ defined in [CZ] (for more details, see Proposition 5.8).

Example 4.6. For any FCGWA category $(\mathcal{C}, \mathcal{W})$ and $\mathcal{C}' \subset \mathcal{C}$ an FCGW subcategory, $(\mathcal{C}', \mathcal{W} \cap \mathcal{C}')$ forms an FCGWA category.

Example 4.7. As explained in Example 3.4, weakly idempotent complete exact categories can be given the structure of an FCGW category. Let $\mathcal{C}$ be such a category, which in addition has a Waldhausen structure. If we denote by $\mathcal{W}$ the class of objects $X \in \mathcal{C}$ such that $0 \longrightarrow X$ is a weak equivalence, then $(\mathcal{C}, \mathcal{W})$ will be an FCGWA category whenever $\mathcal{W}$ has 2-out-of-3.

For example, this will be the case when $\mathcal{C}$ is a Waldhausen category constructed from a cotorsion pair and any such class $\mathcal{W}$ of acyclic objects as in [Sar20], when $\mathcal{C}$ is a biWaldhausen category satisfying the extension and saturation axioms (such as the complicial biWaldhausen categories of [TT90, 1.2.11]), and when $\mathcal{C}$ satisfies the saturation axiom and is both left and right proper (like the complicial exact categories with weak equivalences of [Sch11, Definition 3.2.9]).

Example 4.8. In Section 10, we introduce an FCGW category of chain complexes of finite sets. As we show in Section 11, these admit an FCGWA structure where the class of acyclic objects is given by the exact chain complexes, defined analogously to the classical algebraic setting.

The following results can be easily deduced for any FCGWA category from Definition 4.1.

Lemma 4.9. *All isomorphisms are weak equivalences.*

Lemma 4.10. *Given a weak equivalence $X \xrightarrow{\sim} Y$, if either X or Y is acyclic, then both are.*

Lemma 4.11. *Any map between acyclic objects is a weak equivalence.*

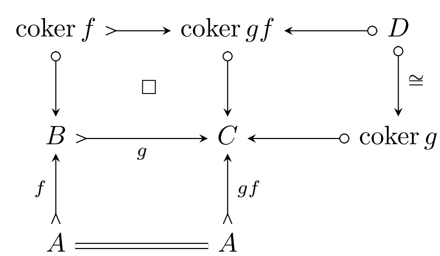
In particular, all morphisms in the full double subcategory $\mathcal{W}$ are weak equivalences, and an object in $\mathcal{C}$ is acyclic if and only if both the m- and e-morphisms from $\varnothing$ are weak equivalences.

Additionally, we can prove the following.

Lemma 4.12. *m- and e-equivalences each satisfy 2-out-of-3. In particular, they form subcategories of $\mathcal{M}$ and $\mathcal{E}$.*

Proof. We prove this for m-morphisms, the argument for e-morphisms is dual.

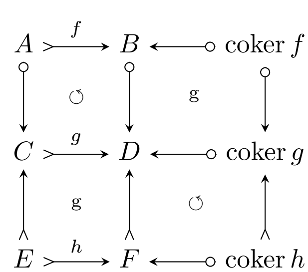
Given m-morphisms $f\colon A \rightarrowtail B$ and $g\colon B \rightarrowtail C$, we consider the following diagram



By Lemma 4.10 D is acyclic if and only if $\operatorname{coker} g$ is, so if any two of f, g, gf are weak equivalences, then two of $\operatorname{coker} f, \operatorname{coker} g, \operatorname{coker} gf$ are acyclic, and hence so is the third by Definition 4.1. Together with Lemma 4.9, this shows that weak equivalences form subcategories of $\mathcal{M}$ and $\mathcal{E}$. ☐

Lemma 4.13. *In a kernel-cokernel pair of squares, if any two of the three parallel maps are weak equivalences then so is the third.*

Proof. Consider the kernel-cokernel pair of squares depicted in the left column of the diagram below, with parallel m-morphisms f, g, h



Taking cokernels of both squares we get a kernel-cokernel sequence

$$\operatorname{coker} f \circ\!\!\longrightarrow \operatorname{coker} g \longleftarrow\!\!< \operatorname{coker} h$$

as shown in the diagram, so by Definition 4.1 if any two of f, g, h are weak equivalences then so is the third. □

Lemma 4.14. *Acyclic objects are closed under $\star$-pushouts (when these exist).*

Proof. Consider a span of m-morphisms $B \longleftarrow\!\!< A \rightarrowtail C$ where A, B, C are acyclic. By Lemma 4.11 these morphisms are weak equivalences, hence B/A is acyclic. By axiom ($\star$), $(B \star_A C)/C \cong B/A$, so the map $C \rightarrowtail B \star_A C$ is a weak equivalence. Therefore, $B \star_A C$ is acyclic by Lemma 4.11.

The same argument holds for spans of e-morphisms whose $\star$-pushout exists. □

Remark 4.15. The definition of acyclicity structures, along with Lemma 4.14 above, imply that $\mathcal{W}$ forms an FCGW category by Lemma 3.11. Conversely, given an FCGW category $\mathcal{C}$, any full FCGW subcategory that is closed under extensions provides an acyclicity structure.

Definition 4.16. An FCGW subcategory $\mathcal{C}'$ of $\mathcal{C}$ is **closed under extensions** if, for any kernel-cokernel sequence

$$A \rightarrowtail B \longleftarrow\!\!\circ\, C$$

in $\mathcal{C}$ such that A, C are in $\mathcal{C}'$, B is also in $\mathcal{C}'$.

An FCGW category often admits more than one natural choice of acyclicity structure; in fact, Section 8 provides a tool for comparing the two resulting FCGWA structures when one is a refinement of the other.

Definition 4.17. A **refinement** of an FCGWA category $(\mathcal{C}, \mathcal{W})$ is a subclass $\mathcal{V} \subseteq \mathcal{W}$ of acyclic objects such that $(\mathcal{C}, \mathcal{V})$ also forms an FCGWA category.

Example 4.18. The poset of refinements of $(\mathcal{C}, \mathcal{W})$ ordered by inclusion has both minimal and maximal elements, given by initial objects in $\mathcal{C}$ and $\mathcal{W}$ itself, respectively.

The following is immediate from our definitions, along with Remark 4.15.

Lemma 4.19. *For any refinement $(\mathcal{C}, \mathcal{V})$ of an FCGWA category $(\mathcal{C}, \mathcal{W})$, there exists an FCGWA subcategory $(\mathcal{W}, \mathcal{V}) \subseteq (\mathcal{C}, \mathcal{W})$.*

Part 2. K-theory of FCGWA categories

5. $S_\bullet$-construction

We are now equipped to define the K-theory of an FCGWA category, which we do by imitating the $S_\bullet$ construction in our setting. The construction is similar to that of [CZ, Definition 7.10], but we also accommodate weak equivalences, and moreover the variants in our construction (mostly, the restriction to good cubes) allow for this process to be iterated. In other words, given an FCGWA category $\mathcal{C}$, we construct a simplicial double category $S_\bullet\mathcal{C}$ which is furthermore a simplicial FCGWA category.

The following double category will be useful for defining our $S_\bullet$ construction.

Definition 5.1. For each n, let $\mathcal{S}_n$ denote the double category generated by the following objects, horizontal morphisms, vertical morphisms, and squares.

$$\begin{array}{ccccccccc}
A_{0,0} & \rightarrowtail & A_{0,1} & \rightarrowtail & A_{0,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{0,n} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & A_{1,1} & \rightarrowtail & A_{1,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{1,n} \\
 & & & & \uparrow & & & & \uparrow \\
 & & & & A_{2,2} & \rightarrowtail & \cdots & \rightarrowtail & A_{2,n} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & \cdots \\
 & & & & & & & & \uparrow \\
 & & & & & & & & A_{n,n}
\end{array}$$

Definition 5.2. Given an FCGWA category $\mathcal{C}$, we define a simplicial double category $S_\bullet\mathcal{C}$ as follows:

- for each n, $S_n\mathcal{C}$ is the full double subcategory of $\mathcal{C}^{\mathcal{S}_n}$ given by the functors F such that $F(A_{i,i}) = \varnothing$ for all i, and that F sends all squares in $\mathcal{S}_n$ to distinguished squares in $\mathcal{C}$.
- for the simplicial structure, the face map $d_i \colon S_n\mathcal{C} \longrightarrow S_{n-1}\mathcal{C}$, $0 \le i \le n$, deletes the objects $F(A_{j,i})$ and $F(A_{i,j})$ for all j, where what remains after discarding or composing the affected squares is a diagram of shape $\mathcal{S}_{n-1}$; the degeneracy map $s_i \colon S_n\mathcal{C} \longrightarrow S_{n+1}\mathcal{C}$ inserts a row and column of identity morphisms above and to the right of $F(A_{i,i})$

We will often refer to the objects of $S_n\mathcal{C}$ as "staircases".

Proposition 5.3. *$S_n\mathcal{C}$ is an FCGWA category, with FCGW structure inherited from that of $\mathcal{C}^{\mathcal{S}_n}$ as described in Theorem 3.15, and acyclic objects defined as the pointwise acyclics in $\mathcal{C}$.*

Proof. We show in Proposition B.2 that $S_n\mathcal{C}$ is an FCGW subcategory of $\mathcal{C}^{\mathcal{S}_n}$, and pointwise acyclic diagrams clearly form an acyclicity structure. □

Definition 5.4. For an FCGWA category $(\mathcal{C}, \mathcal{W})$, define

$$K(\mathcal{C}, \mathcal{W}) = \Omega|wS_\bullet\mathcal{C}|$$

and

$$K_n(\mathcal{C}, \mathcal{W}) = \pi_n K(\mathcal{C}, \mathcal{W}),$$

where $wS_\bullet\mathcal{C}$ is the simplicial double category obtained by restricting the m-morphisms and e-morphisms in $S_\bullet\mathcal{C}$ to the m-equivalences and e-equivalences.

As usual, we start by studying K_0 and showing that it agrees with the intuitive Grothendieck group. Similarly to [CZ, Theorem 4.3], most of the relations will be given by the distinguished squares, except that we get additional relations induced by the weak equivalences.

Proposition 5.5. *For any FCGWA category $(\mathcal{C}, \mathcal{W})$, $K_0(\mathcal{C}, \mathcal{W})$ is the free abelian group generated by the objects of $\mathcal{C}$, modulo the relations that, for any distinguished square*

$$\begin{array}{ccc}
A & \rightarrowtail & B \\
\downarrow & \square & \downarrow \\
C & \rightarrowtail & D
\end{array}$$

we have $[A] + [D] = [B] + [C]$, and that for any horizontal or vertical weak equivalence $A \xrightarrow{\sim} B$ we have $[A] = [B]$.

Proof. By definition, $K_0(\mathcal{C},\mathcal{W}) = \pi_0\Omega|wS_\bullet\mathcal{C}| = \pi_1|wS_\bullet\mathcal{C}|$. Since $|wS_\bullet\mathcal{C}|$ is path-connected (as $|wS_0\mathcal{C}| = *$), it follows from the Van-Kampen Theorem that $\pi_1|wS_\bullet\mathcal{C}|$ is the free group on $\pi_0|wS_1\mathcal{C}|$, modulo the relations $\delta_1(x) = \delta_2(x)\delta_0(x)$ for each $x \in \pi_0|wS_2\mathcal{C}|$.

Let us describe what these conditions entail. The elements of $|wS_1\mathcal{C}|$ are the objects of $\mathcal{C}$, and two objects A, B are in the same connected component precisely when there exists a pseudo-commutative square

$$\begin{array}{ccc} A & \overset{\sim}{\rightarrowtail} & \bullet \\ \circ\!\downarrow\wr & \circlearrowleft & \circ\!\downarrow\wr \\ \bullet & \overset{\sim}{\rightarrowtail} & B \end{array}$$

On the other hand, elements of $|wS_2\mathcal{C}|$ are kernel-cokernel sequences in $\mathcal{C}$, and given $x = A \rightarrowtail B \leftarrow\!\circ\, B/A$, we have that $\delta_1(x) = B$, $\delta_0(x) = B/A$ and $\delta_2(x) = A$.

Note that $K_0(\mathcal{C})$ is abelian because, as explained in Remark 3.3, we have trivial extensions

$$A \rightarrowtail A + B \leftarrow\!\circ\, B, \qquad B \rightarrowtail A + B \leftarrow\!\circ\, A$$

and so $[A][B] = [A + B] = [B][A]$. We will use additive notation from now on.

We first show that $K_0(\mathcal{C},\mathcal{W})$ identifies weakly equivalent objects. If $A \overset{\sim}{\circ\!\!\rightarrow} B$ is an e-equivalence, we can fit it in a pseudo-commutative square

$$\begin{array}{ccc} A & = & A \\ \circ\!\downarrow\wr & \circlearrowleft & \circ\!\downarrow\wr \\ B & = & B \end{array}$$

from which we get that $[A] = [B]$. The argument for m-equivalences is analogous. To show that the distinguished square relation of the statement is always satisfied in $K_0(\mathcal{C},\mathcal{W})$, we recall that distinguished squares induce isomorphisms on cokernels. We can then see that

$$\begin{aligned} [B] &= [A] + [B/A] \\ &= [A] + [D/C] \\ &= [A] + [D] - [C] \end{aligned}$$

which yields the desired relation.

Finally, we assume the relations from distinguished squares and weak equivalences and show how it implies all the relations in our description of $K_0(\mathcal{C},\mathcal{W})$ above. If $x = A \rightarrowtail B \leftarrow\!\circ\, B/A$ is an element of $|wS_2\mathcal{C}|$, then there exists a distinguished square

$$\begin{array}{ccc} \varnothing & \rightarrowtail & B/A \\ \circ\!\downarrow & \square & \circ\!\downarrow \\ A & \rightarrowtail & B \end{array}$$

and this gives us the relation $[B] = [A] + [B/A]$. The fact that objects in the same connected component in $|wS_1\mathcal{C}|$ are identified is a direct consequence of the fact that weakly equivalent objects are identified. $\square$

Having established a new K-theory machinery, we now wish to show that it agrees with the existing ones for all the relevant examples. We start by stating the following, analogous to [Wal85, 1.4.1 Corollary (2)].

Definition 5.6. Given an FCGWA category $\mathcal{C}$, let $s_\bullet\mathcal{C}$ denote the simplicial set given by $s_n\mathcal{C} = \operatorname{ob} S_n\mathcal{C}$.

Lemma 5.7. *For an FCGW category $\mathcal{C}$, we have $iS_\bullet\mathcal{C} \simeq s_\bullet\mathcal{C}$, where i denotes the class of isomorphisms in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ has shared isomorphisms, as does each $S_n\mathcal{C}$ by Proposition 5.3, the double subcategory $iS_n\mathcal{C}$ is isomorphic to the double category of commutative squares in the groupoid $I(S_n\mathcal{C})$ of isomorphisms in $S_n\mathcal{C}$. By Waldhausen's Swallowing Lemma ([Wal85, 1.5.6]), $iS_n\mathcal{C}$ is then homotopy equivalent to the groupoid $I(S_n\mathcal{C})$ itself, and from this point the proof proceeds exactly as in [Wal85, 1.4.1]. □

Using this lemma, we see that the K-theory of an FCGWA category with isomorphisms agrees with its K-theory as constructed in [CZ].

Proposition 5.8. *For an FCGWA category $\mathcal{C}$, $K(\mathcal{C}, i)$ agrees with the K-theory of its underlying CGW category as defined in* [CZ].

Proof. By Lemma 5.7, $K(\mathcal{C}, i)$ is homotopy equivalent to $\Omega|s_\bullet\mathcal{C}|$, which is precisely K^S of the underlying CGW category of $\mathcal{C}$ as defined in [CZ, Definition 7.4]. □

Remark 5.9. In particular, this implies that the K-theory of the FCGW categories given by exact categories, sets, and varieties of Examples 3.4 to 3.7 agree with their existing counterparts in the literature.

Remark 5.10. The only caveat if one wishes to model the K-theory of exact categories through our formalism is that, as explained in Example 3.4, in order for an exact category to define an FCGW structure, it needs to be weakly idempotent complete. However, this does not present a real obstruction for K-theoretic purposes, as any exact category $\mathcal{C}$ satisfies $K(\mathcal{C}) \simeq K(\overline{\mathcal{C}})$, where $\overline{\mathcal{C}}$ denotes the full exact subcategory of the idempotent completion of $\mathcal{C}$ consisting of the objects A such that $[A] \in K_0(\mathcal{C})$. In particular, $\overline{\mathcal{C}}$ is weakly idempotent complete.

It is natural to ask whether our notion of K-theory also agrees with the existing ones when working with an exact category with weak equivalences, such as chain complexes with quasi-isomorphisms. Due to the way it was constructed, our K-theory machinery is only designed to take as input a category whose weak equivalences are defined through a class of acyclics. That is, if there is any hope of a comparison, the exact category must be such that an admissible monomorphism (resp. epimorphism) is a weak equivalence if and only if its cokernel (resp. kernel) is weakly equivalent to 0.

Furthermore, since our double-categorical perspective only deals with admissible monomorphisms and epimorphisms, it must be the case that m- and e-equivalences encode the data of all weak equivalences. This is the case, for example, when weak equivalences can be expressed as composites of admissible monomorphisms and epimorphisms which are themselves weak equivalences.

Fortunately, this seems to be the case for the vast majority of exact categories with weak equivalences that arise in practice.

Proposition 5.11. *Let $\mathcal{C}$ be an exact category with a class of weak equivalences w, and let $\mathcal{W}$ be the class of objects $X \in \mathcal{C}$ such that $0 \longrightarrow X$ is in w. If $(\mathcal{C}, w)$ is either*

- *a complicial exact category with weak equivalences as in* [Sch11, Definition 3.2.9],

- *a complicial biWaldhausen category as in* [TT90, 1.2.11] *closed under canonical homotopy pushouts and pullbacks (*[TT90, 1.1.2]*), or*
- *an exact category with weak equivalences constructed from a cotorsion pair as in* [Sar20] *and such that $\mathcal{W}$ has 2-out-of-3*

then the K-theory of $(\mathcal{C}, w)$ as a Waldhausen category is homotopy equivalent to the K-theory of $(\mathcal{C}, \mathcal{W})$ as an FCGWA category.

Proof. In all the specified cases, there exists a homotopy fiber sequence of K-theory spectra of Waldhausen categories

$$K(\mathcal{W}, i) \longrightarrow K(\mathcal{C}, i) \longrightarrow K(\mathcal{C}, w)$$

However, by Proposition 5.8 the two leftmost terms are equivalent to the K-theory spaces of $(\mathcal{W}, i)$ and $(\mathcal{C}, i)$ regarded as FCGWA categories, and by Theorem 8.1, there exists a homotopy fiber sequence of K-theory spaces of FCGWA categories

$$K(\mathcal{W}, i) \longrightarrow K(\mathcal{C}, i) \longrightarrow K(\mathcal{C}, \mathcal{W})$$

These are furthermore shown to be spectra in Theorem 7.7, and so we conclude that their cofibers must be homotopy equivalent. □

6. Additivity Theorem

The purpose of this section is to show that our K-theory construction satisfies the Additivity Theorem. Aside from being a fundamental result that any K-theory machinery is expected to satisfy, it will be useful in the next sections when we establish the Fibration Theorem and discuss a version of the Gillet–Waldhausen Theorem.

In order to state the Additivity Theorem, we define extension categories in our setting.

Definition 6.1. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ be full FCGW subcategories of an FCGW category $\mathcal{C}$. We define the **extension double category** $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ as the full double category of $S_2(\mathcal{C})$ whose objects are determined by kernel-cokernel sequences in $\mathcal{C}$ of the form

$$A \rightarrowtail C \leftarrow\!\circ\; B$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Explicitly, an m-morphism in $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a triple of pointwise m-morphisms in $\mathcal{A}, \mathcal{C}, \mathcal{B}$ respectively, related by good and pseudo-commutative squares as follows

$$\begin{array}{ccccc} A & \rightarrowtail & C & \leftarrow\!\circ & B \\ \downarrow & \text{g} & \downarrow & \circlearrowleft & \downarrow \\ A' & \rightarrowtail & C' & \leftarrow\!\circ & B' \end{array}$$

and e-morphisms are defined analogously. Pseudo-commutative squares in $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ are given by triples of pseudo-commutative squares in $\mathcal{A}, \mathcal{C}, \mathcal{B}$ respectively, natural in the appropriate sense.

Lemma 6.2. *$E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is an FCGW category, with the structure inherited from $S_2(\mathcal{C})$ of Proposition B.2. Furthermore if $\mathcal{C}$ is FCGWA, then pointwise acyclic objects give $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ an FCGWA structure.*

Proof. When $\mathcal{A} = \mathcal{B} = \mathcal{C}$, we have that $E(\mathcal{C}, \mathcal{C}, \mathcal{C}) = S_2(\mathcal{C})$ and the result is shown in Proposition B.2. It is then straightforward to check that $E(\mathcal{A}, \mathcal{C}, \mathcal{B}) \subseteq E(\mathcal{C}, \mathcal{C}, \mathcal{C})$ is an FCGW(A) subcategory by Lemma 3.11, as $\mathcal{A}, \mathcal{B}$ are FCGW subcategories. □

In several instances, it will be useful to recognize when a certain FCGW category is equivalent (in the sense of Definition 1.10) to an extension category. We study this in the following lemma.

Lemma 6.3. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ be full FCGW subcategories of an FCGWA category $\mathcal{C}$ with inclusion functors $i_{\mathcal{A}}, i_{\mathcal{B}}$. $\mathcal{C}$ is equivalent to $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ if we have the following:*

- *FCGWA functors $F\colon \mathcal{C} \longrightarrow \mathcal{A}$, $G\colon \mathcal{C} \longrightarrow \mathcal{B}$,*
- *an m-natural transformation $\phi\colon i_{\mathcal{A}} F \rightarrowtail 1_{\mathcal{C}}$,*
- *an e-natural transformation $\psi\colon i_{\mathcal{B}} G \circ\!\!\longrightarrow 1_{\mathcal{C}}$,*
- *for each object C in $\mathcal{C}$, $FC \xrightarrow{\phi_C} C \xleftarrow{\varphi_C} GC$ is a kernel-cokernel pair,*
- *every extension in $\mathcal{C}$ is isomorphic to one of the above form*

Proof. The data above, excluding the last property, determine a FCGWA functor $\mathcal{C} \longrightarrow E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ left inverse to the forgetful functor $E(\mathcal{A}, \mathcal{C}, \mathcal{B}) \longrightarrow \mathcal{C}$ so long as ϕ, ψ are good natural transformations. To see that this is always the case, consider an m-morphism $f\colon C \rightarrowtail C'$ in $\mathcal{C}$; we then have the following pair of naturality squares for ϕ, ψ:

$$\begin{array}{ccccc} FC & \xrightarrow{\phi_C} & C & \xleftarrow{\psi_C} & GC \\ {\scriptstyle Ff}\downarrow & & {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle Gf} \\ FC' & \xrightarrow[\phi_{C'}]{} & C' & \xleftarrow[\psi_{C'}]{} & GC' \end{array}$$

As the top and bottom row are kernel-cokernel pairs and the square on the right is pseudo-commutative, there exists a good square in $\mathcal{M}$ which agrees with the left square everywhere except possibly Ff. However, as both squares commute and $\phi_{C'}$ is a monomorphism, the remaining map in the good square must indeed be Ff, so the naturality squares of ϕ for m-morphisms are good. The same is true for the naturality squares of ψ for e-morphisms by a dual argument.

Finally, it remains to show that the functor $\mathcal{C} \longrightarrow E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is an equivalence by checking the conditions of Proposition 1.11. Essential surjectivity holds by our last assumption. Fullness and faithfulness for m-morphisms follows from Lemma 2.16 and the analogous uniqueness of pullback squares, as any m-morphism in $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ as above is uniquely determined by its source and target extensions and the map f. The same properties follow dually for e-morphisms and similarly for pseudo-commutative squares, which are uniquely determined by their boundaries. □

Corollary 6.4. *In the conditions of Lemma 6.3, we get a homotopy equivalence*

$$wS_\bullet \mathcal{C} \simeq wS_\bullet E(\mathcal{A}, \mathcal{C}, \mathcal{B})$$

Proof. It is tedious but straightforward to check that an equivalence $L\colon \mathcal{C} \longrightarrow E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ in the sense of Definition 1.10 induces an equivalence $S_n\mathcal{C} \longrightarrow S_n E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ for each n. Moreover, these restrict to equivalences $wS_n\mathcal{C} \longrightarrow wS_n E(\mathcal{A}, \mathcal{C}, \mathcal{B})$, since isomorphisms are weak equivalences, and a map f in $\mathcal{C}$ is an m-equivalence (resp. e-equivalence) if and only if Lf is an m-equivalence (resp. e-equivalence). □

We now return to the goal of this section: to prove the Additivity Theorem stated below.

Theorem 6.5 (Additivity)**.** *Let $\mathcal{C}$ be an FCGWA category. Then, the map*

$$wS_\bullet E(\mathcal{C}, \mathcal{C}, \mathcal{C}) \longrightarrow wS_\bullet \mathcal{C} \times wS_\bullet \mathcal{C}$$

induced by

$$(A \rightarrowtail C \longleftarrow\!\!\circ\ B) \mapsto (A, B)$$

is a homotopy equivalence.

The proof of Additivity proceeds in a manner almost identical to McCarthy's [McC93]. Just as in [Wal85, Theorem 1.4.2], the first step is to reduce the proof of Additivity to the case when the equivalences considered are isomorphisms. In the classical case, this is done by showing that the bisimplicial set $(m,n) \mapsto s_n\mathcal{C}(m,w)$ is equivalent to the bisimplicial set $(m,n) \mapsto w_m S_n\mathcal{C}$, or, in other words, that staircases of sequences of weak equivalences in $\mathcal{C}$ are the same as sequences of weak equivalences of staircases in $\mathcal{C}$. We now introduce the double categorical version of this statement.

Definition 6.6. Let $(\mathcal{C},\mathcal{W})$ be an FCGWA category, and let $\mathcal{D}$ denote the free double category on an $l \times m$ grid of squares. The **double category of w-grids** $w_{l,m}\mathcal{C}$ is the full double subcategory of $\mathcal{C}^{\mathcal{D}}$ of the grids whose morphisms are all weak equivalences.

Proposition 6.7. *Let $(\mathcal{C},\mathcal{W})$ be an FCGWA category. Then $w_{l,m}\mathcal{C}$ is an FCGW category with structure inherited from that of $\mathcal{C}^{\mathcal{D}}$ of Theorem B.1. Moreover, if $\mathcal{V}$ a refinement of $\mathcal{W}$, then the double subcategory of grids in $\mathcal{V}$ forms an acyclicity structure on $w_{l,m}\mathcal{C}$.*

We defer the proof of this proposition to Proposition B.3. With this structure in hand, we can see the following.

Lemma 6.8. *There is an isomorphism of simplicial sets*

$$s_\bullet w_{l,m}\mathcal{C} \cong w_{l,m}S_\bullet\mathcal{C},$$

simplicial in both l and m. More generally, for any refinement $\mathcal{V} \subseteq \mathcal{W}$,

$$vS_\bullet w_{l,m}\mathcal{C} \cong vw_{l,m}S_\bullet\mathcal{C}.$$

Proof. This follows immediately from the definitions, and it amounts to saying that staircases of w-grids in $\mathcal{C}$ are the same as w-grids of staircases in $\mathcal{C}$. □

Like in the classical case, this allows us to show that weak equivalences are not an integral part of the Additivity Theorem.

Proposition 6.9. *If the map*

$$s_\bullet E(\mathcal{A},\mathcal{A},\mathcal{A}) \longrightarrow s_\bullet\mathcal{A} \times s_\bullet\mathcal{A}$$

is a homotopy equivalence for every FCGW category $\mathcal{A}$, then the map

$$wS_\bullet E(\mathcal{C},\mathcal{C},\mathcal{C}) \longrightarrow wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C}$$

is a homotopy equivalence for every FCGWA category $(\mathcal{C},\mathcal{W})$.

Proof. Let $(\mathcal{C},\mathcal{W})$ be an FCGWA category, and consider the FCGW category of w-grids $w_{l,m}\mathcal{C}$ of Proposition 6.7. Note that for each l,m,n, we have by Lemma 6.8 an isomorphism

$$s_n w_{l,m}\mathcal{C} \cong w_{l,m}S_n\mathcal{C}.$$

Moreover, there is a homotopy equivalence

$$s_\bullet w_{l,m}E(\mathcal{C},\mathcal{C},\mathcal{C}) \simeq s_\bullet E(w_{l,m}\mathcal{C}, w_{l,m}\mathcal{C}, w_{l,m}\mathcal{C})$$

for each l,m arising via Lemma 1.13 from the evident equivalence of double categories. Applying the assumption of the lemma to each $\mathcal{A} = w_{l,m}\mathcal{C}$ gives homotopy equivalences of simplicial sets

$$w_{l,m}S_\bullet E(\mathcal{C},\mathcal{C},\mathcal{C}) \simeq s_\bullet w_{l,m}E(\mathcal{C},\mathcal{C},\mathcal{C}) \simeq s_\bullet E(w_{l,m}\mathcal{C}, w_{l,m}\mathcal{C}, w_{l,m}\mathcal{C})$$
$$\longrightarrow s_\bullet w_{l,m}\mathcal{C} \times s_\bullet w_{l,m}\mathcal{C} \simeq w_{l,m}S_\bullet\mathcal{C} \times w_{l,m}S_\bullet\mathcal{C}$$

which assemble into a levelwise homotopy equivalence of trisimplicial sets, and thus a homotopy equivalence

$$wS_\bullet E(\mathcal{C},\mathcal{C},\mathcal{C}) \longrightarrow wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C}.$$

□

We are now ready to prove Additivity. Our proof is nearly identical to [Cam19, Section 4], which in turn follows McCarthy [McC93]; we outline the details in the proof that require some attention when translated to our setting.

Proof of Theorem 6.5. By Proposition 6.9, it suffices to show that Additivity holds for FCGW categories (with isomorphisms as weak equivalences). Note that all definitions and results up to (and including) [Cam19, Proposition 4.13] can be readily adapted to our setting. Using McCarthy's notation, it remains to show that the map

$$\Gamma_n\colon S_\bullet F|C^2(-,n) \longrightarrow S_\bullet F|C^2(-,n)$$

is homotopic to the identity, where $F\colon E(\mathcal{C},\mathcal{C},\mathcal{C}) \longrightarrow \mathcal{C}\times\mathcal{C}$ denotes the additivity functor.

This is achieved by defining a simplicial homotopy h as follows: for each m, and each $0 \le i \le m$, the map

$$h_i\colon S_\bullet F|C^2(m,n) \longrightarrow S_\bullet F|C^2(m+1,n)$$

takes a generic element $e \in S_\bullet F|C^2(m,n)$ of the form

$$\begin{array}{ccccccc}
\varnothing = A_0 & \rightarrowtail & A_1 & \rightarrowtail & \dots & \rightarrowtail & A_m \\
\downarrow & \text{g} & \downarrow & & & & \downarrow \\
\varnothing = C_0 & \rightarrowtail & C_1 & \rightarrowtail & \dots & \rightarrowtail & C_m \\
\uparrow & \circlearrowleft & \uparrow & & & & \uparrow \\
\varnothing = B_0 & \rightarrowtail & B_1 & \rightarrowtail & \dots & \rightarrowtail & B_m
\end{array}$$

$$\varnothing = A_0 \rightarrowtail A_1 \rightarrowtail \dots \rightarrowtail A_m \rightarrowtail S_0 \rightarrowtail S_1 \rightarrowtail \dots \rightarrowtail S_n$$

$$\varnothing = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \dots \rightarrowtail T_n$$

to the element $h_i(e) \in S_\bullet F|C^2(m+1,n)$ given by

$$\begin{array}{ccccccccccc}
\varnothing = A_0 & \rightarrowtail & A_1 & \rightarrowtail & \dots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = \dots = & S_0 \\
\downarrow & \text{g} & \downarrow & & & & \downarrow & \text{g} & \downarrow & & \downarrow \\
\varnothing = C_0 & \rightarrowtail & C_1 & \rightarrowtail & \dots & \rightarrowtail & C_i & \rightarrowtail & C_i \star_{A_i} S_0 & \rightarrowtail \dots \rightarrowtail & C_m \star_{A_m} S_0 \\
\uparrow & \circlearrowleft & \uparrow & & & & \uparrow & \circlearrowleft & \uparrow & & \uparrow \\
\varnothing = B_0 & \rightarrowtail & B_1 & \rightarrowtail & \dots & \rightarrowtail & B_i & = & B_i & \rightarrowtail \dots \rightarrowtail & B_m
\end{array}$$

$$\varnothing = A_0 \rightarrowtail A_1 \rightarrowtail \dots \rightarrowtail A_i \rightarrowtail S_0 = \dots = S_0 = S_0 \rightarrowtail S_1 \rightarrowtail \dots \rightarrowtail S_n$$

$$\varnothing = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_i = B_i \rightarrowtail \dots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \dots \rightarrowtail T_m$$

where the maps and squares between $\star$-pushouts are given by Proposition A.3.

Even though they are not pictured in the above diagrams, we must make choices of staircases, and verify that the maps pictured above truly give kernel-cokernel pairs in $\mathcal{C}$. Let $A_{k,l}, B_{k,l}$ and $C_{k,l}$ denote the objects in the (non-depicted) staircases of the top, bottom, and middle rows of the extension in $e \in S_\bullet F|C^2(m,n)$. Similarly, denote by $h_i(e)^A_{k,l}$, $h_i(e)^B_{k,l}$ and $h_i(e)^C_{k,l}$ the objects in the staircases of the top, bottom, and middle rows of the extension in $h_i(e)$. Then, we let

$$h_i(e)^A_{k,l} = \begin{cases} A_{k,l} & k, l \leq i \\ S_0/A_{0,k} & k \leq i, l > i \\ \varnothing & \text{otherwise} \end{cases}$$

$$h_i(e)^C_{k,l} = \begin{cases} C_{k,l} & k, l \leq i \\ h_i(e)^A_{k,l} & k = i, l = i+1 \\ h_i(e)^B_{k,l} & l, k \geq i+1 \\ C_{k,l-1} \star_{A_{k,l-1}} h_i(e)^A_{k,l} & \text{otherwise} \end{cases}$$

$$h_i(e)^B_{k,l} = \begin{cases} B_{k,l} & k, l \leq i \\ B_{k,l-1} & k \leq i, l \geq i+1 \\ B_{k-1,l-1} & k \geq i+1, l \geq i+1 \end{cases}$$

First, we must make sure that the data of $h_i(e)^A, h_i(e)^B$ and $h_i(e)^C$ actually form staircases. The first two are immediate, as all the squares involved are squares already present in e. The fact that $h_i(e)^C$ is a staircase is due to the existence of distinguished squares

$$\begin{array}{ccc} C_{k,l} \star_{A_{k,l}} S_{k,0} & \rightarrowtail & C_{k,l+1} \star_{A_{k,l+1}} S_{k,0} \\ \uparrow\circ & \square & \uparrow\circ \\ B_{k,l} & \rightarrowtail & B_{k,l+1} \end{array} \qquad \begin{array}{ccc} C_{k,l} & \rightarrowtail & C_{k,l} \star_{A_{k,l}} S_{k,0} \\ \uparrow\circ & \square & \uparrow\circ \\ C_{k+1,l} & \rightarrowtail & C_{k+1,l} \star_{A_{k+1,l}} S_{k+1,0} \end{array}$$

$$\begin{array}{ccc} C_{k,l} \star_{A_{k,l}} S_{k,0} & \rightarrowtail & C_{k,l+1} \star_{A_{k,l+1}} S_{k,0} \\ \uparrow\circ & \square & \uparrow\circ \\ C_{k+1,l} \star_{A_{k+1,l}} S_{k+1,0} & \rightarrowtail & C_{k+1,l+1} \star_{A_{k+1,l+1}} S_{k+1,0} \end{array}$$

arising from Proposition A.3, Proposition A.4, and Proposition A.12 respectively, where we abbreviate $S_{k,0} \coloneqq S_0/A_{0,k}$.

For each k, l, we have evident choices of maps

$$h_i(e)^A_{k,l} \rightarrowtail h_i(e)^C_{k,l} \leftarrow\!\circ\, h_i(e)^B_{k,l}$$

which form kernel-cokernel sequences. It remains to check that these assemble into maps

$$h_i(e)^A \rightarrowtail h_i(e)^C \leftarrow\!\circ\, h_i(e)^B;$$

that is, that all the squares between the staircases are of the correct form. A careful study reveals that this is ensured by the aforementioned properties of the $\star$-pushout, together with the fact that by Proposition A.4, we have pseudo-commutative squares

$$\begin{array}{ccc} S_{k,0} & \rightarrowtail & C_{k,l} \star_{A_{k,l}} S_{k,0} \\ \uparrow\circ & \circlearrowleft & \uparrow\circ \\ S_{k+1,0} & \rightarrowtail & C_{k+1,l} \star_{A_{k+1,l}} S_{k+1,0} \end{array}$$

for each k, l, whose induced map on cokernels is the map $B_{k+1,l} \circ\!\!\rightarrow B_{k,l}$ found in e.

Just as in [Cam19], one can check that h defines a simplicial homotopy from Γ_n to id. $\square$

It will also be useful to have an equivalent version of the Additivity Theorem at hand.

Theorem 6.10. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$ be full FCGW subcategories of an FCGWA category $(\mathcal{C}, \mathcal{W})$. Then, the map*

$$wS_\bullet E(\mathcal{A}, \mathcal{C}, \mathcal{B}) \longrightarrow wS_\bullet \mathcal{A} \times wS_\bullet \mathcal{B}$$

induced by

$$(A \rightarrowtail C \longleftarrow\!\circ\, B) \mapsto (A, B)$$

is a homotopy equivalence.

Proof. The proof is identical to the relevant part of [Wal85, Proposition 1.3.2], since by Remark 3.3 our FCGW categories always admit trivial extensions of the form

$$A \rightarrowtail A \star_\varnothing B \longleftarrow\!\circ\, B$$

□

7. Relative K-theory and delooping

In this section, we show that for any FCGWA category $(\mathcal{C}, \mathcal{W})$, $K(\mathcal{C}, \mathcal{W})$ is a spectrum. This is done by defining a notion of relative K-theory and following the same outline as in [Wal85, Section 1.5]; we include the proofs here for completeness.

Definition 7.1. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an FCGWA functor between FCGWA categories. For each n, we define the double category $S_n(F)$ as the pullback

$$\begin{array}{ccc} S_n(F) & \longrightarrow & S_{n+1}\mathcal{B} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_0} \\ S_n\mathcal{A} & \xrightarrow[F]{} & S_n\mathcal{B} \end{array}$$

$S_n(F)$ is then the double category of staircases in $S_{n+1}\mathcal{B}$ which are equipped with a lift of all but the top row to $S_n\mathcal{A}$ along F.

Lemma 7.2. *$S_\bullet(F)$ is a simplicial FCGWA category.*

Proof. The fact that each $S_n(F)$ is an FCGWA category follows directly from the FCGWA structures on $S_{n+1}\mathcal{B}$ and $S_n\mathcal{A}$ given by Proposition 5.3. The face and degeneracy maps are given by shifting those of $S_n\mathcal{B}$; that is, $d_i^{S_\bullet(F)} := d_{i+1}^{S_\bullet \mathcal{B}}$, and $s_i^{S_\bullet(F)} := s_{i+1}^{S_\bullet \mathcal{B}}$. □

The above construction allows us to present the following definition.

Definition 7.3. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an FCGWA functor between FCGWA categories. The **relative K-theory** of F is defined as

$$K(F) = \Omega|wS_\bullet(F)|$$

Just as in [Wei13, Chapter IV, 8.5.4], we have the following.

Lemma 7.4. *If $\mathcal{A} = \mathcal{B}$, $wS_\bullet S_\bullet(\mathrm{id}_\mathcal{B})$ is contractible.*

Proof. Note that in this case, $S_n(\mathrm{id}_\mathcal{B})$ is defined via the pullback

$$\begin{array}{ccc} S_n(\mathrm{id}_\mathcal{B}) & \longrightarrow & S_{n+1}\mathcal{B} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_0} \\ S_n\mathcal{B} & = & S_n\mathcal{B} \end{array}$$

and thus $S_n(\mathrm{id}_{\mathcal{B}}) \cong S_{n+1}\mathcal{B}$; in other words, $S_\bullet(\mathrm{id}_{\mathcal{B}})$ is the simplicial path space of $S_\bullet\mathcal{B}$. Similarly, one can see that for each n, $wS_nS_\bullet(\mathrm{id}_{\mathcal{B}})$ is the simplicial path space of $wS_nS_\bullet\mathcal{B}$. Then, we have a homotopy equivalence $wS_nS_\bullet(\mathrm{id}_{\mathcal{B}}) \simeq wS_nS_0\mathcal{B} \simeq *$ for each n, from which we conclude our result. □

Note that, given an FCGWA functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ we have FCGWA functors

$$\mathcal{B} \longrightarrow S_n(F)$$

taking $B \in \mathcal{B}$ to $\varnothing \rightarrowtail B = \dots = B \in S_n(F)$, and

$$S_n(F) \longrightarrow S_n\mathcal{A}$$

given by one of the legs of the pullback. These functors satisfy the following proposition and its corollary, analogous to [Wal85, Proposition 1.5.5, Corollary 1.5.7].

Proposition 7.5. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an FCGWA functor. Then, we have a homotopy fiber sequence*

$$wS_\bullet\mathcal{B} \longrightarrow wS_\bullet S_\bullet(F) \longrightarrow wS_\bullet S_\bullet\mathcal{A}$$

Proof. First, we have a homotopy equivalence $wS_\bullet S_n(F) \simeq wS_\bullet E(\mathcal{B}, S_n(F), S_n\mathcal{A})$, as the conditions in Corollary 6.4 are easily checked. Then, by the Additivity Theorem 6.5, we have a homotopy equivalence

$$wS_\bullet S_n(F) \simeq wS_\bullet\mathcal{B} \times wS_\bullet S_n\mathcal{A}$$

for each n, from which we deduce the existence of the homotopy fiber sequence in the statement. □

Corollary 7.6. *Let $F\colon \mathcal{B} \longrightarrow \mathcal{C}$ be an FCGWA functor. Then, there exists a homotopy fiber sequence*

$$wS_\bullet\mathcal{B} \longrightarrow wS_\bullet\mathcal{C} \longrightarrow wS_\bullet S_\bullet(F)$$

We can finally deduce the main result in this section.

Theorem 7.7. *Let $(\mathcal{C}, \mathcal{W})$ be an FCGWA category. Then, $K(\mathcal{C}, \mathcal{W}) = \Omega|wS_\bullet\mathcal{C}|$ is an infinite loop space.*

Proof. Using Proposition 7.5 for $\mathcal{A} = \mathcal{B} = \mathcal{C}$ yields a homotopy fiber sequence

$$wS_\bullet\mathcal{C} \longrightarrow wS_\bullet S_\bullet(\mathrm{id}_{\mathcal{C}}) \longrightarrow wS_\bullet S_\bullet\mathcal{C}$$

But $wS_\bullet S_\bullet(\mathrm{id}_{\mathcal{C}})$ is contractible by Lemma 7.4, and so we conclude that there exists a homotopy equivalence $|wS_\bullet\mathcal{C}| \simeq \Omega|wS_\bullet S_\bullet\mathcal{C}|$. Iterating this process yields the desired delooping

$$|wS_\bullet\mathcal{C}| \simeq \Omega|wS_\bullet S_\bullet\mathcal{C}| \simeq \Omega\Omega|wS_\bullet S_\bullet S_\bullet\mathcal{C}| \simeq \cdots \simeq \Omega^n|wS_\bullet^{n+1}\mathcal{C}| \simeq \dots$$

□

8. Fibration Theorem

This section is dedicated to our primary tool for comparing FCGWA categories: the analogue of Waldhausen's Fibration Theorem, which relates the K-theory spectra of an FCGW category equipped with two comparable classes of weak equivalences. The statement is as follows.

Theorem 8.1 (Fibration)**.** *Let $\mathcal{V}$ and $\mathcal{W}$ be two acyclicity structures on an FCGW category $\mathcal{C}$, such that $\mathcal{V} \subseteq \mathcal{W}$. Then, there exists a homotopy fiber sequence*

$$K(\mathcal{W}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{V}) \longrightarrow K(\mathcal{C}, \mathcal{W})$$

Our proof largely follows that of Waldhausen, but avoids the rather burdensome assumptions that go into proving that the category of weak equivalences is homotopy equivalent to that of trivial cofibrations. Indeed, the reader might have noticed we do not require any additional conditions on our structures in order for our Fibration Theorem to hold. In contrast, the classical version due to Waldhausen (see [Wal85, Theorem 1.6.4]) asks for the saturation and extension axioms, and for the existence of a cylinder functor satisfying the cylinder axiom. Even the more relaxed version of Waldhausen's Fibration due to Schlichting (see [Sch06, Theorem A.3]) only goes as far as replacing cylinders by factorizations: every map must factor as a cofibration followed by a weak equivalence.

The reason behind this apparent clash is that our FCGWA categories were, in a way, constructed so that all of these properties are already incorporated. Namely, the saturation axiom (in our case, the fact that m- and e-equivalences satisfy 2-out-of-3) is an easy consequence of the definition of m- and e-equivalences, as seen in Lemma 4.12. Similarly, the extension axiom is required in the classical setting in order to prove that trivial cofibrations can be characterized by having acyclic cokernels; this is precisely how all our m-equivalences are defined in Definition 4.3.

As for the absence of a cylinder or factorization requirement, the reason is that all of the maps that our constructions see are already "simple enough" and do not need to be decomposed any further; this is a feature of the double-categorical approach. Concretely, this amounts to considering only admissible monomorphisms and epimorphisms in an exact category as opposed to working with arbitrary morphisms.

As a consequence, our proof departs from Waldhausen's in that it does not need to go through the subcategory of trivial cofibrations, which he denotes $\overline{w}S_\bullet\mathcal{C}$. Instead, we rely on the following result, which exploits the symmetry of our setting, where vertical maps have equally convenient properties to horizontal ones.

Proposition 8.2. *For any refinement $(\mathcal{C},\mathcal{V})$ of $(\mathcal{C},\mathcal{W})$ and any l,m, we have homotopy equivalences of simplicial double categories*

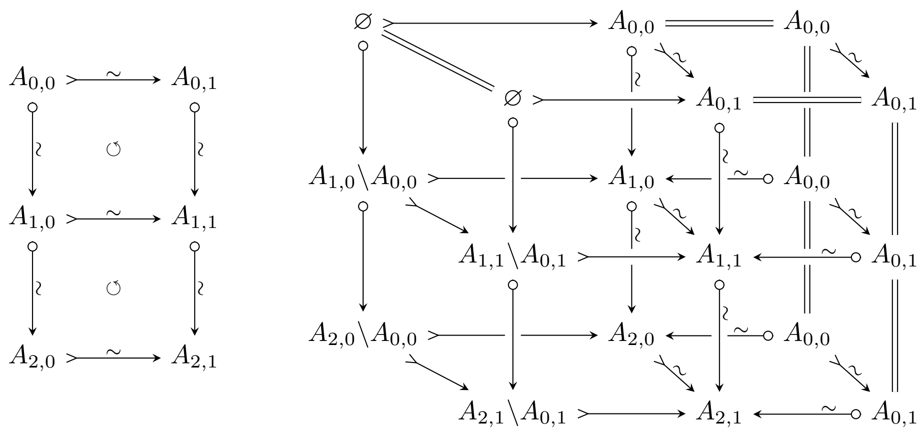
$$vS_\bullet w_{l,m}\mathcal{C} \simeq vS_\bullet w_{0,m}\mathcal{C} \times vS_\bullet w_{l-1,m}\mathcal{W}$$

and

$$vS_\bullet w_{l,m}\mathcal{C} \simeq vS_\bullet w_{l,0}\mathcal{C} \times vS_\bullet w_{l,m-1}\mathcal{W}$$

Proof. We prove the first statement; the second is entirely dual. The strategy will be to show that $w_{l,m}\mathcal{C}$ is equivalent (in the sense of Lemma 6.3) to the extension FCGWA category $E(w_{l-1,m}\mathcal{W}, w_{l,m}\mathcal{C}, w_{0,m}\mathcal{C})$; then, we deduce the desired statement from Corollary 6.4 and the Additivity Theorem 6.5.

For this, consider an object $A_{*,\bullet}$ in $w_{l,m}\mathcal{C}$ pictured below left, and associate to it the object in $E(w_{l-1,m}\mathcal{W}, w_{l,m}\mathcal{C}, w_{0,m}\mathcal{C})$ pictured below right (where l,m are pictured as 2 and 1 respectively for convenience). We henceforth abuse notation and identify $w_{0,m}\mathcal{C}$ with its image under the inclusion $w_{0,m}\mathcal{C} \hookrightarrow w_{l,m}\mathcal{C}$, and similarly for $w_{l-1,m}\mathcal{W}$.

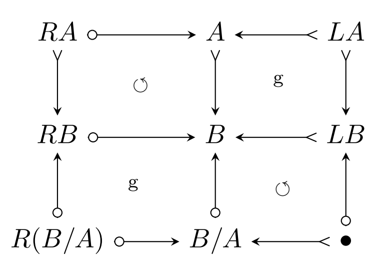


First of all, we check that the diagram above right truly is an object of $E(w_{l-1,m}\mathcal{W}, w_{l,m}\mathcal{C}, w_{0,m}\mathcal{C})$. Indeed, all of the squares are either good or pseudo-commutative, it is clearly a kernel-cokernel pair since these are constructed pointwise, and the grid on the right is an element of $w_{0,m}\mathcal{C}$. Lastly, the grid on the left is comprised of objects in $\mathcal{W}$ since they are all kernels of e-equivalences, and then the maps between them must be w- and e-equivalences by Lemma 4.11; thus, this grid is an object of $w_{l-1,m}\mathcal{W}$.

Now, to use Lemma 6.3, we need to define an FCGWA functor $R\colon w_{l,m}\mathcal{C} \longrightarrow w_{0,m}\mathcal{C}$ together with an e-natural transformation $\eta\colon R \Rightarrow \mathrm{id}$. Let R take an object $A_{*,\bullet}$ in $w_{l,m}\mathcal{C}$ as above to the rightmost grid in the picture, which is an object of $w_{0,m}\mathcal{C}$. This assignment evidently forms an FCGWA functor, as it simply forgets and then repeats part of the structure. Let the components of η be given by the e-morphism we see in the extension above, from the grid on the right ($RA_{*,\bullet}$) to the grid in the middle ($A_{*,\bullet}$). It is then immediate to verify that η is an e-natural transformation; moreover, all its component squares of e-morphisms are good.

Next, we define an FCGWA functor $L\colon w_{l,m}\mathcal{C} \longrightarrow w_{l-1,m}\mathcal{W}$ together with an m-natural transformation $\mu\colon R \Rightarrow \mathrm{id}$ by taking the kernel of the e-natural transformation η. Note that by Theorem B.1, this produces a double functor and an m-natural transformation, and furthermore, that L takes an object $A_{*,\bullet}$ to the leftmost grid pictured in the extension, and the components of μ agree with the m-morphism we see in the picture from the grid on the left to the one in the middle.

To see that L is an FCGWA functor, we must check that it preserves the remaining relevant structure. The fact that L preserves good squares is ensured by the converse in Proposition A.7, and it also preserves $\star$-pushouts, since by Remark A.5 the $\star$-pushout of the kernels is the kernel of the $\star$-pushouts. To see that L preserves cokernels, let $A \rightarrowtail B$ be an m-morphism in $w_{l,m}\mathcal{C}$ and construct the following diagram



where all columns and rows are kernel-cokernel pairs. Then, we have that $\bullet$ must be both the kernel of $R(B/A) \circ\!\!\longrightarrow B/A$ (which is by definition $L(B/A)$) and the cokernel of $LA \rightarrowtail LB$ (which is LB/LA). This shows that L preserves cokernels; the proof for kernels is analogous. Lastly, L preserves acyclic objects, as $\mathcal{V}$ is closed under kernels.

As to the last condition of Lemma 6.3, in order to see that every object $B \rightarrowtail A \longleftarrow\!\!\circ\, C$ in $E(w_{l-1,m}\mathcal{W}, w_{l,m}\mathcal{C}, w_{0,m}\mathcal{C})$ is of the form $LA \rightarrowtail A \longleftarrow\!\!\circ\, RA$ up to isomorphism, note that as $B \in w_{l-1,m}\mathcal{W}$, it has initial objects in the top row, and so the top components of $C \circ\!\!\longrightarrow A$ are necessarily isomorphisms by Lemma 2.14. Hence up to isomorphism, each row C must agree with the top row of A, and we get that $C \cong RA$. As k preserves isomorphisms, this implies that $B \cong LA$, completing the proof. □

We can now proceed to the proof of the Fibration Theorem.

Proof. (Theorem 8.1) To obtain the desired homotopy fiber sequence on K-theory, it is enough to show that

$$vS_\bullet\mathcal{W} \longrightarrow vS_\bullet\mathcal{C} \longrightarrow wS_\bullet\mathcal{C}$$

is a homotopy fiber sequence. For this, let $vwS_\bullet\mathcal{C}$ denote the simplicial triple category which has w-maps in two directions, and v-maps in the other two. Note that we can include $vS_\bullet\mathcal{C}$ into $vwS_\bullet\mathcal{C}$ by considering identities in the w-directions. Similarly, we have an inclusion of $wS_\bullet\mathcal{C}$ into $vwS_\bullet\mathcal{C}$ which, as $\mathcal{V} \subseteq \mathcal{W}$, is furthermore a homotopy equivalence by the 2-dimensional analogue of Waldhausen's Swallowing Lemma ([Wal85, Lemma 1.6.5]), proven easily by applying the original twice. We will abuse notation and write $vwS_\bullet\mathcal{C} \longrightarrow wS_\bullet\mathcal{C}$ for the homotopy inverse, which exists truly at the level of spaces.

In order to show the sequence pictured above is a homotopy fiber sequence, it suffices to prove that the outer square below is a homotopy pullback, as each category $w_{-,m}S_n\mathcal{W}$ has an initial object and so $wS_\bullet\mathcal{W}$ is contractible.

$$\begin{array}{ccccc} vS_\bullet\mathcal{W} & \longrightarrow & vwS_\bullet\mathcal{W} & \longrightarrow & wS_\bullet\mathcal{W} \\ \downarrow & & \downarrow & & \downarrow \\ vS_\bullet\mathcal{C} & \longrightarrow & vwS_\bullet\mathcal{C} & \longrightarrow & wS_\bullet\mathcal{C} \end{array}$$

Since the horizontal maps in the square above right are homotopy equivalences by the Swallowing Lemma, this is equivalent to showing that the square above left is a homotopy pullback.

Up to this point, our proof is virtually identical (albeit higher-dimensional) to [Wal85, Theorem 1.6.4]. The conclusion, however, diverges from Waldhausen's approach and instead exploits the symmetry in our FCGW categories.

Recall that we have homotopy equivalences

$$\begin{aligned} vw_{l,m}S_\bullet\mathcal{C} &\simeq vS_\bullet w_{l,m}\mathcal{C} \\ &\simeq (vS_\bullet w_{0,m}\mathcal{C}) \times (vS_\bullet w_{l-1,m}\mathcal{W}) \\ &\simeq (vS_\bullet w_{0,0}\mathcal{C} \times vS_\bullet w_{0,m-1}\mathcal{W}) \times (vS_\bullet w_{l-1,m}\mathcal{W}) \end{aligned}$$

where the first equivalence (in fact, isomorphism) is due to Lemma 6.8, and the others are obtained from Proposition 8.2. Then, we have

$$vw_{l,m}S_\bullet\mathcal{C} \simeq vS_\bullet\mathcal{C} \times vS_\bullet w_{0,m-1}\mathcal{W} \times vS_\bullet w_{l-1,m}\mathcal{W},$$

and using the same reasoning for the FCGW category $\mathcal{W}$ in place of $\mathcal{C}$, we see that

$$vw_{l,m}S_\bullet\mathcal{W} \simeq vS_\bullet\mathcal{W} \times vS_\bullet w_{0,m-1}\mathcal{W} \times vS_\bullet w_{l-1,m}\mathcal{W}.$$

Writing X for the trisimplicial double category with

$$X_{\bullet,l,m} = vS_\bullet w_{0,m-1}\mathcal{W} \times vS_\bullet w_{l-1,m}\mathcal{W},$$

the argument above shows that the relevant square is homotopy equivalent to the following:

$$\begin{array}{ccc} vS_\bullet\mathcal{W} & \longrightarrow & vS_\bullet\mathcal{W}\times X \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{C} & \longrightarrow & vS_\bullet\mathcal{C}\times X \end{array}$$

which is a homotopy pullback, as the homotopy cofibers of the horizontal maps agree. □

9. Localization Theorem

In the previous section, we saw how the Fibration Theorem 8.1 allows us to compare the K-theory spectra $K(\mathcal{C},\mathcal{W})$ and $K(\mathcal{C},\mathcal{V})$ of an FCGW category $\mathcal{C}$ with two classes of weak equivalences when $\mathcal{V}\subseteq\mathcal{W}$; namely, they differ by a homotopy fiber $K(\mathcal{W},\mathcal{V})$. Interestingly, as an immediate consequence of our Fibration Theorem, we obtain a Localization Theorem that allows us to compare the K-theory spectra of two different FCGW categories $\mathcal{A}\subseteq\mathcal{B}$ by finding a homotopy cofiber.

$$K(\mathcal{A},\mathcal{W}) \longrightarrow K(\mathcal{B},\mathcal{W}) \longrightarrow K(\mathcal{B},\mathcal{A})$$

Theorem 9.1. *Let $\mathcal{A}\subseteq\mathcal{B}$ be a full inclusion of FCGW categories, such that $\mathcal{A}$ is closed under cokernels of m-morphisms, kernels of e-morphisms, and extensions in $\mathcal{B}$. Then, there exists an FCGWA category $(\mathcal{B},\mathcal{A})$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B},\mathcal{A})$$

is a homotopy fiber sequence.

Proof. This is a direct application of Theorem 8.1 for $\mathcal{C}=\mathcal{B}$, $\mathcal{W}=\mathcal{A}$, $\mathcal{V}=\varnothing$, as any full FCGW subcategory $\mathcal{A}\subseteq\mathcal{B}$ which is closed under extensions forms an acyclicity structure in $\mathcal{B}$. □

This generalizes several Localization Theorems in the literature, as we now study.

9.1. Abelian and exact categories.

The original Localization Theorem is due to Quillen, and was introduced in the context of abelian categories.

Theorem. [Qui73, Theorem 5] *Let $\mathcal{A}$ be a Serre subcategory of an abelian category $\mathcal{B}$. Then there exists an abelian category $\mathcal{B}/\mathcal{A}$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A})$$

is a homotopy fiber sequence.

Recall that a subcategory $\mathcal{A}\subseteq\mathcal{B}$ is called *Serre* if it is full, and for every short exact sequence $X\hookrightarrow Y\twoheadrightarrow Z$ in $\mathcal{B}$, we have that $Y\in\mathcal{A}$ if and only if $X,Z\in\mathcal{A}$.

Although immensely useful, this result suffers from an evident limitation: it only applies to abelian categories, while many of the categories of interest to K-theory are not abelian, but exact. Following this line of thought, different authors have generalized Quillen's Localization Theorem to exact categories by requiring additional conditions on the Serre subcategory $\mathcal{A}$. Their results can be stated as follows.

Theorem. [Sch04, Theorem 2.1],[Car98] *Let $\mathcal{A}$ be a Serre subcategory of an exact category $\mathcal{B}$. If in addition $\mathcal{A}$ is left or right s-filtering* ([Sch04])*, or localizes $\mathcal{B}$* ([Car98])*, then there exists an exact category $\mathcal{B}/\mathcal{A}$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A})$$

is a homotopy fiber sequence.

We omit the definitions of s-filtering and localizing subcategories, but as the authors show, these conditions are automatically satisfied when the categories in question are both abelian.

In a different vein, a Localization Theorem was proved by the first author in [Sar20] that weakens the Serre requirement on the subcategory $\mathcal{A}$ in favor of more algebraic conditions.

Theorem. [Sar20, Theorem 6.1] *Let $\mathcal{B}$ be an exact category closed under kernels of epimorphisms and with enough injective objects, and $\mathcal{A} \subseteq \mathcal{B}$ a full subcategory having 2-out-of-3 for short exact sequences and containing all injective objects. Then there exists a Waldhausen category $(\mathcal{B}, w_{\mathcal{A}})$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}, w_{\mathcal{A}})$$

is a homotopy fiber sequence.

To compare these results to Theorem 9.1, assume the exact category $\mathcal{B}$ is weakly idempotent complete (as is the case, for example, if $\mathcal{B}$ is abelian); hence, so is a subcategory $\mathcal{A}$ satisfying the hypotheses of any of the theorems above. As explained in Remark 5.10, this assumption on $\mathcal{B}$ is harmless for K-theoretical purposes. However, it provides a more convenient model since, as detailed in Example 3.4, $\mathcal{A}$ and $\mathcal{B}$ can be given a structure of FCGW categories.

When specialized to the case of FCGW categories coming from exact categories, our Localization Theorem reads as follows.

Theorem. *Let $\mathcal{B}$ be an exact category and $\mathcal{A} \subseteq \mathcal{B}$ a full subcategory having 2-out-of-3 for short exact sequences in $\mathcal{B}$. Then there exists an FCGWA category $(\mathcal{B}, \mathcal{A})$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}, \mathcal{A})$$

is a homotopy fiber sequence.

If the inclusion $\mathcal{A} \subseteq \mathcal{B}$ satisfies the conditions of any of the previous Localization theorems above, then $\mathcal{A}$ is in particular closed under cokernels of $\mathcal{B}$-admissible monomorphisms and kernels of $\mathcal{B}$-admissible epimorphisms; thus $\mathcal{A}$ is a full FCGWA subcategory of $\mathcal{B}$. Moreover, $\mathcal{A}$ is closed under extensions in $\mathcal{B}$, and so our Localization Theorem can be used to produce an FCGWA category $(\mathcal{B}, \mathcal{A})$ such that

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}, \mathcal{A})$$

is a homotopy fiber sequence. Since the K-theory spectra of $\mathcal{A}$ and $\mathcal{B}$ as exact and as FCGW categories agree by Remark 5.9, it must be that $K(\mathcal{B}, \mathcal{A}) \simeq K(\mathcal{B}/\mathcal{A})$ or $K(\mathcal{B}, \mathcal{A}) \simeq K(\mathcal{B}, w_{\mathcal{A}})$; then, our theorem provides an FCGWA model for the cofibers constructed through the existing Localization Theorems.

Notably, Section 9.1 only requires that $\mathcal{A}$ has 2-out-of-3 for short exact sequences in $\mathcal{B}$, and thus provides a wider field for applications than the previously existing results. Much like [Sar20, Theorem 6.1], our Localization Theorem constructs a model for the cofiber which is no longer an exact category, though the FCGWA category we construct cannot in general be modeled by a Waldhausen category as the conditions on weak equivalences between the two are not compatible. This suggests that FCGWA categories are ideally suited to model localizations of exact categories.

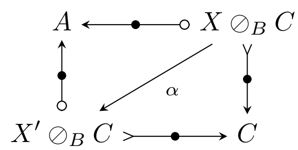
9.2. **ACGW categories.** We recall the Localization Theorem for ACGW categories, introduced in [CZ].

Theorem. [CZ, Theorem 8.6] *Suppose that $\mathcal{B}$ is an ACGW category and $\mathcal{A}$ is an ACGW subcategory satisfying the following conditions:*

(W) *$\mathcal{A}$ is m-well-represented or m-negligible in $\mathcal{B}$ and $\mathcal{A}$ is e-well-represented or e-negligible in $\mathcal{B}$.*

(CGW) *$\mathcal{B}\backslash\mathcal{A}$ is a CGW-category.*

(E) *For two diagrams $A \leftarrow\!\bullet\!\circ X \rightarrowtail\!\bullet\!\to B$ and $A \leftarrow\!\bullet\!\circ X' \rightarrowtail\!\bullet\!\to B$ which represent the same morphism in $\mathcal{B}\backslash\mathcal{A}$ there exists a diagram $B \leftarrow\!\bullet\!\circ C$ and an isomorphism $\alpha\colon X \oslash_B C \longrightarrow X' \oslash_B C$ such that the induced diagram*



commutes. The same statement holds with e-morphisms and m-morphisms swapped.

Then,

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}\backslash\mathcal{A})$$

is a homotopy fiber sequence.

We omit the definitions of m- and e-well-represented subcategories ([CZ, Definition 8.4]), m- and e-negligible subcategories ([CZ, Definition 8.5]), and of the CGW category $\mathcal{B}\backslash\mathcal{A}$ itself ([CZ, Definition 8.1]).

In [CZ], the authors check the hypotheses of this theorem in two different contexts: that of abelian categories with an inclusion $\mathcal{A} \subseteq \mathcal{B}$ of a Serre subcategory, and that of reduced schemes of finite type of bounded dimension where they consider the inclusion $\mathsf{Sch}_{rf}^{d-1} \subseteq \mathsf{Sch}_{rf}^{d}$. The example of abelian categories agrees with the classical case, and was compared to our result in the previous subsection. As to the inclusion $\mathsf{Sch}_{rf}^{d-1} \subseteq \mathsf{Sch}_{rf}^{d}$, we note that Sch_{rf}^{d-1} is closed under cokernels of m-morphisms, kernels of e-morphisms, and extensions in Sch_{rf}^{d}, and so our Localization Theorem recovers this example as well.

Notably, adding weak equivalences as additional structure rather than strictly inverting them lets us avoid the often tedious process of checking that the double category $\mathcal{B}\backslash\mathcal{A}$ is CGW.

Part 3. Chain complexes of finite sets

10. Chain complexes

One of the main motivations for developing the theory of FCGW categories is to allow for more general mathematical objects to be analyzed "algebraically" in the mold of exact categories. A very powerful tool in the algebraic world is that of chain complexes; these provide a convenient model one can use to do homological algebra, homotopy theory, and even K-theory. In short, chain complexes over an exact category generalize its objects and allow for more combinatorial manipulations, without changing its K-theory, according to the classical Gillet–Waldhausen Theorem.

In this section, we seek to generalize this approach, and use the unifying language of FCGW categories to motivate a definition of chain complexes in a new setting: the FCGW category of finite sets. While much of the theory of chain complexes can be imitated for general FCGW categories, these chain complexes do not themselves form an FCGW category without introducing additional information, particularly for the construction of $\star$-pushouts of spans in $\mathcal{M}$. In future work with Inna Zakharevich, we expect this approach to generalize to other examples of interest, such as varieties.

We begin by recalling the usual definition of a chain complex on an abelian category, cast in the light of FCGW categories.

Definition 10.1. Let $\mathcal{A}$ be an abelian category, considered as an FCGW category in the standard way. A **chain complex over the abelian category** $\mathcal{A}$ is a diagram in $\mathcal{A}$ of the form

$$\cdots X_{i+1} \longleftarrow\!\!\circ\ \overline{X}_{i+1} \rightarrowtail X_i \longleftarrow\!\!\circ\ \overline{X}_i \rightarrowtail X_{i-1} \cdots$$

where i ranges over the integers, satisfying the **chain condition**: for each i, the following is a pseudo-commutative square.

$$\begin{array}{ccc} \varnothing & \rightarrowtail & \overline{X}_i \\ \circ\!\!\downarrow & \circlearrowleft & \circ\!\!\downarrow \\ \overline{X}_{i+1} & \rightarrowtail & X_i \end{array}$$

A **monomorphism** (resp. **epimorphism**) **of chain complexes** is a collection $\{f_i, \overline{f}_i\}$ of monomorphisms (resp. epimorphisms) in $\mathcal{A}$ that form commutative diagrams

$$\begin{array}{ccccc} X_i & \longleftarrow\!\!\circ & \overline{X}_i & \rightarrowtail & X_{i-1} \\ f_i \downarrow & \circlearrowleft & \downarrow \overline{f}_i & & \downarrow f_{i-1} \\ Y_i & \longleftarrow\!\!\circ & \overline{Y}_i & \rightarrowtail & Y_{i-1} \end{array} \qquad \begin{array}{ccccc} X_i & \longleftarrow\!\!\circ & \overline{X}_i & \rightarrowtail & X_{i-1} \\ f_i \downarrow & & \downarrow \overline{f}_i & \circlearrowleft & \downarrow f_{i-1} \\ Y_i & \longleftarrow\!\!\circ & \overline{Y}_i & \rightarrowtail & Y_{i-1} \end{array}$$

Note that this notion of chain complex agrees with the classical one. Here, a differential $X_i \longleftarrow\!\!\circ\ \overline{X}_i \rightarrowtail X_{i-1}$ is simply the epi-mono factorization of a general map $d_i\colon X_{i+1} \longrightarrow X_i$, and we have $\overline{X}_{i+1} = \mathrm{im} d_i$. Furthermore, given a diagram $\overline{X}_{i+1} \rightarrowtail X_i \longleftarrow\!\!\circ\ \overline{X}_i$, we can complete it to a pseudo-commutative square as done in Lemma 2.16. In this case, the pseudo-commutative completion always exists since abelian categories have pullbacks of monomorphisms and epimorphisms, and the process yields the epi-mono factorization of the composite $\overline{X}_{i+1} \hookrightarrow X_i \twoheadrightarrow \overline{X}_i$ in the abelian category. Then, the chain condition says that this composite must factor through the zero object, which is equivalent to the classical condition on differentials $d^2 = 0$. We henceforth refer to these pseudo-commutative completions as "mixed pullbacks", following the convention in [CZ].

As for the morphisms, recall that pseudo-commutative squares are the commutative squares, and so the maps $\overline{f}_i$ simply denote the induced maps on the images of the differentials.

Since the FCGW category of finite sets of Examples 2.8 and 3.6 also has all mixed pullbacks, we could easily use the above definition to obtain a notion of chain complex of sets. These admit a simple notion of homology where H_i is defined as the total complement in X_i of the pair of injections $\overline{X}_{i+1} \rightarrowtail X_i \longleftarrow\!\!\circ\ \overline{X}_i$; that is, $H_i = X_i \backslash (\overline{X}_i \cup \overline{X}_{i+1})$. Moreover, we recover classical results from homological algebra, such as the Snake Lemma and the long exact sequence in homology.

However, these chain complexes of sets do not form an FCGW category, as they fail to have the necessary $\star$-pushouts. The reason for this obstruction is that, even though any span of injections between finite sets admits a pushout, a natural transformation between two such spans induces a function between their pushouts which is not in general an injection, even if the transformation is objectwise injective.

In order to remedy this, we relax the m-morphisms in our differentials to instead include all functions of sets in that direction, which we denote by $\longrightarrow$. First, let us comment on how the relevant features in the FCGW category of finite sets can be extended to include arbitrary functions in the m-direction.

Lemma 10.2. *Let $\overline{\mathcal{M}}$ denote the category of finite sets and all functions; $\mathrm{Ar}_{\mathrm{g}}\,\overline{\mathcal{M}}$ denote the category with objects m-morphisms and morphisms pullback squares between them in $\overline{\mathcal{M}}$; and $\mathrm{Ar}_{\overline{\circlearrowleft}}\,\mathcal{E}$ denote the category with objects e-morphisms and morphisms pullback squares between them in $\overline{\mathcal{M}}$. Then, the following hold:*

- *$\mathcal{M}$ (resp. $\mathcal{E}$) is closed under base change in $\overline{\mathcal{M}}$; that is, the pullback of a span*

$$B \rightarrowtail A \longleftarrow C \qquad (\text{resp. } B \circ\!\!\longrightarrow A \longleftarrow C)$$

  exists and we get $B \times_A C \rightarrowtail C$ (resp. $B \times_A C \circ\!\!\rightarrow C$),
- *$k\colon \mathrm{Ar}_{\circlearrowleft}\,\mathcal{E} \longrightarrow \mathrm{Ar}_{\mathrm{g}}\,\mathcal{M}$ extends to an equivalence $\mathrm{Ar}_{\overline{\circlearrowleft}}\,\mathcal{E} \longrightarrow \mathrm{Ar}_{\mathrm{g}}\,\overline{\mathcal{M}}$ between squares as below*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \circ\!\!\downarrow & \circlearrowleft & \circ\!\!\downarrow \\ C & \longrightarrow & D \end{array} \qquad\qquad \begin{array}{ccc} C & \longrightarrow & D \\ \uparrow\!\!\curlyvee & \mathrm{g} & \uparrow\!\!\curlyvee \\ E & \longrightarrow & F \end{array}$$

- *any cospan as below can be completed to a unique mixed pullback as below right*

$$\begin{array}{ccc} & & B \\ & & \circ\!\!\downarrow \\ C & \longrightarrow & D \end{array} \qquad\qquad \begin{array}{ccc} A & \longrightarrow & B \\ \circ\!\!\downarrow & \circlearrowleft & \circ\!\!\downarrow \\ C & \longrightarrow & D \end{array}$$

Proof. The proof is immediate, once we recall that both m- and e-morphisms are injections, pseudo-commutative and good squares are pullbacks, and k takes complements. $\square$

We can now define our chain complexes of finite sets.

Definition 10.3. A **chain complex of finite sets** is a diagram in FinSet of the form

$$\cdots X_{i+1} \longleftarrow\!\!\circ\ \overline{X}_{i+1} \longrightarrow X_i \longleftarrow\!\!\circ\ \overline{X}_i \longrightarrow X_{i-1} \cdots$$

where i ranges over the integers, satisfying the **chain condition**: for each i, the following is a pseudo-commutative square.

$$\begin{array}{ccc} \varnothing & \longrightarrow & \overline{X}_i \\ \circ\!\!\downarrow & \circlearrowleft & \circ\!\!\downarrow \\ \overline{X}_{i+1} & \longrightarrow & X_i \end{array}$$

The objects $\{X_i\}$ are called the **degrees** of X, $\{\overline{X}_i\}$ are called the **images** of X, and each $X_i \longleftarrow\!\!\circ\ \overline{X}_i \longrightarrow X_{i-1}$ is called a **differential** of X. The differentials in a chain complex therefore agree with partial functions, where $X_i \longleftarrow\!\!\circ\ \overline{X}_i$ represents the inclusion of the domain into X_i.

As we will show, it is these complexes which form an FCGW category satisfying our version of the Gillet–Waldhausen Theorem. The remainder of this subsection is devoted to the construction of the FCGW structure.

Definition 10.4. An m-morphism f of chain complexes over FinSet, or **chain m-morphism**, is a collection $\{f_i, \overline{f}_i\}$ of m-morphisms in FinSet that form diagrams as below left, where the square in $\overline{\mathcal{M}}$ commutes.

$$\begin{array}{ccccc} X_i & \longleftarrow\!\circ & \overline{X}_i & \longrightarrow & X_{i-1} \\ {\scriptstyle f_i}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \overline{f}_i} & & \downarrow{\scriptstyle f_{i-1}} \\ Y_i & \longleftarrow\!\circ & \overline{Y}_i & \longrightarrow & Y_{i-1} \end{array} \qquad \begin{array}{ccccc} X_i & \longleftarrow\!\circ & \overline{X}_i & \longrightarrow & X_{i-1} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle \overline{g}_i} & \circlearrowleft & \downarrow{\scriptstyle g_{i-1}} \\ Y_i & \longleftarrow\!\circ & \overline{Y}_i & \longrightarrow & Y_{i-1} \end{array}$$

Similarly, a **chain e-morphism** is a collection $\{g_i, \overline{g}_i\}$ of e-morphisms in $\mathcal{A}$ that form diagrams as above right, where the square in $\mathcal{E}$ commutes.

A **pseudo-commutative square** between such morphisms is a levelwise pseudo-commutative square, meaning a pseudo-commutative square at each degree and each image, which commutes with all the squares in the surrounding m- and e-morphisms.

Similarly, a **good square** of chain m-morphisms (resp. e-morphisms) is a levelwise good commuting square of chain m-morphisms (resp. e-morphisms).

Example 10.5. For any chain complex X, there are unique chain m- and e-morphisms from the constant complex at $\varnothing$:

$$\begin{array}{ccccc} \varnothing & \longleftarrow\!\circ & \varnothing & \longrightarrow & \varnothing \\ \downarrow & \circlearrowleft & \downarrow & & \downarrow \\ X_i & \longleftarrow\!\circ & \overline{X}_i & \longrightarrow & X_{i-1} \end{array} \qquad \begin{array}{ccccc} \varnothing & \longleftarrow\!\circ & \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow & \circlearrowleft & \downarrow \\ X_i & \longleftarrow\!\circ & \overline{X}_i & \longrightarrow & X_{i-1} \end{array}$$

Before discussing chain complexes in more detail, we prove two basic results that will be useful for checking the chain condition in different situations.

Lemma 10.6. *Given a cospan $B \xrightarrow{f} A \xleftarrow{g} C$, its mixed pullback has $\varnothing$ in the remaining corner if and only if f factors through the kernel of g (up to isomorphism).*

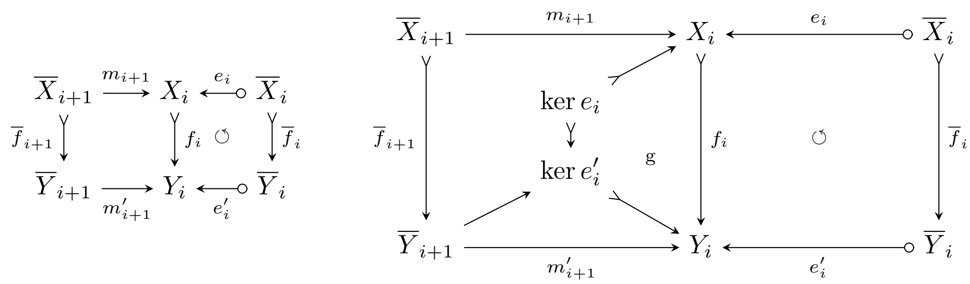
Proof. We find the mixed pullback by taking the pullback of f and $k(g)$ in $\overline{\mathcal{M}}$, and applying k^{-1}.

$$\begin{array}{ccc} B/P & \longrightarrow & C \\ \downarrow & \circlearrowleft & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \\ \uparrow & \lrcorner & \uparrow{\scriptstyle k(g)} \\ P & \longrightarrow & A\backslash C \end{array}$$

Then, $D = \varnothing$ if and only if the map $P \rightarrowtail B$ is an isomorphism by Lemma 2.14, which happens if and only if f is equal (up to isomorphism) to the composite $P \longrightarrow A\backslash C \overset{k(g)}{\rightarrowtail} A$. $\square$

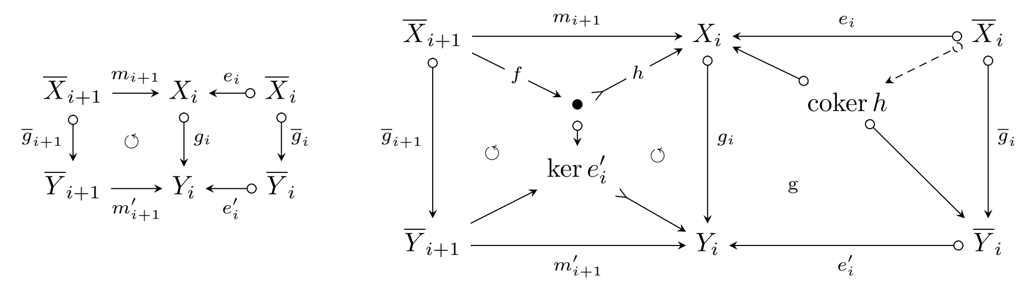
Proposition 10.7. *Let Y be a chain complex, and X be a diagram containing the data of a chain complex, possibly without the chain condition. If we have either the data of a chain m-morphism $X \rightarrowtail Y$ or that of a chain e-morphism $X \circ\!\!\longrightarrow Y$, then X must satisfy the chain condition.*

Proof. Assume we have the data of a chain m-morphism $f\colon X \longrightarrow Y$ as pictured below left.



Applying k to the given pseudo-commutative square, we get an induced good square on kernels as pictured above right. Since Y is a chain complex, it satisfies the chain condition, and so Lemma 10.6 ensures that m'_{i+1} factors through $\ker e'_i$. Finally, since good squares are pullbacks, we get an induced map $m\colon \overline{X}_{i+1} \longrightarrow \ker e_i$ such that $m_{i+1} = k(e_i)m$, and using Lemma 10.6 again we conclude that X satisfies the chain condition.

Now assume instead that we have the data of a chain e-morphism $g\colon X \circ\!\!\longrightarrow Y$ as below left.



We describe the steps that need to be taken to construct the diagram above right. First, take the mixed pullback of g_i and $k(e'_i)$ to produce $\bullet$ and the pseudo-commutative square on the right. Taking the mixed pullback of the new map $\bullet \circ\!\!\longrightarrow \ker e'_i$ and of the map $\overline{Y}_{i+1} \longrightarrow \ker e'_i$ from Lemma 10.6, we produce the left pseudo-commutative square whose new e-morphism must agree with $\overline{X}_{i+1} \circ\!\!\longrightarrow \overline{Y}_{i+1}$ by the uniqueness of pseudo-commutative squares of Lemma 10.2, since the original square involving the vertices $\overline{X}_{i+1}$, X_i, Y_i, $\overline{Y}_{i+1}$ is pseudo-commutative.

Now apply c to the first pseudo-commutative square we constructed, to produce the good square on its right. Since good squares are pullbacks, we get an induced map $\overline{X}_i \circ\!\!\longrightarrow \operatorname{coker} h$. This in turn induces a map $m\colon \bullet \rightarrowtail \ker e_i$ such that $h = k(e_i)m$ by Lemma 2.12, which concludes the proof, as now $m_{i+1} = hf = k(e_i)mf$ and we can apply Lemma 10.6. □

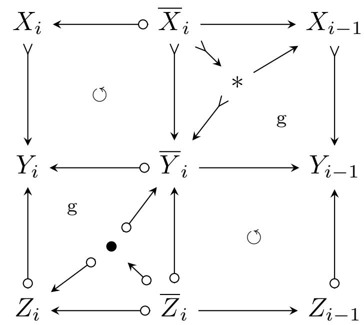
The chain m- and e-morphisms between chain complexes, together with the pseudo-commutative and good squares of Definition 10.4, form a double category $\mathsf{Ch}(\mathsf{FinSet}) = (\mathcal{M}_{\mathsf{Ch}}, \mathcal{E}_{\mathsf{Ch}})$, which we now endow with the structure of a pre-FCGW category. First, we deal with isomorphisms.

Lemma 10.8. *A chain m-morphism (resp. e-morphism) is an isomorphism in $\mathcal{M}_{\mathsf{Ch}}$ (resp. $\mathcal{E}_{\mathsf{Ch}}$) if and only if it is a degreewise isomorphism.*

Proof. An isomorphism in $\mathcal{M}_{\mathsf{Ch}}$ necessarily consists of isomorphisms on each degree and each image, as the identity in $\mathcal{M}_{\mathsf{Ch}}$ is given by levelwise identity m-morphisms. For the converse, we note that if $f_i\colon X_i \rightarrowtail Y_i$ is an isomorphism, then $\overline{f}_i\colon \overline{X}_i \longrightarrow \overline{Y}_i$ is an isomorphism by Lemma 2.13, and the chain m-morphism has an inverse by the condition on pseudo-commutative squares in Definition 1.7. □

We now construct the cokernel (resp. kernel) of a chain m-morphism (resp. e-morphism).

Proposition 10.9. *Given a chain m-morphism $X \rightarrowtail Y$ (or e-morphism $Z \circ\!\!\rightarrow Y$) as in the diagram below, we can construct the pictured e-morphism (resp. m-morphism) and its domain complex as the cokernel (resp. kernel) in each degree.*



Proof. Given such an m-morphism of chain complexes, we can apply c to the top left pseudo-commutative square to get the good square as pictured, with vertices $\bullet$, Z_i, $\overline{Y}_i$, Y_i. We then take the pullback of the top right commuting square, and apply k^{-1} to get the pseudo-commutative square below right (thus defining $\overline{Z}_i \coloneqq \operatorname{coker}(* \rightarrowtail \overline{Y}_i)$). The map from $\overline{X}_i$ to the pullback is necessarily in $\mathcal{M}$ as it composes to a monomorphism, and by Lemma 2.12 it induces a map from $\overline{Z}_i$ to the bottom right pullback, completing the construction of the differential in Z. That Z satisfies the chain condition follows immediately from Proposition 10.7.

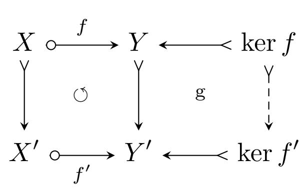
The converse construction is entirely dual. □

These constructions define the kernel and cokernel of chain e- and m-morphisms (respectively), which are inverse to one another by construction (up to isomorphism). We further show that these correspondences extend to functors k and c by defining their action on maps.

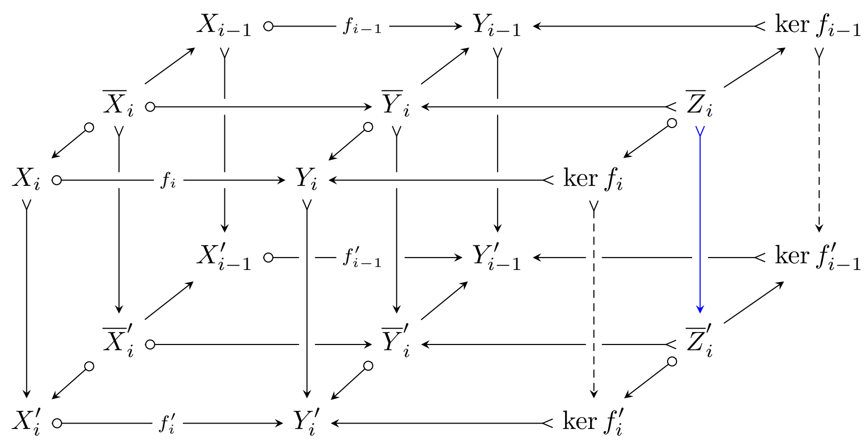
Lemma 10.10. *The kernel and cokernel constructions defined above above extend to equivalences of categories*

$$k\colon \operatorname{Ar}_{\circlearrowleft} \mathcal{E}_{\mathsf{Ch}} \longrightarrow \operatorname{Ar}_{\mathrm{g}} \mathcal{M}_{\mathsf{Ch}} \qquad\qquad c\colon \operatorname{Ar}_{\circlearrowleft} \mathcal{M}_{\mathsf{Ch}} \longrightarrow \operatorname{Ar}_{\mathrm{g}} \mathcal{E}_{\mathsf{Ch}}$$

Proof. Start with a morphism in $\operatorname{Ar}_{\circlearrowleft} \mathcal{E}_{\mathsf{Ch}}$, that is, a pseudo-commutative square between two chain e-morphisms; we show that there exists an induced chain m-morphism between their kernels which forms a good square.



To do this, consider the more detailed picture below, where the chain complexes $\ker f$ and $\ker f'$ are constructed as in Proposition 10.9.



Since pseudo-commutative squares in chain complexes are levelwise pseudo-commutative, and cokernels are constructed degreewise, we immediately get the dashed morphisms in the picture above, which form good squares in FinSet.

To construct the blue map, take the mixed pullback of the cospan

$$\ker f_i \rightarrowtail \dashrightarrow \ker f_i' \longleftarrow\!\circ\ \overline{Z}_i$$

By the uniqueness of mixed pullbacks of Lemma 10.2, the composite of this new pseudo-commutative square with the pseudo-commutative square of vertices $\overline{Z}_i'$, $\ker f_i'$, $\overline{Y}_i'$, Y_i' must agree with the composite of the pseudo-commutative squares $\overline{Z}_i$, $\ker f_i$, $\overline{Y}_i$, Y_i and $\overline{Y}_i$, Y_i, $\overline{Y}_i'$, Y_i', since they are both mixed pullbacks of the cospan

$$\ker f_i \rightarrowtail Y_i' \longleftarrow\!\circ\ \overline{Y}_i'$$

Thus, the mixed pullback we constructed must have $\overline{Z}_i$ as the new vertex, and a map $\overline{Z}_i \circ\!\!\longrightarrow \overline{Z}_i'$ which is the blue map we desired.

To show that the morphisms we constructed form a chain m-morphism and that the square of m-morphisms is good, we must check that the involved squares are of the correct type. By construction, the square involving $\overline{Z}_i$, $\overline{Z}_i'$, $\ker f_i$, $\ker f_i'$ is pseudo-commutative, and the square $\overline{Z}_i$, $\overline{Z}_i'$, $\overline{Y}_i'$, Y_i' commutes; in a moment we will show that it is actually good. Note that the square $\overline{Z}_i$, $\overline{Z}_i'$, $\ker f_{i-1}$, $\ker f_{i-1}'$ now commutes, as it does when post-composed with the monic $\operatorname{coker} f_{i-1}' \rightarrowtail Y_{i-1}'$. Finally, the mentioned square is good by appealing to the pullback lemma, since good squares in FinSet are pullbacks.

This proves we have a functor $k\colon \mathrm{Ar}_{\circlearrowleft}\, \mathcal{E}_{\mathsf{Ch}} \longrightarrow \mathrm{Ar}_{\mathrm{g}}\, \mathcal{M}_{\mathsf{Ch}}$, as the construction above is evidently functorial. Furthermore, this functor is faithful since the maps constructed are unique. To see that it is full, and thus conclude that k is an equivalence as desired, it suffices to start with a good square of chain m-morphisms and prove that we get an induced pseudo-commutative square after taking c on objects; this proof is entirely dual to the one above, as is the statement about the functor c. □

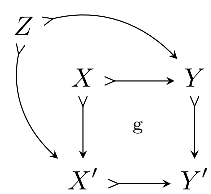
The above construction also reveals the following.

Lemma 10.11. *A pseudo-commutative square of chain complexes induces an isomorphism on kernels (and cokernels) if and only if it is degreewise distinguished.*

Proof. Since chain isomorphisms are characterized by being degreewise isomorphisms by Lemma 10.8, we obtain the desired correspondence by recalling that pseudo-commutative squares of chain complexes are in particular degreewise pseudo-commutative in FinSet, and that the induced morphisms on degrees on (co)kernels are the induced maps from these pseudo-commutative squares in FinSet, as we can see in the proof of Lemma 10.10. □

Lemma 10.12. *Good squares in $\mathcal{M}_{\mathsf{Ch}}$ (resp. $\mathcal{E}_{\mathsf{Ch}}$) are pullbacks.*

Proof. We prove the statement for $\mathcal{M}_{\mathsf{Ch}}$; the one for $\mathcal{E}_{\mathsf{Ch}}$ is identical. Suppose that we have a good square in $\mathcal{M}_{\mathsf{Ch}}$, and another commutative square as depicted below; we wish to show there exists a unique chain m-morphism $Z \rightarrowtail X$ making the diagram commute.



Recall that good squares of chain complexes are levelwise good by definition, and so we get induced maps $Z_i \rightarrowtail X_i$ and $\overline{Z}_i \rightarrowtail \overline{X}_i$ for every i. It remains to show that these form a chain m-morphism, but this is immediate: the required squares will be pseudo-commutative by appealing to axiom (PBL) in FinSet, and the remaining squares in $\overline{\mathcal{M}}$ commute, since they do when post-composed with the monics $X_i \rightarrowtail X'_i$. □

Theorem 10.13. Ch(FinSet) *is a pre-FCGW category.*

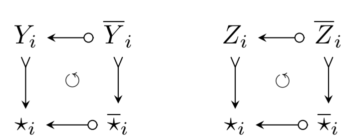
Proof. Ch(FinSet) has shared isomorphisms by Lemma 10.8. Good squares are pullbacks by Lemma 10.12, and they include weak triangles by definition as these are levelwise pullbacks. In addition, the functors $k\colon \mathrm{Ar}_{\circlearrowleft} \mathcal{E}_{\mathsf{Ch}} \longrightarrow \mathrm{Ar}_{\mathrm{g}} \mathcal{M}_{\mathsf{Ch}}$ and $c\colon \mathrm{Ar}_{\circlearrowleft} \mathcal{M}_{\mathsf{Ch}} \longrightarrow \mathrm{Ar}_{\mathrm{g}} \mathcal{E}_{\mathsf{Ch}}$ are equivalences by Lemma 10.10.

For the axioms, note that $\mathcal{M}_{\mathsf{Ch}}$ and $\mathcal{E}_{\mathsf{Ch}}$ have a shared initial object $\varnothing$ by Example 10.5, and all morphisms monic by the same property of the levelwise morphisms. Axiom (D) follows from Lemma 10.11, and axiom (K) follows from the same property in each degree. □

In order to upgrade this pre-FCGW structure to a full FCGW structure, we need to construct $\star$-pushouts of chain complexes. We do so in the next two results.

Proposition 10.14. *Any span of chain m-morphisms admits a $\star$-pushout, which is a levelwise $\star$-pushout. Furthermore, it has a universal property with respect to good squares in $\mathcal{M}_{\mathsf{Ch}}$.*

Proof. Let $Y \leftarrowtail X \rightarrowtail Z$ be a span of chain m-morphisms, and consider the levelwise $\star$-pushouts in FinSet, which we denote by $\star_i$, $\overline{\star}_i$. For each i, there exists a map $\overline{\star}_i \circ\!\!\longrightarrow \star_i$ such that the squares below are pseudo-commutative, by Proposition A.4.



Also, since $\star$-pushouts in FinSet are (categorical) pushouts, there exists a map $\overline{\star}_i \longrightarrow \star_{i-1}$ such that the squares below commute, by the universal property of $\overline{\star}_i$.

$$\begin{array}{ccc} \overline{Y}_i & \longrightarrow & Y_{i-1} \\ \downarrow & & \downarrow \\ \overline{\star}_i & \longrightarrow & \star_{i-1} \end{array} \qquad \begin{array}{ccc} \overline{Z}_i & \longrightarrow & Z_{i-1} \\ \downarrow & & \downarrow \\ \overline{\star}_i & \longrightarrow & \star_{i-1} \end{array}$$

One can check that the data above determines a chain complex $\star$, together with chain m-morphisms $Y \rightarrowtail \star$ and $Z \rightarrowtail \star$ that complete the span to a good square, since it is levelwise good.

For the universal property, consider a good square as below.

$$\begin{array}{ccc} X & \rightarrowtail & Y \\ \downarrow & \text{g} & \downarrow \\ Z & \rightarrowtail & W \end{array}$$

By construction, we get induced levelwise maps $\star_i \rightarrowtail W_i$ and $\overline{\star}_i \rightarrowtail \overline{W}_i$ that make the relevant levelwise diagrams commute. It remains to show that they assemble into a chain m-morphism; that is, that in the diagram

$$\begin{array}{ccccc} \star_i & \longleftarrow\!\circ & \overline{\star}_i & \longrightarrow & \star_{i-1} \\ \downarrow & & \downarrow & & \downarrow \\ W_i & \longleftarrow\!\circ & \overline{W}_i & \longrightarrow & W_{i-1} \end{array}$$

the square on the left is pseudo-commutative, and the one on the right commutes in $\overline{\mathcal{M}}$. But the first assertion is the content of Corollary A.9, and the second is a consequence of the uniqueness in the universal property of the pushout for $\overline{\star}_i$. Clearly the map $\star \rightarrowtail W$ is unique (up to unique isomorphism), since it is constructed using the levelwise universal properties in FinSet. □

Proposition 10.15. *Any span of chain e-morphisms which is part of a good square in $\mathcal{E}_{\mathsf{Ch}}$ admits a $\star$-pushout, which is a levelwise $\star$-pushout. Furthermore, it has a universal property with respect to good squares in $\mathcal{E}_{\mathsf{Ch}}$.*

Proof. Consider the following good square in $\mathcal{E}_{\mathsf{Ch}}$

$$\begin{array}{ccc} X & \circ\!\!\longrightarrow & Y \\ \downarrow & \text{g} & \downarrow \\ Z & \circ\!\!\longrightarrow & W \end{array}$$

If we take degreewise $\star$-pushouts, we get induced maps $\star_i \circ\!\!\longrightarrow W_i$. For the images, let P_i denote the mixed pullback of the cospan

$$\overline{W}_i \longrightarrow W_{i-1} \longleftarrow\!\circ \star_{i-1}.$$

One can in fact check that $P_i = \overline{Y}_i \star_{\overline{X}_i} \overline{Z}_i$, as (in sets) this reduces to the fact that taking preimages preserves unions. Then, P_i is a pushout, and we get an induced map $P_i \longrightarrow \star_i$.

It is immediate, either by construction or by applying axiom (PBL) for FinSet, that we get the data of chain e-morphisms $Y \circ\!\!\!\longrightarrow \star$, $Z \circ\!\!\!\longrightarrow \star$ and $\star \circ\!\!\!\longrightarrow W$; in addition, the latter is unique by construction. Finally, we see that $\star$ is indeed a chain complex by Proposition 10.7. □

Theorem 10.16. $\mathsf{Ch}(\mathsf{FinSet})$ *is an FCGW category.*

Proof. By Theorem 10.13, we know that $\mathsf{Ch}(\mathsf{FinSet})$ forms a pre-FCGW category. We now check the axioms of Definition 3.1. Axiom (PO) holds by Proposition 10.14. Similarly, axiom ($\star$) holds by Propositions 10.14 and 10.15, where the isomorphism on (co)kernels is a consequence of the fact that $\star$-pushouts of chain complexes are degreewise $\star$-pushouts in FinSet, together with Lemma 10.8. Finally, axioms (GS), (PBL) and (POL) follow immediately from the same properties for FinSet, as all structures involved are defined or constructed levelwise. □

Remark 10.17. Although all of the constructions in this section have been for chain complexes indexed in the integers, it is easy to see that every result holds if one restricts to **bounded** chain complexes; that is, chain complexes of sets with a finite number of non-empty degrees and images; we denote this FCGW category by $\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}$. Similarly, we denote by $\mathsf{Ch}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}$ the FCGW category of chain complexes X such that $X_i = \varnothing$ for $i \notin [a, b]$, for any $a \leq b$.

11. Exact complexes

Classically, the class of weak equivalences between chain complexes we consider are the quasi-isomorphisms. Using homological algebra methods, one can characterize the monomorphisms (resp. epimorphisms) that are quasi-isomorphisms as the ones whose cokernel (resp. kernel) are exact complexes. We now define exact chain complexes of finite sets in analogy with the classical algebraic case, and show that they form a class of acyclic objects in $\mathsf{Ch}(\mathsf{FinSet})$, thus providing us with a notion of quasi-isomorphism in this setting.

Definition 11.1. A chain complex of finite sets is **exact** if it is of the form

$$X_{i+1} \longleftarrow\!\!\circ\; \overline{X}_{i+1} \rightarrowtail X_i \longleftarrow\!\!\circ\; \overline{X}_i \rightarrowtail X_{i-1}$$

in the sense that all of the maps in $\overline{\mathcal{M}}$ are m-morphisms, and additionally each mixed cospan $\overline{X}_{i+1} \circ\!\!\!\longrightarrow X_i \longleftarrow\!\!< \overline{X}_i$ is a kernel-cokernel pair. In other words, for all i the pseudo-commutative square expressing the chain condition has the form:

$$\begin{array}{ccc} \varnothing & \rightarrowtail & \overline{X}_i \\ \circ\!\!\downarrow & \square & \circ\!\!\downarrow \\ \overline{X}_{i+1} & \rightarrowtail & X_i \end{array}$$

We write $\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})$ for the full double subcategory of exact complexes in $\mathsf{Ch}(\mathsf{FinSet})$.

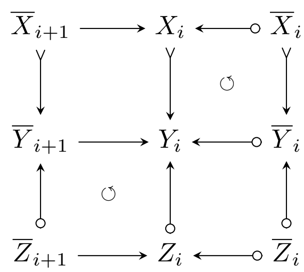
Remark 11.2. Just like with general chain complexes of sets, there is a direct comparison between this definition and the one for exact complexes in abelian categories. In fact, in this case the comparison is completely direct: this definition could have been formulated for an abelian category $\mathcal{A}$ instead of FinSet, and it would recover the classical notion.

An exact complex then amounts to a partition $X_i \cong \overline{X}_{i+1} \sqcup \overline{X}_i$ for all i, by both restricting the maps $\overline{X}_{i+1} \longrightarrow X_i$ to be inclusions and insisting by the exactness condition that the homology set H_i mentioned previously is empty.

As expected, exact complexes form a class of acyclic objects in our chain complexes of sets.

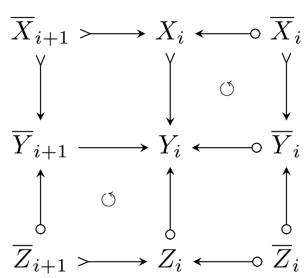
Proposition 11.3. $(\mathsf{Ch}(\mathsf{FinSet}), \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet}))$ *forms an FCGWA category.*

Proof. As the constant complex at $\varnothing$ is always exact, it remains only to show that exact complexes are closed under kernels, cokernels, and extensions. To see that they are closed under kernels and cokernels, consider the following kernel-cokernel pair in $\mathsf{Ch}(\mathsf{FinSet})$:



If X and Y are exact, then the top left square is necessarily good, as it is the kernel of the top right square. By the construction of the cokernel in Proposition 10.9, this implies the leftmost column above is a kernel-cokernel sequence (which in particular means that the map $\overline{Z}_{i+1} \longrightarrow Z_i$ is in $\mathcal{M}$), and by the same argument with indices shifted, so is the rightmost column. This shows that the bottom left and right squares form a kernel-cokernel pair, so Z is exact. The dual argument shows that kernels also preserve exact complexes, so it only remains to show that they are closed under extensions.

Consider an extension of exact complexes in $\mathsf{Ch}(\mathsf{FinSet})$ as follows.



It follows from the definition of kernel and cokernel of chain morphisms that $Y_i \cong X_i \sqcup Z_i$ and $\overline{Y}_i \cong \overline{X}_i \sqcup \overline{Z}_i \sqcup V_i$ for all i and some sets V_i, where the components of the differential of Y agree with those of X and Z on $\overline{X}_i$ and $\overline{Z}_i$ and the inclusions from X and Z are the canonical coproduct inclusions at each level. As the top right square is a pullback, the e-morphism in the differential of Y must map V_i entirely into Z_i. But as $Z_i \cong \overline{Z}_{i+1} \sqcup \overline{Z}_i$ by exactness of Z, and Y satisfies the chain condition, V_i must map entirely into $\overline{Z}_i$ as does $\overline{Z}_i$ itself. As e-morphisms are monic, V_i must then be empty, so Y is isomorphic to $X \sqcup Z$ and therefore exact as X and Z are. □

Remark 11.4. From the first part of the proof of Proposition 11.3, we can also observe that in the special case of exact chain complexes, the (co)kernel construction of Proposition 10.9 is done by taking (co)kernels levelwise, not just degreewise.

Exact chain complexes determine classes of m- and e-equivalences which, mirroring the classical algebraic setting, we call **quasi-isomorphisms**. In particular, these are chain maps

$$\begin{array}{ccccc} \overline{X}_{i+1} & \longrightarrow & X_i & \longleftarrow\!\circ & \overline{X}_i \\ \downarrow & & \downarrow & \circlearrowleft & \downarrow \\ \overline{Y}_{i+1} & \longrightarrow & Y_i & \longleftarrow\!\circ & \overline{Y}_i \end{array} \qquad \begin{array}{ccccc} \overline{X}_{i+1} & \longrightarrow & X_i & \longleftarrow\!\circ & \overline{X}_i \\ \downarrow & \circlearrowleft & \downarrow & & \downarrow \\ \overline{Y}_{i+1} & \longrightarrow & Y_i & \longleftarrow\!\circ & \overline{Y}_i \end{array}$$

such that both squares (for both m- and e-maps) are pushouts.

If both squares are pullbacks, it is straightforward to check that the homology set H_i of X includes into that of Y. A chain map is then a quasi-isomorphism if and only if both squares are pullbacks, the induced inclusions of homology sets are isomorphisms, and each element of Y_i with more than one element in its preimage in $\overline{Y}_{i+1}$ is in the image of X_i. This latter condition, which says that the "non-injective part" of $\overline{Y}_{i+1} \longrightarrow Y_i$ is covered by $\overline{X}_{i+1} \longrightarrow X_i$, is a consequence of the condition that the maps in the differentials of an exact complex are monic.

12. Gillet–Waldhausen Theorem

The aim of this final section is to prove a version of the Gillet–Waldhausen Theorem; this will show that our new notion of chain complexes of finite sets with quasi-isomorphisms provide an alternate model for the K-theory of finite sets.

Our proof of the Gillet-Waldhausen Theorem follows the same outline as the classical proof in [TT90, Theorem 1.11.7]; nevertheless, we include it here, adapted to our setting. We first show two lemmas that will be crucial for the proof of the theorem. In both lemmas, whenever we allude to the K-theory of a category of chain complexes, we do so by considering chain complexes as an FCGW category (with isomorphisms).

Lemma 12.1. *The FCGW functor*

$$\mathsf{Ch}(\mathsf{FinSet})_{[a,b]} \longrightarrow \prod^{b-a+1} \mathsf{FinSet}$$

sending a chain complex X to the tuple $(X_{b-1}, X_{b-2}, \dots, X_a, X_b)$ induces a homotopy equivalence on K-theory.

Proof. First of all, note that this correspondence (the projection of a chain complex to its degrees) is indeed an FCGW functor, as all the structure on chain complexes is defined degreewise.

The proof then proceeds by induction on $b-a$. If $b = a$, the assertion is trivial since the two FCGW categories in question are the same. For the inductive step, it suffices to show that the FCGW functor

$$\mathsf{Ch}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{Ch}(\mathsf{FinSet})_{[a,b-1]} \times \mathsf{FinSet}$$

sending a chain complex X to the tuple

$$(X_{b-1} \longleftarrow\!\circ\ \overline{X}_{b-1} \longrightarrow X_{b-2} \longleftarrow\!\circ \dots \longrightarrow X_a, X_b)$$

induces a homotopy equivalence on K-theory. By the Additivity Theorem 6.5, we have a homotopy equivalence

$$K(E(\mathsf{Ch}(\mathsf{FinSet})_{[a,b-1]}, \mathsf{Ch}(\mathsf{FinSet})_{[a,b]}, \mathsf{FinSet})) \simeq K(\mathsf{Ch}(\mathsf{FinSet})_{[a,b-1]}) \times K(\mathsf{FinSet}).$$

On the other hand, we can consider the FCGW functors

$$F\colon \mathsf{Ch}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{Ch}(\mathsf{FinSet})_{[a,b-1]}, \qquad G\colon \mathsf{Ch}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{FinSet}$$

that truncate a chain complex, where F removes X_b and G removes everything except for X_b. Clearly these satisfy the hypotheses of Corollary 6.4, as every extension in $\mathsf{Ch}(\mathsf{FinSet})_{[a,b]}$ is, up to isomorphism, of the form

$$\begin{array}{ccccccccccccccccc}
FX & & \varnothing & \leftarrow\!\circ & \varnothing & \rightarrow & X_{b-1} & \leftarrow\!\circ & \overline{X}_{b-1} & \rightarrow & X_{b-2} & \leftarrow\!\circ \cdots \rightarrow & X_{a+1} & \leftarrow\!\circ & \overline{X}_{a+1} & \rightarrow & X_a \\
\downarrow & & \downarrow & \circlearrowleft & \downarrow & & \| & \circlearrowleft & \| & & \| & & \| & \circlearrowleft & \| & & \| \\
X & & X_b & \leftarrow\!\circ & \overline{X}_b & \rightarrow & X_{b-1} & \leftarrow\!\circ & \overline{X}_{b-1} & \rightarrow & X_{b-2} & \leftarrow\!\circ \cdots \rightarrow & X_{a+1} & \leftarrow\!\circ & \overline{X}_{a+1} & \rightarrow & X_a \\
\uparrow & & \| & & \uparrow & \circlearrowleft & \uparrow & & \uparrow & \circlearrowleft & \uparrow & & \uparrow & & \uparrow & \circlearrowleft & \uparrow \\
GX & & X_b & \leftarrow\!\circ & \varnothing & \rightarrow & \varnothing & \leftarrow\!\circ & \varnothing & \rightarrow & \varnothing & \leftarrow\!\circ \cdots \rightarrow & \varnothing & \leftarrow\!\circ & \varnothing & \rightarrow & \varnothing
\end{array}$$

and so we get a homotopy equivalence

$$K(\mathsf{Ch}(\mathsf{FinSet})_{[a,b]}) \simeq K(E(\mathsf{Ch}(\mathsf{FinSet})_{[a,b-1]}, \mathsf{Ch}(\mathsf{FinSet})_{[a,b]}, \mathsf{FinSet})),$$

which proves the claim. □

Lemma 12.2. *The FCGW functor*

$$\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a,b]} \longrightarrow \prod^{b-a} \mathsf{FinSet}$$

sending an exact chain complex X to the tuple $(\overline{X}_b, \overline{X}_{b-1}, \dots, \overline{X}_{a+1})$ induces a homotopy equivalence on K-theory.

Proof. First of all, note that this correspondence (the projection of an exact chain complex to its images) is an FCGW functor, since all the structure on exact chain complexes is defined levelwise, as noted in Remark 11.4.

The proof then proceeds by induction on $b-a$. If $b = a$, the result follows trivially as an exact complex concentrated in a single degree is trivial. For the inductive step, it suffices to show that the FCGW functor

$$\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a+1,b]} \times \mathsf{FinSet}$$

sending an exact chain complex X to the tuple

$$(X_b \xleftarrow{\cong}\!\circ\, \overline{X}_b \rightarrowtail X_{b-1} \longleftarrow\!\circ \dots \rightarrowtail X_{a+2} \longleftarrow\!\circ\, \overline{X}_{a+2} \xrightarrowtail{\mathrm{id}} \overline{X}_{a+2}, \overline{X}_{a+1})$$

induces a homotopy equivalence on K-theory. Consider the FCGW functors

$$F\colon \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a+1,b]}, \qquad G\colon \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a,b]} \longrightarrow \mathsf{FinSet}$$

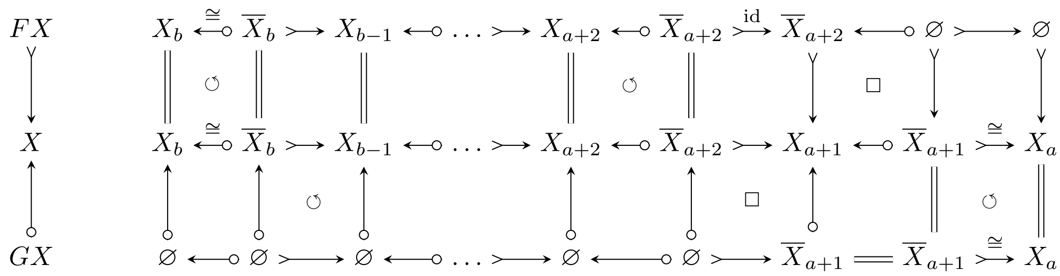
that respectively send an exact chain complex to

$$X_b \xleftarrow{\cong}\!\circ\, \overline{X}_b \rightarrowtail X_{b-1} \longleftarrow\!\circ \dots \rightarrowtail X_{a+2} \longleftarrow\!\circ\, \overline{X}_{a+2} \xrightarrowtail{\mathrm{id}} \overline{X}_{a+2}$$

and

$$\overline{X}_{a+1} = \overline{X}_{a+1} \xrightarrowtail{\cong} X_a$$

Clearly these satisfy the hypotheses of Corollary 6.4, as every extension in $\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[a,b]}$ is, up to isomorphism, of the form



and so we get a homotopy equivalence

$$K(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}) \simeq K(E(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a+1},\mathsf{b}]}, \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}, \mathsf{FinSet})),$$

which proves the claim by the Additivity Theorem 6.5. □

We now use the lemmas above to prove the main result of this section: the Gillet–Waldhausen Theorem.

Theorem 12.3 (Gillet–Waldhausen)**.** *There exists a homotopy equivalence*

$$K(\mathsf{FinSet}) \simeq K(\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}, \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})^{\mathsf{b}})$$

between the K-theory of finite sets with isomorphisms, and the K-theory of the FCGWA category of bounded chain complexes with quasi-isomorphisms.

Proof. Our goal is to show that for all $a \leq b$,

$$K(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}) \longrightarrow K(\mathsf{Ch}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}) \longrightarrow K(\mathsf{FinSet})$$

is a homotopy fiber sequence, and then take colimits on all intervals of the form $[a-i, a+i]$ to obtain the fiber sequence

$$K(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})) \longrightarrow K(\mathsf{Ch}(\mathsf{FinSet}) \longrightarrow K(\mathsf{FinSet}).$$

Recalling that the Localization Theorem 9.1 gives a homotopy fiber sequence

$$K(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})^{\mathsf{b}}) \longrightarrow K(\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}) \longrightarrow K(\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}, \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})^{\mathsf{b}}),$$

whose terms are spectra by Proposition 7.5, we must have a homotopy equivalence

$$K(\mathsf{FinSet}) \simeq K(\mathsf{Ch}(\mathsf{FinSet})^{\mathsf{b}}, \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})^{\mathsf{b}}),$$

as in the stable case the homotopy fiber sequences are also homotopy cofiber sequences in which the cofibers are uniquely determined up to homotopy.

By Lemmas 12.1 and 12.2, we have the following diagram whose vertical maps are homotopy equivalences

$$\begin{array}{ccc} K(\mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}) & \longrightarrow & K(\mathsf{Ch}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}) \\ \simeq \big\downarrow & & \big\downarrow \simeq \\ \prod^{b-a} K(\mathsf{FinSet}) & & \prod^{b-a+1} K(\mathsf{FinSet}) \end{array}$$

We now define the map

$$\prod^{b-a} \mathsf{FinSet} \longrightarrow \prod^{b-a+1} \mathsf{FinSet}$$

$$(A_b, A_{b-1}, \ldots, A_{a+1}) \mapsto (A_b, A_b \sqcup A_{b-1}, A_{b-1} \sqcup A_{b-2}, \ldots, A_{a+2} \sqcup A_{a+1}, A_{a+1})$$

which makes the following diagram commute (up to isomorphism).

$$\begin{array}{ccc} \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]} & \longrightarrow & \mathsf{Ch}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]} \\ \downarrow & & \downarrow \\ \prod^{b-a} \mathsf{FinSet} & \longrightarrow & \prod^{b-a+1} \mathsf{FinSet} \end{array}$$

Indeed, if we start with a chain complex $X \in \mathsf{Ch}^{\mathrm{E}}(\mathsf{FinSet})_{[\mathsf{a},\mathsf{b}]}$, the left vertical map sends X to the tuple $(\overline{X}_b, \overline{X}_{b-1}, \ldots, \overline{X}_{a+1})$, which is then sent to

$$(\overline{X}_b, \overline{X}_b \sqcup \overline{X}_{b-1}, \overline{X}_{b-1} \sqcup \overline{X}_{b-2}, \ldots, \overline{X}_{a+2} \sqcup \overline{X}_{a+1}, \overline{X}_{a+1})$$

in $\prod^{b-a+1} \mathsf{FinSet}$. On the other side, the composite of the top inclusion with the right vertical map sends X to $(X_b, X_{b-1}, X_{b-2}, \ldots, X_{a+1}, X_a)$. As X is an exact chain complex, these two are isomorphic.

Then, in order to obtain the desired homotopy fiber sequence, it suffices to show that

$$\prod^{b-a} K(\mathsf{FinSet}) \longrightarrow \prod^{b-a+1} K(\mathsf{FinSet}) \longrightarrow K(\mathsf{FinSet})$$

is a homotopy cofiber sequence. This is a standard result, but for convenience we provide a prove in the following lemma. □

Lemma 12.4. *The homotopy cofiber of the map*

$$\prod^{b-a} K(\mathsf{FinSet}) \longrightarrow \prod^{b-a+1} K(\mathsf{FinSet})$$

is equivalent to $K(\mathsf{FinSet})$*.*

Proof. Consider the diagonal map $\Delta\colon K(\mathsf{FinSet}) \longrightarrow K(\mathsf{FinSet}) \times K(\mathsf{FinSet})$. By Corollary 7.6, we can model the homotopy cofiber of Δ as the K-theory spectrum of the simplicial FCGW category $S_\bullet(\Delta)$, and by a classical argument this is homotopy equivalent to $K(\mathsf{FinSet})$ itself.

Letting $n = b - a$, Δ is the case $n = 1$ of the map in the statement of the lemma. We now proceed by induction to show that this map has homotopy cofiber equivalent to hocofib(Δ) for any n. Assume the map

$$F_{n-1}\colon \prod_{1}^{n-1} K(\mathsf{FinSet}) \longrightarrow \prod_{0}^{n-1} K(\mathsf{FinSet})$$

as defined in the proof of Theorem 12.3 has cofiber equivalent to hocofib(Δ).

Consider the commuting diagram

$$\begin{array}{ccccc}
\prod_{j=1}^{n-1} K(\mathsf{FinSet}) & \xrightarrow{F_{n-1}} & \prod_{j=0}^{n-1} K(\mathsf{FinSet}) & \longrightarrow & \mathrm{hocofib}(\Delta) \\
\downarrow{\scriptstyle q^0} & & \downarrow{\scriptstyle q^1} & & \downarrow{\scriptstyle q} \\
\prod_{j=1}^{n} K(\mathsf{FinSet}) & \xrightarrow{F_{n}} & \prod_{j=0}^{n} K(\mathsf{FinSet}) & \longrightarrow & \mathrm{hocofib}(F_n)
\end{array}$$

where q^0 and q^1 are the inclusions inserting $\varnothing$ in the last component, and q is the induced map on homotopy cofibers.

As F_n, F_{n-1} are inclusions on the K-groups (acting exactly as in their definition with $\sqcup$ replaced by $+$), the long exact sequences of homotopy groups associated to the fiber sequences in the rows above break into short exact sequences:

$$\begin{array}{ccccc}
\prod_{j=1}^{n-1} K_i(\mathsf{FinSet}) & \xhookrightarrow{F_{n-1}} & \prod_{j=0}^{n-1} K_i(\mathsf{FinSet}) & \twoheadrightarrow & K_i(\mathrm{hocofib}(\Delta)) \\
\big\downarrow{\scriptstyle q_i^0} & & \big\downarrow{\scriptstyle q_i^1} & & \downarrow{\scriptstyle q_i} \\
\prod_{j=1}^{n} K_i(\mathsf{FinSet}) & \xhookrightarrow{F_{n}} & \prod_{j=0}^{n} K_i(\mathsf{FinSet}) & \twoheadrightarrow & K_i(\mathrm{hocofib}(F_n))
\end{array}$$

By the snake lemma, we have an exact sequence

$$0 \longrightarrow \ker(q_i) \longrightarrow K_i(\mathsf{FinSet}) \longrightarrow K_i(\mathsf{FinSet}) \longrightarrow \mathrm{coker}(q_i) \longrightarrow 0.$$

But the map induced by F_n between the cokernels of q_i^0, q_i^1 is the identity on $K_i(\mathsf{FinSet})$, as F_n sends the last component of $\prod_{j=1}^{n} K_i(\mathsf{FinSet})$ to the last component of $\prod_{j=0}^{n} K_i(\mathsf{FinSet})$. q_i is then an isomorphism, having trivial kernel and cokernel, and so $\mathrm{hocofib}(F_n) \simeq \mathrm{hocofib}(\Delta)$. ☐

Appendix. Functoriality constructions

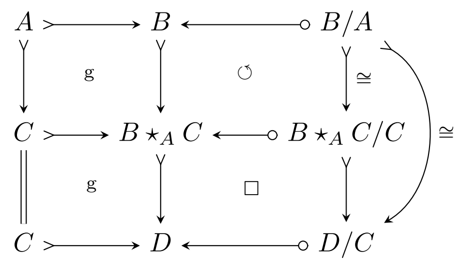
Appendix A. Properties of $\star$-pushouts

We establish some technical results concerning $\star$-pushouts. All of the results in this section assume an FCGW category.

Lemma A.1. *For any good square in $\mathcal{M}$ as below inducing an isomorphism on cokernels, the induced map $B \star_A C \rightarrowtail D$ is an isomorphism.*

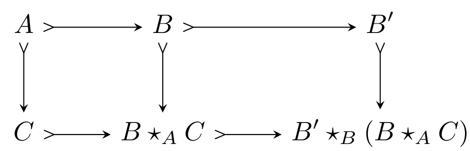
$$\begin{array}{ccccc}
A & \rightarrowtail & B & \longleftarrow\!\circ & B/A \\
\downarrow & \mathrm{g} & \downarrow & \circlearrowleft & \downarrow{\scriptstyle \cong} \\
C & \rightarrowtail & D & \longleftarrow\!\circ & D/C
\end{array}$$

Proof. By the definition of $\star$-pushouts, we have the following diagram



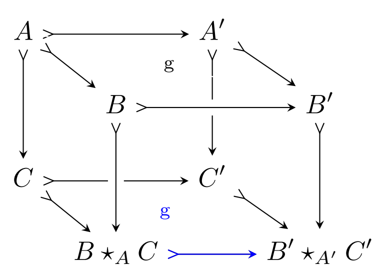
where the map $B \star_A C/C \rightarrowtail D/C$ is an isomorphism as the composite $B/A \cong B \star_A C/C \rightarrowtail D/C$ is an isomorphism. Then, since distinguished squares induce isomorphisms on cokernels, Lemma 2.14 implies that the map $B \star_A C \rightarrowtail D$ is an isomorphism. □

Corollary A.2. *Given a diagram $C \leftarrowtail A \rightarrowtail B \rightarrowtail B'$, we have $B' \star_B (B \star_A C) \cong B' \star_A C$. In other words, the composite of $\star$-pushouts below is the $\star$-pushout of the outer span.*



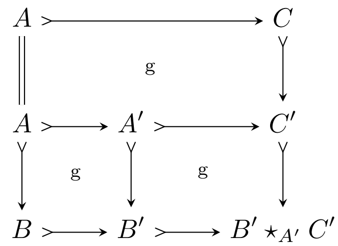
Proof. The induced map on cokernels of the vertical m-morphisms is a composite of isomorphisms, so by Lemma A.1 the composite is a $\star$-pushout. □

Proposition A.3. *Given a black commutative diagram as below, where the top face is a good square, there exists an induced blue m-morphism between $\star$-pushouts such that the two squares created commute, and the bottom one is a good square*



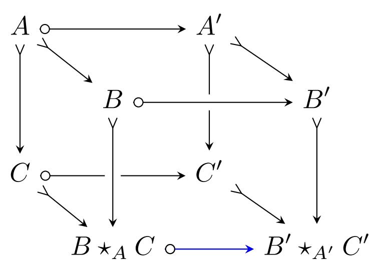
Moreover, this assignment is functorial, and if all the original faces are good squares then the two squares created are good, and this is a good cube. The analogous statement for e-morphisms also holds, if both $\star$-pushouts exist.

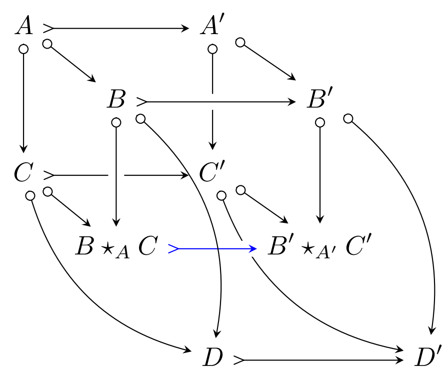
Proof. In order to obtain the desired blue m-morphism such that the two squares created commute, it suffices to note that the square



is good, and invoke the universal property of the $\star$-pushout $B \star_A C$. The bottom square is good by axiom (POL), and functoriality follows from uniqueness of the maps induced by the $\star$-pushout. Finally, if all faces are good, then this is a good cube, since the southern square is an identity square. □

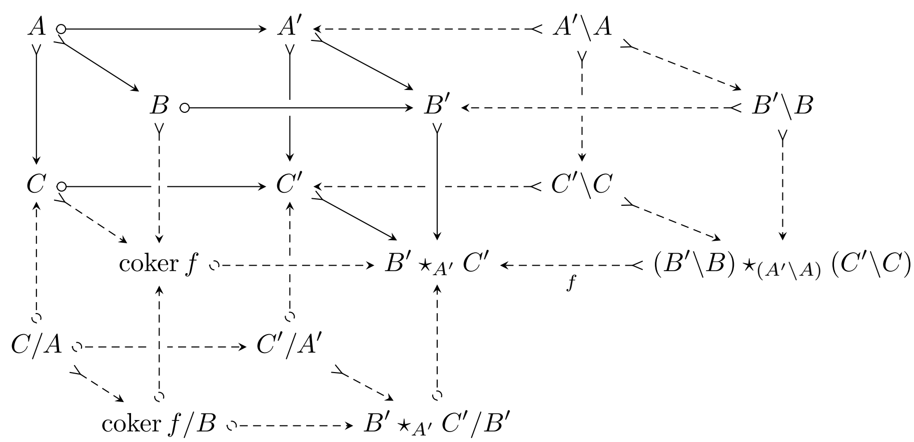
Proposition A.4. *Given a black diagram as below left, where all faces are either good or pseudo-commutative squares, there exists an induced blue e-morphism between $\star$-pushouts such that the two squares created are pseudo-commutative.*





Moreover, this assignment is functorial, and if one of the pseudo-commutative squares is distinguished, then so is the parallel new square. The analogous statement for e-morphisms also holds, if we start from a black diagram as above right.

Proof. The constructions necessary for the proof are represented in the diagram below, where the black arrows are given in the data, and the ones we construct are dashed. We proceed to explain the steps in order.



First, consider the kernels of the given horizontal e-morphisms, and construct the $\star$-pushout of the induced span between them. By Proposition A.3, there exists an m-morphism

$$(B'\backslash B) \star_{(A'\backslash A)} (C'\backslash C) \overset{f}{\rightarrowtail} B' \star_{A'} C'$$

such that all squares on the top right cube are good.

We can now consider coker f and form the cube on the top left, which uses all of the original data except for $B \star_A C$, placing coker f in its stead. Note that all the squares in this cube are either good or pseudo-commutative (by construction, together with axiom (PBL)).

Taking cokernels of the vertical m-morphisms yields the bottom left cube, where all squares are either good or pseudo-commutative (again by construction, together with axiom (PBL)). By definition of $B' \star_{A'} C'$, the map $C'/A' \rightarrowtail B' \star_{A'} C'/B'$ is an isomorphism. Then, by Lemma 2.13, the map $C/A \rightarrowtail \operatorname{coker} f/B$ is an isomorphism as well, and by Lemma A.1 we get that the induced m-morphism $B \star_A C \rightarrowtail \operatorname{coker} f$ must also be an isomorphism, which concludes the proof of the first statement.

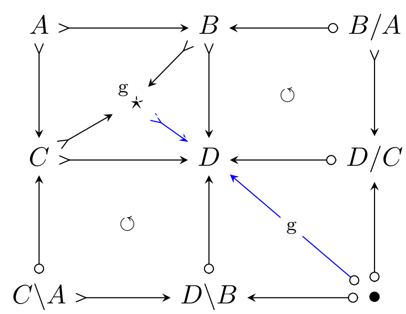
Now suppose the given top square is distinguished. This implies that the map $A'\backslash A \rightarrowtail B'\backslash B$ is an isomorphism; then, so is $C'\backslash C \rightarrowtail (B'\backslash B) \star_{(A'\backslash A)} (C'\backslash C)$, and thus the bottom square of the top left cube must be distinguished as well. □

Remark A.5. From the kernel-cokernel sequence

$$B \star_A C \cong \operatorname{coker} f \hookrightarrow B' \star_{A'} C' \overset{f}{\longleftarrow\!\!\prec} (B'\backslash B) \star_{(A'\backslash A)} (C'\backslash C)$$

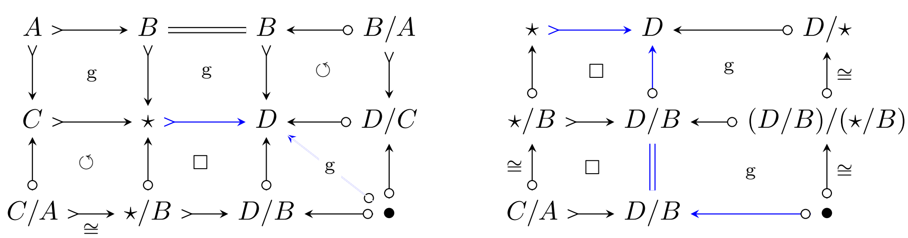
constructed in the proof above, we see that the kernel of the induced e-morphism is precisely the $\star$-pushout of the kernels of the three given e-morphisms in the data.

Lemma A.6. *Given a good square between objects A, B, C, D as in the diagram below, where $\star$ denotes $B \star_A C$, the maps in blue form a kernel-cokernel pair.*



Proof. First, note that both maps are unique, as the blue m-morphism is the unique map from the $\star$-pushout from axiom (PO), and the blue e-morphism is the composite of the good square formed by applying k^{-1} followed by c to the original good square, equivalently in either direction by Lemma 2.16.

Now, we can factor the left column of the diagram above as below left:

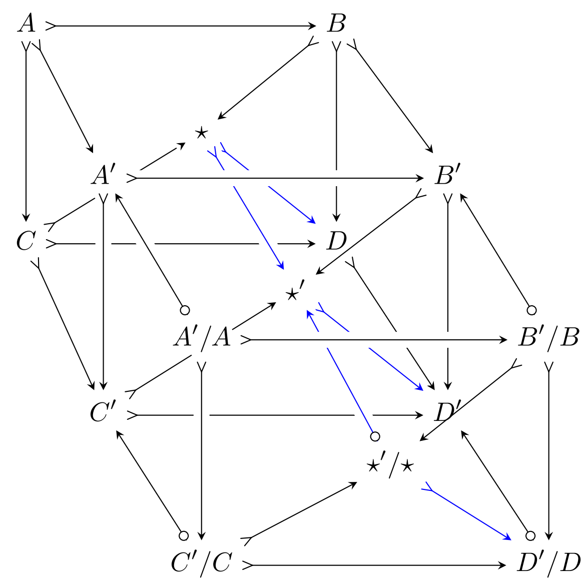


We then have the diagram of horizontal kernel-cokernel pairs above right, where the lower square is pseudo-commutative by Definition 1.7 and distinguished by Lemma 2.14. Therefore, $D/\star \cong \bullet$, so $\star \rightarrowtail D \leftarrow\!\!\circ\ \bullet$ is a kernel-cokernel sequence. □

Let us say a cube is an m-m-e cube if it has m-morphisms in two directions and e-morphisms in the remaining direction; similarly, we have e-e-m cubes, m-m-m cubes, etc.

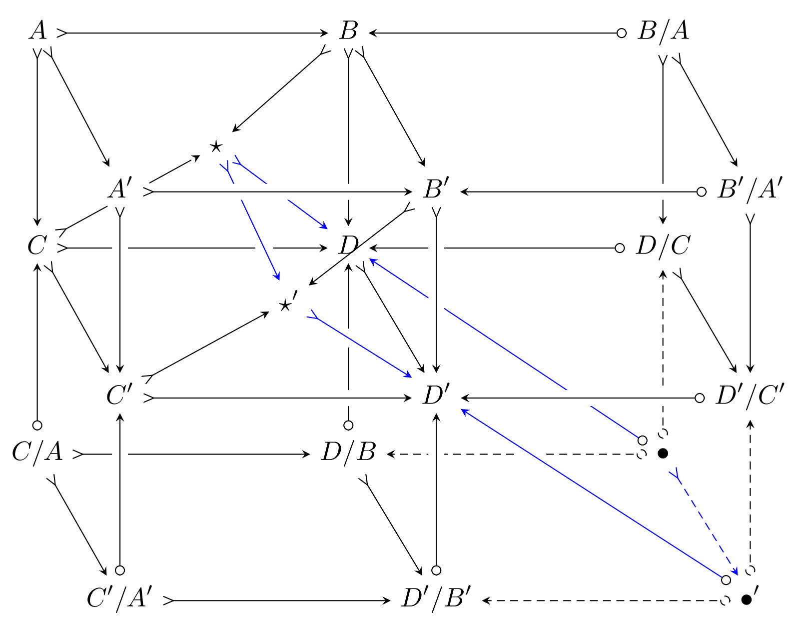
Proposition A.7. *Given a good m-m-m cube, taking cokernels of the m-morphisms and squares in any of the three directions produces an m-m-e cube whose faces are all good or pseudo-commutative squares. Conversely, given such an m-m-e cube, taking kernels produces a good m-m-m cube. The same is also true with the roles of m- and e-morphisms reversed.*

Proof. Consider a good m-m-m cube, whose faces and a choice of southern square are all good squares, and let $\star, \star'$ denote the $\star$-pushouts of the relevant spans. We first take cokernels in the direction of the southern square, as pictured below.



By Remark A.5, $\star'/\star$ is the $\star$-pushout of $B'/B \longleftarrow\!\prec A'/A \rightarrowtail C'/C$, so Remark 3.2 ensures that the square involving A'/A, B'/B, C'/C, D'/D is good. As all of the mixed squares in this m-m-e cube are pseudo-commutative by construction, we have showed that the cokernel cube in this direction is of the desired form.

We now take cokernels of the m-m-m cube in the remaining two directions, as depicted below. This diagram can be further completed by taking cokernels of the m-m-e cubes and producing the black dashed e-morphisms; note that both squares of e-morphisms created are good.



Now, these m-m-e cubes are such that their remaining face is a good square if and only if there exists an induced dashed blue m-morphism as in the picture such that the square

$$\bullet,\ \bullet',\ D,\ D'$$

is pseudo-commutative. Indeed, the square with vertices

$$B/A,\ B'/A',\ D/C,\ D'/C'$$

is a good square if and only if taking its cokernel produces the induced dashed blue m-morphism such that the square

$$\bullet,\ \bullet',\ D/C,\ D'/C'$$

is pseudo-commutative. This, by axiom (PBL), is equivalent to the square

$$\bullet,\ \bullet',\ D,\ D'$$

being pseudo-commutative, which again by axiom (PBL) is equivalent to the square

$$\bullet,\ \bullet',\ C'/A',\ D'/B'$$

being pseudo-commutative. But that, in turn, happens if and only if its kernel square

$$C/A,\ D/B,\ C'/A',\ D'/B'$$

is good.

Finally, as $\star$ denotes $B \star_A C$ and $\star'$ denotes $B' \star_{A'} C'$, the existence of the induced dashed blue m-morphism such that the square

$$\bullet,\ \bullet',\ D,\ D'$$

is pseudo-commutative is equivalent to the southern square of the m-m-m cube being good, since these squares form a kernel-cokernel pair by Lemma A.6.

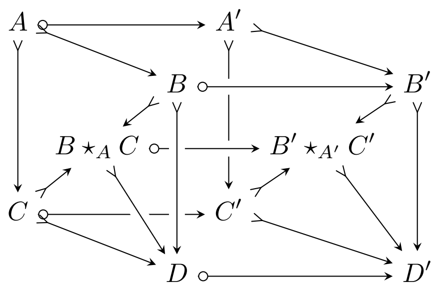
For the converse, to show that the kernel of an m-m-e cube with all faces good or pseudo-commutative is always good, first observe that given such an m-m-e cube pictured as the lower left cube in the diagram above, taking cokernels we get the lower right cube with all faces good or pseudo-commutative, either by construction or in the case of the rightmost face by axiom (PBL). This shows, by Lemma A.6, that in the kernel m-m-m cube pictured as the top left cube in the diagram, the southern square is good.

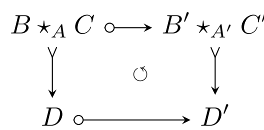
It then remains only to show that the topmost square of the m-m-m cube is good. This follows by constructing the top right m-m-e cube as the kernel of the bottom right cube. Its topmost square is pseudo-commutative by axiom (PBL), and forms a kernel-cokernel pair with the topmost square of the m-m-m cube, which is therefore good. □

Remark A.8. In particular, this implies that there is no need to specify a direction for the good southern square when dealing with good cubes, as claimed in Remark 3.14, since the "goodness" of an m-m-m cube can be equivalently determined from any of its m-m-e cokernel cubes.

We can further deduce the following, which can be interpreted as the statement that all m-m-e and e-e-m cubes with good and pseudo-commutative faces are "good cubes".

Corollary A.9. *Consider an m-m-e cube whose faces are either good or pseudo-commutative squares, together with the induced cube to the $\star$-pushouts as constructed in Proposition A.4,* *depicted below left. Then the square below right is pseudo-commutative.*





The analogous statement holds for e-e-m cubes when the $\star$-pushouts exist.

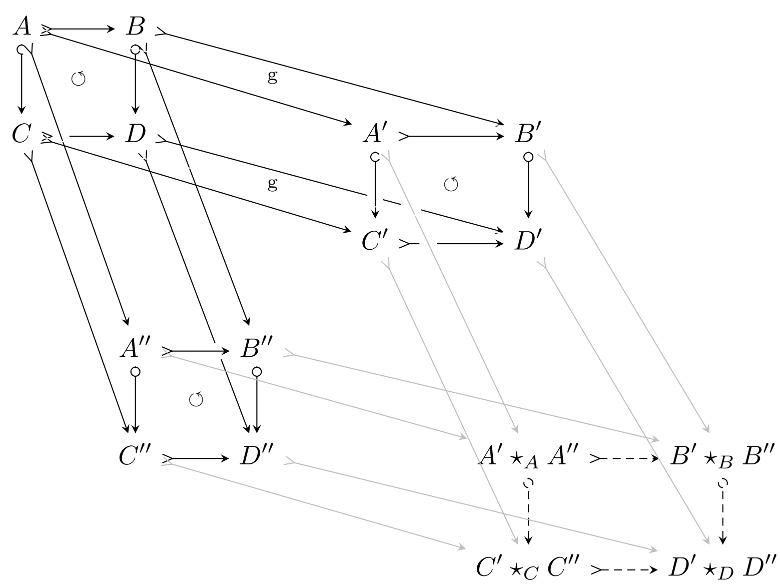
By analogy with m-m-m cubes, we call this pseudo-commutative square the *southern square* of the m-m-e cube.

Proof. The kernel of the outer cube is a good m-m-m cube by Proposition A.7, so the statement is easily deduced from Remark A.8 together with the first picture in the proof of Proposition A.7. ☐

Example A.10. This result illustrates an interesting difference between our motivating examples. In a weakly idempotent complete exact category, where pseudo-commutative squares are simply commuting squares between admissible monomorphisms and epimorphisms, this follows immediately from the universal property of the pushout. In finite sets, however, where the pseudo-commutative squares are pullbacks, this result is precisely the distributivity of intersections over unions among subsets of D': $D \cap (B' \cup C') = (D \cap B') \cup (D \cap C')$.

We now show that $\star$-pushouts preserve pseudo-commutative and distinguished squares.

Proposition A.11. *Given an m-span of pseudo-commutative squares, where all the other mixed squares involved are pseudo-commutative and the squares in one of the cube-legs of the span are good, the induced square between the $\star$-pushouts is pseudo-commutative.*



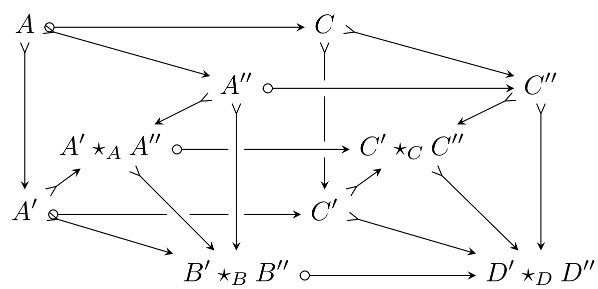
The same statement holds for e-spans when the $\star$-pushouts exist.

Proof. The gray and dashed m-morphisms are obtained from applying Proposition A.3 to the diagrams of m-morphisms on the "top" and "bottom" rows respectively in the diagram above. In turn, the dashed e-morphism $A' \star_A A'' \circ\!\!\longrightarrow C' \star_C C''$ is obtained by applying Proposition A.4 to the sub-diagram involving the objects

$$A,\ C,\ A',\ C',\ A'',\ C'',\ A' \star_A A'',\ C' \star_C C''.$$

Similarly, we get a map $B' \star_B B'' \circ\!\!\longrightarrow D' \star_D D''$.

The result then follows from applying Corollary A.9 to the following cube of good and pseudo-commutative squares, where the resulting pseudo-commutative southern square is precisely the desired induced square of $\star$-pushouts.



$\square$

Proposition A.12. *If the three initial squares in Proposition A.11 are distinguished, then so is the induced square between the $\star$-pushouts.*

Proof. By Proposition A.11, we know that the square between the $\star$-pushouts is pseudo-commutative. To show it is distinguished, first consider the particular case where $A = A' = A'' = \varnothing$; note that then we have $A' \star_A A'' = \varnothing$. In this case, we see that $C \rightarrowtail D$ is the kernel of $B \circ\!\!\longrightarrow D$ (and similarly for the other two distinguished squares). Then, by Remark A.5, $C' \star_C C'' \rightarrowtail D' \star_D D''$ must be the kernel of $B' \star_B B'' \circ\!\!\longrightarrow D' \star_D D''$, which shows that the desired square is distinguished.

For the general case, we paste distinguished squares besides the given squares as follows

$$\begin{array}{ccccc} \varnothing & \rightarrowtail & A & \rightarrowtail & B \\ \downarrow & \square & \downarrow & \square & \downarrow \\ C\backslash A & \rightarrowtail & C & \rightarrowtail & D \end{array} \qquad \begin{array}{ccccc} \varnothing & \rightarrowtail & A' & \rightarrowtail & B' \\ \downarrow & \square & \downarrow & \square & \downarrow \\ C'\backslash A' & \rightarrowtail & C' & \rightarrowtail & D' \end{array} \qquad \begin{array}{ccccc} \varnothing & \rightarrowtail & A'' & \rightarrowtail & B'' \\ \downarrow & \square & \downarrow & \square & \downarrow \\ C''\backslash A'' & \rightarrowtail & C'' & \rightarrowtail & D'' \end{array}$$

and obtain a diagram between $\star$-pushouts

$$\begin{array}{ccccc} \varnothing & \rightarrowtail & A' \star_A A'' & \rightarrowtail & B' \star_B B'' \\ \downarrow & \square & \downarrow & & \downarrow \\ (C'\backslash A') \star_{(C\backslash A)} (C''\backslash A'') & \rightarrowtail & C' \star_C C'' & \rightarrowtail & D' \star_D D'' \end{array}$$

The particular case guarantees that both the left square and the composite are distinguished; then, by Lemma 2.15, the desired square on the right is also distinguished. $\square$

Appendix B. FCGW categories of functors

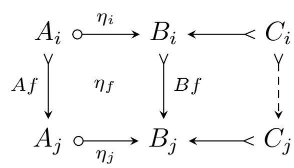
The aim of this subsection is to show that double categories of functors over an FCGW category $\mathcal{C}$ admit an FCGW structure themselves. In particular, this allows us to restrict to the special cases of interest: the double categories of staircases $S_n\mathcal{C}$ and the double categories of w-grids $w_{l,m}\mathcal{C}$.

Theorem B.1. *For any FCGW category $\mathcal{C}$ and double category $\mathcal{D}$, the double category $\mathcal{C}^{\mathcal{D}}$ with structure described in Definition 3.12 and Theorem 3.15 is an FCGW category.*

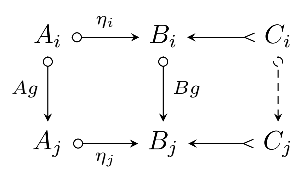
Proof. We begin by checking the conditions in Definition 2.4. First of all, note that $\mathcal{C}^{\mathcal{D}}$ is a double category with shared isomorphisms, since these are defined pointwise, and $\mathcal{C}$ has shared isomorphisms.

We now show that $k\colon \mathrm{Ar}_{\circlearrowleft}\,\mathcal{E} \longrightarrow \mathrm{Ar}_{\mathrm{g}}\,\mathcal{M}$ is well-defined; the argument for c proceeds analogously. To see that k takes an object in $\mathrm{Ar}_{\circlearrowleft}\,\mathcal{E}$ to an object in $\mathrm{Ar}_{\mathrm{g}}\,\mathcal{M}$, we must check that taking pointwise kernels of an e-natural transformation $\eta\colon A \Rightarrow B$ whose squares between e-morphisms are good produces a functor $C \in \mathcal{C}^{\mathcal{D}}$, together with an m-natural transformation $\mu\colon C \Rightarrow B$ whose squares between m-morphisms are good.

For an object $i \in \mathcal{D}$, C_i and μ_i are defined as the kernel of $\eta_i\colon A_i \circ\!\!\longrightarrow B_i$. For an m-morphism $f\colon i \rightarrowtail j$ in $\mathcal{D}$, let Cf be the induced morphism on kernels

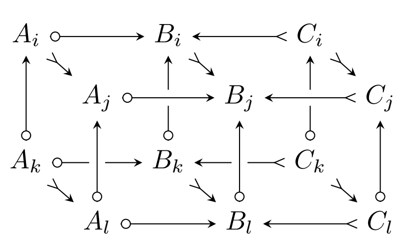


where the pseudo-commutative square on the left exists since η is an e-natural transformation. Similarly, given an e-morphism $g\colon i \circ\!\!\longrightarrow j$ in $\mathcal{D}$, let Cg be the induced morphism on kernels



and μ_g be the induced pseudo-commutative square on the right, where the square on the left commutes by naturality of η, and is good by the additional assumption on η.

Finally, we must check that taking pointwise kernels of the leftmost cube below (whose faces are all good or pseudo-commutative) produces a cube as the one on the right (whose faces are all good or pseudo-commutative).

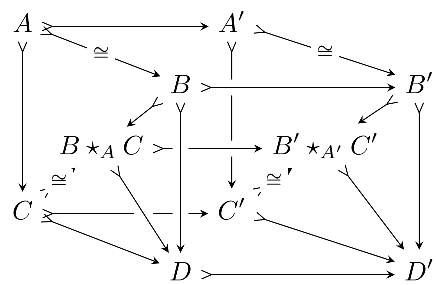


Most of these faces are of the correct type by construction; indeed, the only face one needs to check is the rightmost square between the C's, which is pseudo-commutative by axiom (PBL).

The fact that k takes a morphism in $\mathrm{Ar}_{\circlearrowleft}\,\mathcal{E}$ to a morphism in $\mathrm{Ar}_{\mathrm{g}}\,\mathcal{M}$ is further ensured by Proposition A.7.

Since k is defined pointwise from the kernel functor in $\mathcal{C}$, it is clear that it is faithful. Furthermore, the fact that k and c are inverses on objects up to isomorphism, together with Proposition A.7, show that k is essentially surjective and full.

Axioms (Z) and (M) are trivially satisfied, since m- and e-morphisms in $\mathcal{C}^{\mathcal{D}}$ are pointwise m- and e-morphisms in $\mathcal{C}$. For axiom (G), note that good squares in $\mathcal{C}^{\mathcal{D}}$ are composed of faces which are good squares in $\mathcal{C}$; in particular, all faces are pullbacks in $\mathcal{C}$, and so they are pullbacks in $\mathcal{C}^{\mathcal{D}}$. To see that $\mathrm{Ar}_{\triangle}\,\mathcal{M} \subseteq \mathrm{Ar}_{\mathrm{g}}\,\mathcal{M}$, it suffices to note that the southern square of a cube in $\mathrm{Ar}_{\triangle}\,\mathcal{M}$ agrees (up to isomorphism) with one of the faces of the cube, which is a good square.



Finally, axioms (D) and (K) are immediate, since the functors k and c are defined pointwise. This shows that $\mathcal{C}^{\mathcal{D}}$ is a pre-FCGW category.

We now check the axioms in Definition 3.1. Axiom (GS) holds, as it is true pointwise in $\mathcal{C}$, and good cubes are symmetric by Remark A.8. Axiom (PBL) is satisfied, since a square in $\mathcal{C}^{\mathcal{D}}$ is pseudo-commutative precisely if it is pointwise pseudo-commutative in $\mathcal{C}$. For axiom ($\star$), given a span of m-morphisms $B \leftarrowtail A \rightarrowtail C$ in $\mathcal{C}^{\mathcal{D}}$, we can construct their pointwise $\star$-pushots using axiom ($\star$) for $\mathcal{C}$. By Propositions A.3 and A.4, $\star$-pushouts preserve m- and e-morphisms in the appropriate manner. Furthermore, by Proposition A.11, they preserve pseudo-commutative squares. Thus, pointwise $\star$-pushouts are double functors $\mathcal{D} \longrightarrow \mathcal{C}$.

Propositions A.3 and A.4 also imply that the induced maps $B \longrightarrow B \star_A C$ and $C \longrightarrow B \star_A C$ are m-morphisms in $\mathcal{C}^{\mathcal{D}}$, and that the square below is good.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \star_A C \end{array}$$

Similarly, we can construct the $\star$-pushout of a span of e-morphisms $B \longleftarrow\!\circ\, A \circ\!\longrightarrow C$ in $\mathcal{C}^{\mathcal{D}}$ when we already know the span is part of some good square.

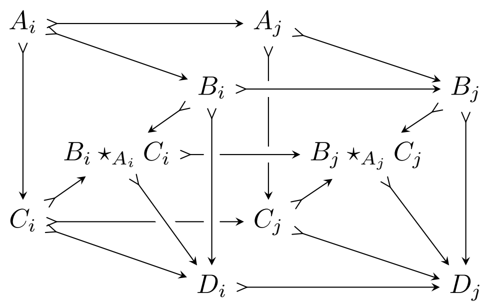
It remains to show the universal property in axiom (PO), since $\star$-pushouts will preserve (co)kernels as $\star$, k, and c are all defined pointwise. Consider a good square in $\mathcal{C}^{\mathcal{D}}$ as below left.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \mathrm{g} & \downarrow \\ C & \rightarrowtail & D \end{array} \qquad \begin{array}{ccc} A_i & \rightarrowtail & B_i \\ \downarrow & \mathrm{g} & \downarrow \\ C_i & \rightarrowtail & D_i \end{array}$$

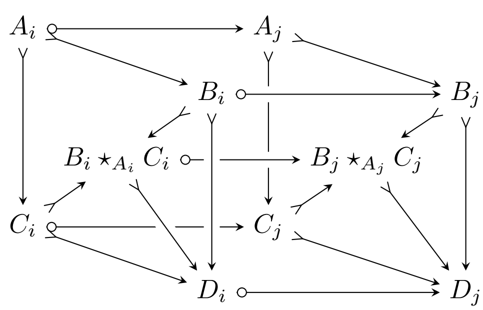
In particular, for each $i \in \mathcal{D}$ we have a good square in $\mathcal{C}$ as above right, which induce pointwise maps $B_i \star_{A_i} C_i \rightarrowtail D_i$, which are unique up to unique isomorphism. We need to show that for each $i \rightarrowtail j$ and $i \circ\!\!\rightarrow j$ in $\mathcal{D}$, the induced squares below are either good or pseudo-commutative.

$$\begin{array}{ccc} B_i \star_{A_i} C_i & \rightarrowtail & D_i \\ \downarrow & & \downarrow \\ B_j \star_{A_j} C_j & \rightarrowtail & D_j \end{array} \qquad \begin{array}{ccc} B_i \star_{A_i} C_i & \rightarrowtail & D_i \\ \circ\!\!\downarrow & & \circ\!\!\downarrow \\ B_j \star_{A_j} C_j & \rightarrowtail & D_j \end{array}$$

For the first statement, note that the square above left is the southern square of the cube

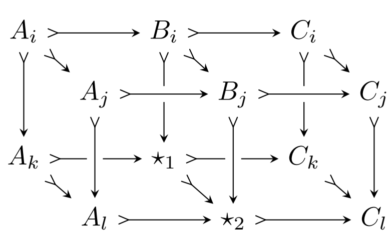


which was assumed to be a good cube; thus, the square must be good. For the second, note that the square above right is the "southern square" of the cube



which, by Corollary A.9, is always pseudo-commutative.

Finally, for axiom (POL), it suffices to check that in any diagram



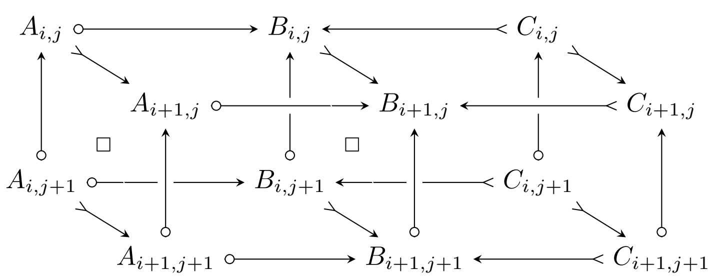
whose outer cube is good, the right cube must be good. Here $\star_1$ denotes $B_i \star_{A_i} A_k$, and $\star_2$ denotes $B_j \star_{A_j} A_l$. Indeed, the back and front faces of the right cube must be good squares due to $\mathcal{C}$ satisfying axiom (POL), and the southern square of the right cube can easily be seen to agree with the southern square of the outer cube, which is good. □

We can further show that we get an FCGW structure when restricting the squares in our $\mathcal{D}$-shaped diagrams to be distinguished in $\mathcal{C}$ and requiring certain objects in $\mathcal{D}$ to be sent to $\varnothing$, as in the double subcategory $S_n\mathcal{C} \subset \mathcal{C}^{\mathcal{S}_n}$ of Definition 5.2.

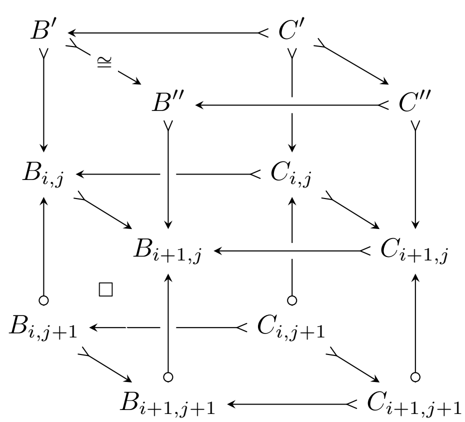
Proposition B.2. *$S_n\mathcal{C}$ is an FCGW subcategory of $\mathcal{C}^{\mathcal{S}_n}$.*

Proof. By Lemma 3.11, in order to show that this is an FCGW subcategory, it suffices to prove that it is closed under k, c, $\star$, and that it contains the initial object. The latter is trivial, as any square whose boundary consists of isomorphisms is distinguished. Furthermore, since k, c and $\star$ are computed pointwise, it is clear that they preserve the condition of sending the objects $A_{i,i}$ to $\varnothing$. It remains to show that each of these preserves distinguished squares.

We first show that k preserves distinguished squares; for this, we show that in the following diagram, where the right cube is the kernel of the left one, the rightmost square is distinguished in $\mathcal{C}$.

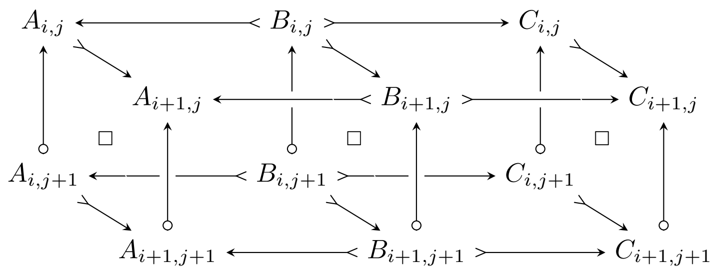


Note that the square is known to be pseudo-commutative, since it is a face in a kernel cube in the FCGWA category $\mathcal{C}^{\mathcal{S}_n}$. To prove it is distinguished, we take the kernel of the right cube in the vertical direction

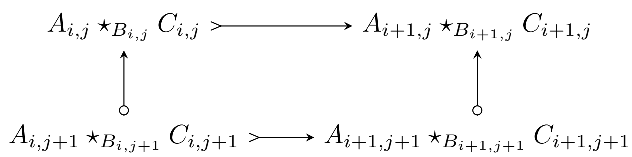


Since the indicated square is distinguished, the induced m-morphism on kernels is an isomorphism. But the top cube is a good cube; in particular, the top face is good, and thus a pullback. This implies that the m-morphism $C' \rightarrowtail C''$ must be an isomorphism, which in turn proves that the desired square is distinguished. The proof that $S_n\mathcal{C}$ is closed under c proceeds dually.

Finally, we prove that $S_n\mathcal{C}$ is closed under $\star$. For this, we need to show that for any span of m-morphisms



the resulting square of $\star$-pushouts below is distinguished,



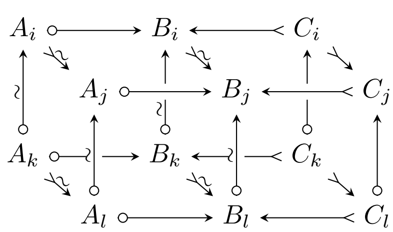
which is ensured by Proposition A.12. □

Lastly, we show that the double category of w-grids $w_{l,m}\mathcal{C} \subset \mathcal{C}^{\mathcal{D}}$ of Definition 6.6 is also an FCGW category.

Proposition B.3. *$w_{l,m}\mathcal{C}$ is an FCGW subcategory of $\mathcal{C}^{\mathcal{D}}$, where $\mathcal{D}$ denotes the free double category on an $l \times m$ grid of squares. Moreover, if $\mathcal{V}$ a refinement of $\mathcal{W}$, then the double subcategory of grids in $\mathcal{V}$ forms an acyclicity structure on $w_{l,m}\mathcal{C}$.*

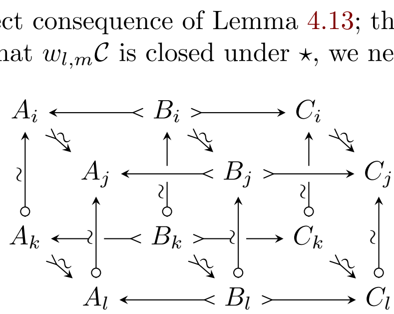
Proof. Once again, by Lemma 3.11, it suffices to prove that $w_{l,m}\mathcal{C}$ is closed under k, c, $\star$, and that it contains the initial object. The latter is trivial, as identity morphisms are always m- and e-equivalences.

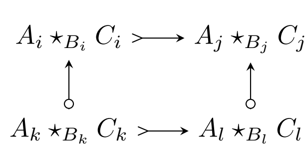
In order to prove that $w_{l,m}\mathcal{C}$ is closed under k, we must show that in the following diagram, where the right cube is the kernel of the left one, the maps in the rightmost square are m- and e-equivalences.



This is a direct consequence of Lemma 4.13; the statement for c is analogous.

To show that $w_{l,m}\mathcal{C}$ is closed under $\star$, we need to prove that for any m-span as below left





the resulting square of $\star$-pushouts pictured above right is distinguished. But by Proposition A.12, we know that $\star$-pushouts preserve kernel-cokernel sequences; in other words, we have that

$$k(A_k \star_{B_k} C_k \circ\longrightarrow A_i \star_{B_i} C_i) = (A_k \backslash A_i) \star_{B_k \backslash B_i} (C_k \backslash C_i),$$
$$c(A_i \star_{B_i} C_i \rightarrowtail A_j \star_{B_j} C_j) = (A_j / A_i) \star_{B_j / B_i} (C_j / C_i),$$

and similarly for the other two maps. We then conclude that the square above right is made of m- and e-equivalences due to Lemma 4.14. □

References

[BÏ0] Theo Bühler. Exact categories. *Expo. Math.*, 28(1):1–69, 2010.

[Cam19] Jonathan A. Campbell. The K-theory spectrum of varieties. *Trans. Amer. Math. Soc.*, 371(11):7845–7884, 2019.

[Car98] Manuel Enrique Cardenas. Localization for exact categories, 1998. Thesis (Ph.D.)–State University of New York at Binghamton.

[CZ] Jonathan A. Campbell and Inna Zakharevich. Dévissage and localization for the Grothendieck spectrum of varieties. Preprint available at.

[FP10] Thomas M. Fiore and Simona Paoli. A Thomason model structure on the category of small n-fold categories. *Algebr. Geom. Topol.*, 10(4):1933–2008, 2010.

[Gra20] Marco Grandis. *Higher dimensional categories*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020. From double to multiple categories.

[McC93] Randy McCarthy. On fundamental theorems of algebraic K-theory. *Topology*, 32(2):325–328, 1993.

[MSV20] Lyne Moser, Maru Sarazola, and Paula Verdugo. A 2Cat-inspired model structure for double categories. Preprint on arXiv:2004.14233, 2020.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Rie16] Emily Riehl. *Category theory in context*. Courier Dover Publications, 2016.

[Sar20] Maru Sarazola. Cotorsion pairs and a K-theory localization theorem. *J. Pure Appl. Algebra*, 224(11):106399, 29, 2020.

[Sch04] Marco Schlichting. Delooping the K-theory of exact categories. *Topology*, 43(5):1089–1103, 2004.

[Sch06] Marco Schlichting. Negative K-theory of derived categories. *Math. Z.*, 253(1):97–134, 2006.

[Sch11] Marco Schlichting. Higher algebraic K-theory. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 2011.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Wal85] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K-theory.

[Zak18] Inna Zakharevich. The category of Waldhausen categories is a closed multicategory. In *New directions in homotopy theory*, volume 707 of *Contemp. Math.*, pages 175–194. Amer. Math. Soc., Providence, RI, 2018.