

SHAPE INDEPENDENT CATEGORY THEORY

A Dissertation

Presented to the Faculty of the Graduate School
of Cornell University

in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by

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May 2022

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Cornell University 2022

Categories, n -categories, bicategories, double categories, multicategories, monoidal categories, and monoids are all examples of algebraic structures on diagrams of combinatorial cells. Many of these structures have features in common with categories, such as a nerve functor, a theory of enrichment, a notion of (co)limits, or a version of the Yoneda lemma. We begin here a program to unify these and other common features into general constructions to form a “shape independent category theory” that can apply to a wide variety of algebraic “higher” category structures.

We start with a technical treatment of “familial monads” on presheaf categories, where each of the structures above form the category of algebras of such a monad. Using a relationship between familial functors and polynomial diagrams in **Cat**, we establish an equivalence between familial monads and the combinatorial data of how arrangements of cells are composed in an algebraic higher category. This data provides a language for describing different types of higher categories, which we use to describe existing results on nerves of familial monad algebras and discuss the algebraic nature of their underlying cell shapes. We also construct new examples of familial monads with a focus on cubical cells.

Finally, we build a theory of enrichment for any type of higher category with top-dimensional cell shapes. This shape independent construction generalizes many existing forms of enrichment, and produces new types of higher categories. The theory relies on a generalization of the wreath product of categories, which provides simple definitions of various universal constructions on categories and an elegant description of the cells in the nerve of an enriched higher category.

BIOGRAPHICAL SKETCH

Brandon Shapiro (correctly pronounced as either Shap-EE-ro or Shap-AYE-ro, though the latter is mandatory when located in eastern Pennsylvania, northern Delaware, southern New Jersey, or a federal courthouse) plays with shapes all day, and aspires to do so professionally. He does not always believe in numbers, but when he does they only go up to like 3. Anything higher might as well be ∞

To Shruthi

ACKNOWLEDGEMENTS

I have had more amazing mentors than I could possibly list here, and am grateful for the teachers, professors, supervisors, and friends that have been a part of my journey through mathematics. Among many others, Susan Wildstrom, Kenny Foner, Colin Adams, Aru Ray, and David Spivak played essential roles in developing my love of math and the topics I research.

This project was greatly aided and improved by conversations with Marcelo Aguiar, Yuri Berest, Hongyi Chu, tsilil clingman, Carmen Constantin, Tobias Fritz, Richard Garner, Chris Kapulkin, Joachim Kock, Tom Leinster, Paolo Perrone, Emily Riehl, Shruthi Sridhar, David Spivak, Kimball Strong, Walter Tholen, Mark Weber, and others. I also appreciate the Cornell Math Department, NDSEG fellowship, and Topos Institute for supporting me and/or this work at various points in time.

I was incredibly fortunate to share my experience in graduate school with David Mehrle, Elise McMahon, John Whelan, Sumun Iyer, Kimball Strong, Chase Vogeli, and countless other awesome Cornell graduate students. I would especially like to thank Maru Sarazola for being the collaborator, commiserator, and copilot I can hardly imagine completing a PhD without.

Neither the length of this thesis nor the time it took to prepare would suffice to express my gratitude to fate or Inna Zakharevich for the privilege of having her as my advisor. Never could I have imagined a more wonderful mentor, role model, and friend as a guide in this part of my life and career. You're the best.

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CHAPTER 0

INTRODUCTION

The theory of categories includes, among other concepts: categories, functors, natural transformations, isomorphisms, limits, colimits, adjunctions, equivalences, the Yoneda lemma, nerves, enrichment, internalization, and fibrations. Whenever a new categorical structure is defined, such as n -categories, double categories, or multicategories, the first steps in developing the new theory are typically to define analogues of some of the concepts above that apply to the new structure. Especially as applications routinely inspire new types of categorical structures, the need to redefine each concept from scratch for a new structure is an undesirable barrier to applying compositional reasoning more broadly in mathematics and in modeling real world phenomena.

The goal of this thesis is to begin the process of automating these definitions, so that each new categorical structure satisfying appropriate conditions immediately inherits a specialized version of each of these concepts. We provide a language for describing a type of algebraic categorical structure or “higher category” and use it to describe new examples and existing generalizations of categories, functors, nerves, and internalizations for any type of higher category. We then move on to define a general theory of enrichment for any structure with top dimensional cell shapes, such as the arrow in categories or the square in double categories, and show how enrichment produces new types of higher categories and interacts nicely with nerves.

Along the way, we make two major technical digressions in support of this story. The first is to further develop the relationship between familiarly representable (a.k.a. familial) functors between presheaf categories $\widehat{\mathcal{C}}$ and polynomial functors between the categories \mathbf{Cat}/\mathcal{C} , defining new bicategories of polynomial diagrams in \mathbf{Cat} with bifunctors into \mathbf{CAT} landing in familial and polynomial functors respectively. The categorical structures we consider are defined as algebras for familial monads, and a comparison between familial and polynomial functors lets us characterize a familial monad T on $\widehat{\mathcal{C}}$ by the cell shapes in \mathcal{C} , the arrangements of cells that can be composed in a T -algebra, and equations between those composition operations.

The second is to generalize the wreath product of categories and analyze its properties, which allows for convenient new definitions of several existing constructions from category theory. We ultimately use the wreath product as a means of combining together the cell shapes and composition operations of two different categorical structures to form a new familial monad with enriched structures as algebras.

In numerous examples, we explore how these generalized constructions recover a wide variety of familiar features of “higher” and “lower” category theory and extend those features to new types of structures. Here is a sample of some of the new statements we can make with these general constructions:

- Cyclic sets are algebras for a familial monad on simplicial sets
- Cubical sets with symmetries are algebras for a familial monad on semicubical sets

- Cubical ω -categories are algebras for a familial monad on semicubical sets
- Multicategories can be enriched in any **Cat**-enriched operad
- The wreath product of **Set** and a category \mathcal{A} is the free coproduct completion of \mathcal{A}
- The wreath product of the category \mathbf{L}_{emon} of free finitely generated commutative monoids with \mathcal{A} is the free finite biproduct completion of \mathcal{A}
- $(n + m)$ -categories are n -categories enriched in $m - \mathbf{Cat}$
- A monoid object in a monoidal category V is a monoid enriched in V
- Monoidal categories are algebras for a familial monad on graphs
- Double categories or multicategories enriched in $2 - \mathbf{Cat}$ are algebras for a familial monad
- Both of the above have fully faithful nerve functors to presheaf categories

Before we proceed to more detailed summaries of each chapter, it is worth placing this program in the context of others that have unified the theories of other types of higher categorical structures. In [33], Riehl and Verity develop a “model independent” theory of geometric $(\infty, 1)$ -categories. “Geometric” means that the objects they consider are generally defined as presheaves with properties rather than the algebraic composition operations in the sort of higher categories considered here. Many of their results including unified definitions of limits, adjunctions, and the Yoneda lemma are among those we hope to one day provide for algebraic higher categories. Globular algebraic notions of $(\infty, 1)$ -categories do fit into our framework, but only as

a subclass of (∞, ∞) -categories (also called weak ω -categories), as familial monads are not expressive enough to enforce invertibility conditions.

Leinster’s theory of globular operads [28] describes different types of weak n - and ω -categories, and more broadly generalized operads provide a formalism for studying weak and lax versions of any fixed (usually strict) algebraic structure that can be defined as algebras for a cartesian monad (including all familial monads). Our focus is instead on unifying results across different strict higher category structures, as while weak higher categories are also algebras for familial monads, their theory relative to the strict versions is rather well understood.

Familial and Polynomial Functors

The categorical structures we are interested in are algebras for a *familially representable* monad on a presheaf category $\widehat{\mathcal{C}} := \mathbf{Set}^{\mathcal{C}^{op}}$. Familially representable monads, which we call simply *familial*, include the free category monad on graphs, whose algebras are categories, the free n -category monad on n -globular sets, and similarly the free construction of each structure mentioned above from an underlying diagram of cells (modeled as a presheaf on a category \mathcal{C} of cell shapes).

As the first step of unifying the different studies of algebras of different familial monads, we provide a combinatorial language for defining familial monads in terms of the “composition operations” in their algebras. This language lets us extract the key properties of these monads for extending concepts of category theory to their

algebras. The end result of this characterization is as follows:

Theorem. (Theorem 5.1) *A familial monad T on a presheaf category $\widehat{\mathcal{C}}$ is completely specified by the following data:*

- *A functor $S: \mathcal{C}^{op} \rightarrow \mathbf{Set}$*
- *A functor $E: \int S \rightarrow \widehat{\mathcal{C}}$*
- *For each c in \mathcal{C} , an element $\eta(c) \in Sc$ with an isomorphism $E\eta(c) \cong y(c)$*
- *For each $t \in Sc$ and $f: Et \rightarrow S$ in $\widehat{\mathcal{C}}$, an element $\mu(t, f)$ with an isomorphism*

$$E\mu(t, f) \cong \operatorname{colim}_{x \in Et} Ef(x)$$

subject to several equations. T sends a presheaf X to the presheaf TX with

$$TX_c = \prod_{t \in Sc} \operatorname{Hom}(Et, X).$$

An algebra for T is a “higher category” with cells that form a presheaf on \mathcal{C} , like the underlying graph of a category, and for each $t \in Sc$ an “operation” sending an Et -shaped arrangement of cells to a c -cell, like the operation sending a string of arrows in a category to its composite. The operations $\eta(c)$ and $\mu(t, f)$ ensure that, respectively, there is always an operation sending a c -cell to itself and applying one operation to the outputs of others is itself an operation. Labeling these is how equations are defined, like the associativity equation in a category that ensures composing the first of three arrows with the composite of the second two agrees with composing them in

the opposite order: both composites of operations are equal to the single operation which composes three arrows at once.

It has been shown in [34, Theorem 8.1] and [19, Proposition 3.8] that *familial functors* are equivalent to a category of pairs (S, E) as above, which we call a *familial representation*. It follows intuitively that familial monads should admit a similar description, with a translation of their cartesian unit and multiplication transformations into the language of familial representations. Examples of familial monads are defined in [34, Section 9], [35, Example 2.14], and [19, Proposition 2.9] in manners similar to the above, but the only general description of the data determining a familial monad on $\widehat{\mathcal{C}}$ is that of [28, Section C.3] in the case when \mathcal{C} is a discrete category, and a complete list of the coherence equations this data must satisfy does not appear in the existing literature.

A direct proof of this characterization, showing how to specify η, μ for a tuple (S, E, η, μ) and checking that they satisfy the unit and associativity equations, would require several enormous diagrams and scores of tedious naturality proofs. Instead, we provide a more conceptual argument, taking advantage of the machinery of polynomial functors as developed in [17] and [36]. In particular, we define a bicategory **Rep** with 1-cells familial representations (S, E) and show that it is biequivalent to the 2-category **Fam** of presheaf categories, familial functors, and cartesian natural transformations by passing through a bicategory of polynomial diagrams in **Cat** (Theorem 4.1). Restricting this biequivalence to formal monads in these bicategories yields an equivalence between tuples (S, E, η, μ) and familial monads.

However, the existing constructions of the bicategory of polynomial diagrams are not sufficient for this proof, which requires polynomial diagrams in **Cat** and both cartesian and vertical morphisms between them. The construction in [17] includes both cartesian and vertical morphisms of polynomials but only applies in a category which is locally cartesian closed, unlike **Cat**, while the bicategory of polynomial diagrams in [36] applies in any category with pullbacks but only includes cartesian morphisms. To remedy this, we define a new bicategory of polynomial diagrams in **Cat** which, by slightly restricting the polynomials which are included, admits both cartesian and vertical morphisms of these polynomials. We also construct bifunctors to **CAT** sending a polynomial diagram to the polynomial functor it induces on either categories or presheaves.

Organization

In Section 1, we introduce familial functors and monads along with our main running examples of n -categories, double categories, multicategories, and monoids. A reader primarily interested in our results for higher category theories and willing to accept on faith the characterization of familial monads above (or its more precise formulation in Theorem 5.1) could skip the remaining sections in Part 1, which are devoted to the proof of this result.

In Section 2 we recall the various notions of fibration in **Cat** and provide proofs of the pullback-stability, composability, and exponentiability of split opfibrations directly in terms of their classifying functors to **Cat**. In Section 3 we introduce

several special classes of polynomial diagrams in **Cat** and develop their relationship with familial representations and familial functors. Finally in Section 4 we define the bicategory structure **Rep** on familial representations and complete the proof that it is biequivalent to **Fam**.

Higher Category Theories

The characterization of familial monads as a tuple $(\mathcal{C}, S, E, \eta, \mu)$ provides a general strategy for defining a familial monad T on $\widehat{\mathcal{C}}$ for any small category \mathcal{C} directly in terms of the composition operations in its algebras, along with their arities and equations between them. This is the combinatorial language we use to describe the properties of the type of higher categories that arise as algebras of T , and define how classical concepts of category theory generalize to those higher categories.

This language allows us to efficiently define several new examples of familial monads beyond those mentioned above. The algebraic structure in a higher category typically involves a combination of degeneracies, symmetries, and compositions, and we describe examples illustrating each of these. A common theme in these examples is familial monads on *semicubical sets*, presheaves with minimally structured cubical cells in each dimension. These cubical cell shapes admit a rich variety of degeneracies, symmetries, and compositions which illustrate the broad scope of higher category structures defined by familial monads. In particular we define monads for cubical sets over semicubical sets, symmetric cubical sets over cubical sets, and cubical ω -

categories over semicubical sets. We also discuss more general monads for adding degeneracies indexed by any Reedy category and adding symmetries to presheaves over \mathcal{C} indexed by a crossed \mathcal{C} -group in the sense of [6, Definition 2.1].

We also cast into this language various results from the literature generalizing the nerve functor to any type of higher category. The classical nerve functor from categories to simplicial sets is based on the idea that all of the algebraic structure of a category is contained in how finite strings of adjacent arrows are composed. These compositions are exhibited by the ordinal categories $[n]$ with n adjacent arrows and all of their composites. The simplex category Δ can be defined as the category of these ordinals.

For any type of higher category that arises as T -algebras, where T is represented in our language by $(\mathcal{C}, S, E, \eta, \mu)$, the role of Δ can be replaced by the “theory category” Θ_T of free higher categories on the arity presheaves Et . Weber’s “Nerve Theorem” ([35, Theorem 4.10]) shows¹ that the functor sending a higher category A to the presheaf on Θ_T whose cells are composable diagrams in A of shape Et is fully faithful. This means that T -algebras can be equivalently regarded as presheaves on Θ_T satisfying certain conditions. When T is a monad on **Set**, this nerve functor sends an algebra to the corresponding model in sets of the Lawvere theory of T .

The objects of Θ_T can be thought of as the arity diagrams Et “equipped with all compositions,” as in the following examples:

¹Weber actually proves this more generally for any “monad with arities”

- for the theory of categories Θ_T is Δ , where the ordinals $[n]$ are strings of composable arrows
- for n -categories Θ_T is Joyal's category Θ_n of pasting diagrams of n -cells and their composites [23]
- for multicategories Θ_T is the category Ω of trees with compositions [30]
- for double categories Θ_T is the product $\Delta \times \Delta$ whose objects can be viewed as grids of composable squares

The morphisms in Θ_T include both the maps between the arity diagrams from $\widehat{\mathcal{C}}$ and maps built out of “cocompositions” $Ty(c) \rightarrow TEt$ for $t \in Sc$ which identify the total composite of the Et -shaped diagram in TEt and provide the composition operations in nerves of T -algebras.

Organization

In Section 5, we state the characterization of familial monads, discuss how it works in our recurring examples and in higher categories with top-dimensional cell shapes, then show how higher category structures behave under restrictions of their cell shapes. In Section 6, we review Lawvere theories and discuss their generalization to the categories Θ_T for the purpose of defining and recognizing nerves of T -algebras. We also describe a factorization system on Θ_T which will help facilitate the definition of enriched T -algebras in Part 3. Finally in Section 7 we show how to construct using

this language familial monads for adding degeneracies, symmetries, and composites to cell diagrams with a focus on cells shaped like cubes in every dimension.

Shape Independent Enrichment

Enriched categories replace the sets $\text{Hom}(a, b)$ of arrows in a category with objects $\mathbf{Hom}(a, b)$ of a monoidal category V , which provides a rigorous compositional framework for settings where $\text{Hom}(a, b)$ naturally forms a space, abelian group, or other mathematical object. Composition in an enriched category is encoded by morphisms $\mathbf{Hom}(a, b) \otimes \mathbf{Hom}(b, c) \rightarrow \mathbf{Hom}(a, c)$ in V and identities by morphisms $I \rightarrow \mathbf{Hom}(a, a)$. Only the sets of arrows are modeled by V , not the objects, and for higher categories enrichment replaces the sets of top-dimensional cells with objects in a category V that has extra structure which varies based on the type of higher category.

Definition. (Theorem 11.5, Theorem 11.14) For a higher category theory T with top-dimensional cell shapes e , and V a (T, e) -structured category, a *higher category enriched in V* consists of:

- the data of a T -algebra without the e -cells
- an object in V for each possible e -cell position
- morphisms in V corresponding to each e -operation which satisfy coherence equations

A (T, e) -structured category is a category (or several categories) with extra structure corresponding to the e -operations of T encoded by the cocomposition maps out of $y(e)$ in Θ_T . For instance when T is the free category monad or the free monoid monad V is a monoidal category, when T is the free double category or 2-category monad V is a braided monoidal category, and when T is the free multicategory monad V consists of categories V_0, V_1, V_2, \dots with functors

$$V_n \times V_{m_1} \times \cdots \times V_{m_n} \rightarrow V_{m_1 + \cdots + m_n}$$

that behave like composition of operations in an operad.

A classical way to define n -categories is as categories enriched in $(n-1)$ -categories. Using this general notion enrichment, we show that under light conditions T -algebras enriched in the category of T' algebras also form a new type of higher category.

Theorem. (Theorem 5.1) *For suitable familial monads T and T' with cell shapes \mathcal{C} and \mathcal{C}' , T -algebras enriched in $\text{alg}(T')$ are equivalent to algebras for a familial monad $T \wr T'$ with cell shapes $\mathcal{C} \wr \mathcal{C}'$.*

This recovers the classical definition of n -categories as well as an elegant variation on it: n -categories enriched in m -categories are precisely $(n+m)$ -categories. It also describes higher categories whose cell shapes are a combination of the classical ones, as well as (strict) monoidal categories, which arise as monoids enriched in categories.

The cell shapes in $\mathcal{C} \wr \mathcal{C}'$ look like those in \mathcal{C} with the top dimensional cell(s) “stuffed” with the cell shapes from \mathcal{C}' . When $\mathcal{C}, \mathcal{C}'$ both contain the shapes of a

vertex and an arrow between two vertices (as in the cell shapes of a category), $\mathcal{C} \wr \mathcal{C}'$ consists of a vertex, an arrow, and a 2-cell which resembles an arrow stuffed inside an arrow:

$$\bullet \longrightarrow \bullet \quad \wr \quad \begin{array}{c} \cdot \\ \downarrow \\ \cdot \end{array} = \begin{array}{c} \cdot \\ \curvearrowright \downarrow \curvearrowleft \\ \cdot \end{array}$$

Underlying this result is an elegant theory generalizing the wreath product of categories ([5, Definition 3.1, Remark 3.4], itself a generalization of the wreath product of groups). The wreath product $\mathcal{A} \wr_{\gamma} \mathcal{B}$ has objects given by objects of \mathcal{A} equipped with a set of objects in \mathcal{B} , and morphisms given by a morphism in \mathcal{A} along with an arrangement of morphisms in \mathcal{B} between the adorning objects. The sizes of the sets and shapes of the arrangements of morphisms are controlled by a functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ to the category of sets and spans between them.

Not only does this generalized wreath product help define the cell shape categories $\mathcal{C} \wr \mathcal{C}'$ and the operations for composing diagrams of those cells, but it also recovers simple definitions of universal constructions from category theory. The wreath product $\mathbf{Set} \wr \mathcal{A}$ is the free coproduct completion of \mathcal{A} , $\mathbf{Set}^{op} \wr \mathcal{A}$ is the free product completion of \mathcal{A} , and $\mathbf{Span} \wr \mathcal{A}$ is the free biproduct completion of \mathcal{A} , with corresponding finite variants given by restricting to finite sets.

The classical relationship between enrichment and nerves is that n -categories are categories enriched in $(n - 1)$ -categories, and their nerves are presheaves over Θ_n , which is the wreath product $\Delta \wr \Theta_{n-1}$. Using our generalized wreath product, this relationship holds for a wide variety of higher categories.

Theorem 0.1. (Theorem 11.28) *For suitable familial monads T, T' , we have $\Theta_{T \wr T'} \cong \Theta_T \wr \Theta_{T'}$.*

In most of the examples discussed here the usual wreath product of categories suffices, but the generalized version allows for more diverse inputs. For instance, monoidal categories have a nerve functor to presheaves over $L_{mon} \bar{\wr} \Delta$ for L_{mon} the category of free finitely generated monoids, which cannot be defined using the usual wreath product.

Organization

In Section 8 we define (T, e) -structured categories and T -algebras enriched in such a category V . In Section 9 we define the generalized wreath product of categories and discuss several examples and properties of the construction. In Section 10 we use the wreath product to construct new cell shapes via “stuffing” and study presheaves over stuffed cell shapes. Finally in Section 11 we prove our main results on enrichment of T -algebras in $alg(T')$, using the wreath product to define the monad $T \wr T'$ and its theory category.

Notation and Terminology

For a small category \mathcal{C} , we write $\widehat{\mathcal{C}}$ for the category $\mathbf{Set}^{\mathcal{C}^{op}}$ of presheaves over \mathcal{C} . For X a presheaf in $\widehat{\mathcal{C}}$, we write X_c for its set of “ c -cells” and $X_i: X_c \rightarrow X_{c'}$ for its action

on a morphism $i: c' \rightarrow c$ of \mathcal{C} .

We write fX for its category of elements, whose objects are pairs $(c \in \text{Ob}(\mathcal{C}), x \in X_c)$, often abbreviated as simply x , and morphisms of the form $i_x: X_i(x) \rightarrow x$ for $i: c' \rightarrow c$ in \mathcal{C} and $x \in X_c$.

We write $*$ for the terminal presheaf in $\widehat{\mathcal{C}}$, and $\{*_c\}$ for its singleton set of c -cells for each $c \in \text{Ob}(\mathcal{C})$.

We say that a natural transformation between functors is *cartesian* if all of its naturality squares are pullbacks.

We write \mathbb{N} for the set of natural numbers $0, 1, 2, \dots$, and \underline{n} for the set $\{1, \dots, n\}$, where $\underline{0}$ is the empty set.

We denote by **Set**, **Cat**, and **CAT** the categories of sets, small categories, and locally small categories respectively, where the latter two are often regarded as 2-categories.

We will sometimes refer to “presheaves over \mathcal{C} ” as “cell diagrams over \mathcal{C} ,” as we prefer to think of them as diagrams/complexes of cells with shapes coming from \mathcal{C} (and these presheaves are not used here in connection to any sort of sheaves).

CHAPTER 1
FAMILIAL AND POLYNOMIAL FUNCTORS

1 Familial Functors and Monads

A functor from a category \mathcal{A} to \mathbf{Set} is representable if it is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(A, -)$ for some object A in \mathcal{A} , called a *representation* of the functor. $\mathbf{Set} \cong \widehat{\mathcal{C}}$ when \mathcal{C} is the terminal category and an object A in \mathcal{A} is equivalently a functor from the terminal category to \mathcal{A} , which suggests how to define representable functors $\mathcal{A} \rightarrow \widehat{\mathcal{C}}$ for general \mathcal{C} . Given a functor $S: \mathcal{C} \rightarrow \mathcal{A}$, we get a functor $\mathcal{A} \rightarrow \widehat{\mathcal{C}}$ given by $c \mapsto \mathrm{Hom}(Sc, -)$.

In algebra, one often encounters functors to \mathbf{Set} which are not representable but instead disjoint unions of representables, represented by a family of objects instead of just one.

Example 1.1. The free monoid functor $\mathbf{Set} \rightarrow \mathbf{Set}$ sends a set X to $* \sqcup X \sqcup X^2 \sqcup X^3 \sqcup \dots$. Each X^n is isomorphic to the set of functions $\mathrm{Hom}_{\mathbf{Set}}(\underline{n}, X)$, so this functor is represented by the family of sets $\{\underline{n}\}_{n \in \mathbb{N}}$. Each \underline{n} corresponds to the unique n -ary operation in a monoid; this operation has *arity* \underline{n} .

In the free monoid example, the representation of the functor consists of the set \mathbb{N} and a functor $\mathbb{N} \rightarrow \mathbf{Set}$, regarding \mathbb{N} as a discrete category. In higher dimensional algebra though, we encounter functors into $\widehat{\mathcal{C}}$ more general than disjoint unions of representables.

1.1 Categories as Algebras

Example 1.2. When $\mathcal{C} = \mathbf{G}_1$, the category $0 \overset{s}{\underset{t}{\rightrightarrows}} 1$, $\widehat{\mathcal{C}}$ is the category of directed graphs. For a graph X , $X_s: X_1 \rightarrow X_0$ identifies the source vertex of each edge and X_t identifies the target. We write $\overset{n}{\rightarrow}$ for the graph

$$\bullet \rightarrow \bullet \rightarrow \cdots \overset{n}{\rightarrow} \bullet \rightarrow \bullet$$

consisting of a single path of length n : $n+1$ vertices and n successive edges connecting them, for $n \geq 0$. When $n = 0, 1$, these “walking paths” include the single vertex and the single edge graphs.

The free category functor $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ sends X to the graph with the same vertices and an edge for every (finite, directed) path in X , including length 0 paths which consist of just a vertex. The set of paths in X of fixed length n is precisely $\text{Hom}(\overset{n}{\rightarrow}, X)$, and the set of all paths in X is then

$$\coprod_{n \in \mathbb{N}} \text{Hom}(\overset{n}{\rightarrow}, X).$$

However, the free category functor is not a disjoint union of representables, as paths of all lengths have the same original set of vertices as their sources and targets. But the vertex part of the functor is representable, as $X_0 \cong \text{Hom}(\overset{0}{\rightarrow}, X)$. This functor is then a disjoint union of representables only in each type of cell separately, and the structure maps are also representable: the source vertex of a length n path is the first vertex in the path, and the function $\text{Hom}(\overset{n}{\rightarrow}, X) \rightarrow \text{Hom}(\overset{0}{\rightarrow}, X)$ identifying this source is represented by the map of graphs from $\overset{0}{\rightarrow}$ to $\overset{n}{\rightarrow}$ picking out the first

vertex.

The data of the free category functor then amounts to the sets $\{\xrightarrow{n}\}_{n \in \mathbb{N}}$ and $\{\xrightarrow{0}\}$ of graphs and the source/target maps from $\xrightarrow{0}$ to \xrightarrow{n} . This can be described as a functor $S: \mathcal{C}^{op} \rightarrow \mathbf{Set}$ sending 0 to $\{0\}$ and sending 1 to \mathbb{N} , along with a functor $E: \int S \rightarrow \widehat{\mathcal{C}}$ sending n to \xrightarrow{n} and $s, t: 0 \rightarrow n$ to the inclusions of the source and target vertices in \xrightarrow{n} .

Functors $\widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ of this form are called *familiably representable*, or just *familial*, and describe a wide variety of freely generated higher category structures, as we discuss below. The functor $S: \mathcal{C}^{op} \rightarrow \mathbf{Set}$ describes the operations which output each cell type in \mathcal{C} , and $E: \int S \rightarrow \widehat{\mathcal{C}}$ identifies the arity of each operation, which for the composition of n arrows in a category is the graph \xrightarrow{n} . The equations such as unit and associativity laws for categories are then between operations of the same arity: the composite of an arrow with the composite of two more arrows, in either order, agrees with the operation composing three arrows all at once.

Our main result describes how to represent the data of a familial functor equipped with the structure of a cartesian monad, giving a direct and simplified method for defining new types of higher categories that fit this pattern.

1.2 Familial Functors

Definition 1.3. For $\mathcal{C}, \mathcal{C}'$ small categories, a *familial representation from \mathcal{C}' to \mathcal{C}* is a pair (S, E) where

- S is a functor $\mathcal{C}^{op} \rightarrow \mathbf{Set}$
- E is a functor $\int S \rightarrow \widehat{\mathcal{C}'}$

Associated to (S, E) is a functor $H_{(S,E)}: \widehat{\mathcal{C}'} \rightarrow \widehat{\mathcal{C}}$ given by, for X a presheaf over \mathcal{C}' ,

- $H_{(S,E)}(X)_c = \coprod_{t \in Sc} \text{Hom}_{\widehat{\mathcal{C}'}}(Et, X)$ for each $c \in \text{Ob}(\mathcal{C})$
- $H_{(S,E)}(X)_i: \coprod_{t \in Sc} \text{Hom}_{\widehat{\mathcal{C}'}}(Et, X) \rightarrow \coprod_{t' \in S_{c'}} \text{Hom}_{\widehat{\mathcal{C}'}}(Et', X)$ for $i: c' \rightarrow c$ is given by

$$E(i_t)^*: \text{Hom}_{\widehat{\mathcal{C}'}}(Et, X) \rightarrow \text{Hom}_{\widehat{\mathcal{C}'}}(E(Si(t)), X).$$

We say that $t \in Sc$ is an *operation* with *output* c and *arity* Et , as every diagram of shape Et in X contributes a c -cell to $H_{(S,E)}(X)$. Note that we treat S as a functor rather than a presheaf over \mathcal{C} , as we prefer to think of presheaves as geometric objects while S plays more of a bookkeeping role, tracking the relationships between the various operations. That said, as a presheaf S is isomorphic to $H_{(S,E)}(*)$.

Definition 1.4. A functor $T: \widehat{\mathcal{C}'} \rightarrow \widehat{\mathcal{C}}$ is *familially representable*, or simply *familial*, if it is naturally isomorphic to a functor of the form $H_{(S,E)}$, where (S, E) is called a *familial representation of T* . We write $\mathbf{Fam}(\widehat{\mathcal{C}'}, \widehat{\mathcal{C}})$ for the category of familial functors from $\widehat{\mathcal{C}'}$ to $\widehat{\mathcal{C}}$ and cartesian natural transformations between them.

Remark 1.5. Familial functors into **Set** were introduced in [12] as “locally representable functors”. They are first called “familiably representable” in [21]. In [28], familiably representable functors $\mathcal{A} \rightarrow \widehat{\mathcal{C}}$ are defined using a slight variation on the notion of familial representations: instead of a pair $(S, E: \int S \rightarrow \mathcal{A})$, they are equivalently represented by a functor from \mathcal{C}^{op} into a category of “families” of objects in \mathcal{A} . In [34], functors which admit “strict generic factorizations” are (nontrivially) equivalent to familial functors in the setting of presheaf categories, with (S, E) called the “spectrum” and “exponent” of a “parametric representation” of a functor. In [35], these functors are called “parametric right adjoints” or “p.r.a. functors.” Most of this overlapping terminology remains in current use, so we choose “familial functors” and “familial representations” as the most suitable for our purposes.

We now describe morphisms of familial representations, which will correspond precisely to cartesian natural transformations of the associated familial functors.

Definition 1.6. For $(S, E), (S', E')$ familial representations from \mathcal{C}' to \mathcal{C} , a morphism $\phi: (S, E) \rightarrow (S', E')$ consists of a morphism $\phi^S: S \rightarrow S'$ in $\widehat{\mathcal{C}}$ and a natural isomorphism ϕ^E

$$\begin{array}{ccc}
 \int S & \xrightarrow{\int \phi^S} & \int S' \\
 & \searrow E & \swarrow E' \\
 & \xrightarrow{\phi^E} & \\
 & \swarrow & \searrow \\
 & \widehat{\mathcal{C}} &
 \end{array}$$

We write $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$ for the category of familial representations from \mathcal{C}' to \mathcal{C} and morphisms of this form between them.

Proposition 1.7. *For small categories $\mathcal{C}', \mathcal{C}$, the assignment $(S, E) \mapsto H_{(S, E)}$ extends to an equivalence of categories $H: \mathbf{Rep}(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{Fam}(\widehat{\mathcal{C}'}, \widehat{\mathcal{C}})$.*

This is proven in [19, Proposition 3.8] for the equivalent notion of pointwise familial functors, but for clarity we give a proof here as well. The ideas in this proof are closely related to [34, Theorem 7.6].

Proof. The functor H sends a morphism $\phi: (S, E) \rightarrow (S', E')$ to the natural transformation given on c -cells ($c \in \text{Ob}(\mathcal{C})$) by

$$\coprod_{t \in S_c} \text{Hom}_{\widehat{\mathcal{C}}'}(Et, -) \rightarrow \coprod_{t' \in S'_c} \text{Hom}_{\widehat{\mathcal{C}}'}(E't', -)$$

mapping $\text{Hom}_{\widehat{\mathcal{C}}'}(Et, -)$ to $\text{Hom}_{\widehat{\mathcal{C}}'}(E'\phi^S(t), -)$ by precomposition with the isomorphism

$$\phi_t^E: Et \cong E'\phi^S(t).$$

This assignment is natural in c precisely because ϕ^E is natural in t .

H is essentially surjective by Definition 1.4, so it remains only to show it is fully faithful. First, we observe that given any cartesian natural transformation $\psi: H_{(S,E)} \rightarrow H_{(S',E')}$ we have for each presheaf X over \mathcal{C}' the following diagram, natural in c , where the vertical maps are given on each component by postcomposition with the unique map $X \rightarrow *$, the upper square is a naturality pullback square, and ψ_c is the unique map making the lower square commute.

$$\begin{array}{ccc} \coprod_{t \in S_c} \text{Hom}_{\widehat{\mathcal{C}}'}(Et, X) & \longrightarrow & \coprod_{t' \in S'_c} \text{Hom}_{\widehat{\mathcal{C}}'}(E't', X) \\ \downarrow & \lrcorner & \downarrow \\ \coprod_{t \in S_c} \text{Hom}_{\widehat{\mathcal{C}}'}(Et, *) & \longrightarrow & \coprod_{t' \in S'_c} \text{Hom}_{\widehat{\mathcal{C}}'}(E't', *) \\ \cong \downarrow & & \downarrow \cong \\ S_c & \xrightarrow{\psi_c} & S'_c \end{array}$$

As the composite square is also a pullback, the top map above restricts to an isomorphism $\text{Hom}_{\widehat{\mathcal{C}}'}(Et, X) \rightarrow \text{Hom}_{\widehat{\mathcal{C}}'}(E'\psi_c(t))$ for each $t \in S_c$, natural in X and t (as an object of fS). By the Yoneda lemma, this isomorphism must be given by precomposition with an isomorphism $\psi_t: E'\psi_c(t) \cong Et$, natural in t . We can then define a morphism of representations $H^{-1}(\psi): (S, E) \rightarrow (S', E')$ with $H^{-1}(\psi)_c^S = \psi_c$ and $H^{-1}(\psi)_t^E = \psi_t^{-1}$, and it is straightforward to check that the assignments

$$H: \mathbf{Rep}(\mathcal{C}', \mathcal{C})\left((S, E), (S', E')\right) \rightleftharpoons \mathbf{Fam}(\widehat{\mathcal{C}}', \widehat{\mathcal{C}})\left(H_{(S, E)}, H_{(S', E')}\right) : H^{-1}$$

are inverse to one another, completing the proof that H is fully faithful. \square

We now describe how familial functors form a sub-2-category **Fam** of **CAT**.

Proposition 1.8. *Categories of the form $\widehat{\mathcal{C}}$ for small \mathcal{C} , familial functors, and cartesian natural transformations form a 2-category.*

Proof. It suffices to show that familial functors are closed under identities and composites; the corresponding properties of cartesian transformations follow immediately, noting that familial functors preserve pullbacks ([34, Theorem 8.1]).

That the identity functor is familial is a consequence of the Yoneda lemma, as for each X in $\widehat{\mathcal{C}}$ we have $X_c \cong \text{Hom}(y(c), X)$, for each object c in \mathcal{C} . Given two familial functors $F: \widehat{\mathcal{C}}' \rightarrow \widehat{\mathcal{C}}$ and $G: \widehat{\mathcal{C}}'' \rightarrow \widehat{\mathcal{C}}'$ represented by (S, E) and (S', E') respectively, [19, Propositions 3.11, 3.12] show that

$$FG(X)_c \cong \coprod_{\substack{t \in S_c, \\ f: Et \rightarrow S'}} \text{Hom}_{\widehat{\mathcal{C}}''}\left(\text{colim}_{x: y(c') \rightarrow Et} E'f(x), X\right),$$

which can also be shown directly using basic properties of limits and colimits. \square

These representations for identities and composites are discussed in more detail in Section 4, where they are shown to form the identities and composites of the bicategory \mathbf{Rep} whose morphism categories are given by $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$.

Definition 1.9. The familial representation of $\text{id}_{\widehat{\mathcal{C}}}$ is given by (S^0, E^0) , where $S_c^0 = \{*_c\}$ be the terminal functor and $E^0: \int S^0 \cong \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}}$.

Definition 1.10. For F, G familial functors as above represented by $(S, E), (S', E')$, the familial representation of FG is given by (SS', EE') , where

$$SS'_c = \coprod_{t \in S_c} \text{Hom}_{\widehat{\mathcal{C}}}(Et, S) \quad \text{and} \quad EE'(t, f) = \text{colim}_{x: y(c') \rightarrow Et} E'f(x).$$

Definition 1.11. A *familial monad* is a monad on a presheaf category $\widehat{\mathcal{C}}$ whose functor part is familial and whose unit and multiplication transformations are cartesian.

A familial monad is the same as a formal monad in the 2-category \mathbf{Fam} , and this description will facilitate our characterization of familial monads in Theorem 5.1.

Familial monads are of interest for their precision in describing algebraic structures on categories $\widehat{\mathcal{C}}$ of presheaves with operations taking as input an “arity diagram,” such as strings of composable edges in a graph, and outputting a single cell, like the composite arrow in a category. These kinds of algebras include most familiar higher category structures, whose operations typically encode the structure of unit cells, composition of cells, or symmetries where the various “sources” and/or “targets” of a cell can be permuted to form a new cell.

The unit and multiplication transformations are expected to be cartesian to encode (as in the proof of Proposition 1.7) that the equations in these algebraic structures are always between operations with the same arity. For instance, associativity for categories asserts that any binary parenthesization of the same string of arrows has the same total composite; this is an equation between two potentially different operations with the same arity diagram.

1.3 Examples of familial monads

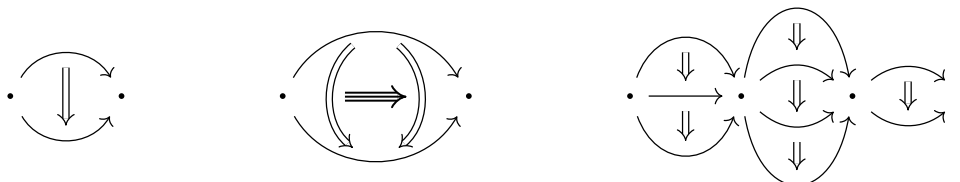
Throughout this work, we will exhibit each construction for each of the following fundamental higher category theories: monoids, ordinary categories, n -categories, multicategories, and double categories. Familial representations for the free monoid monad on sets (Example 1.1) and free category monad on graphs (Example 1.2, also [28, Example C.3.3]) are described above, and so we now provide the same for the free n -category, multicategory, and double category monads.

Example 1.12. (See also [28, Proposition F.2.3].) n -categories are algebras for a familial monad on n -globular sets, which are presheaves over

$$\mathbf{G}_n = 0 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \cdots \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} n$$

with $s \circ s = t \circ s$ and $s \circ t = t \circ t$ at each level. For an n -globular set X , we call the elements of X_n n -cells, which for $n = 0, 1$ look like a vertex and edge, respectively,

and for $n = 2, 3$ look like those below left.



The operations and arities of the free n -category monad outputting an m -cell correspond to the m -dimensional free globular pasting diagrams of composable n -cells, which are a single vertex for $m = 0$, the strings \xrightarrow{k} of composable arrows for $m = 1$, and for $m = 2$ include diagrams such as that above right.

Globular sets are the arbitrarily-high-dimensional analogue of n -globular sets, defined as diagrams over the *glob category*

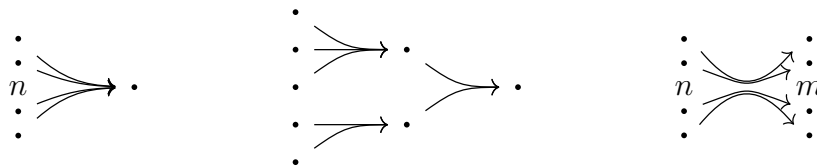
$$\mathbf{G} = 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} n \underset{t}{\overset{s}{\rightrightarrows}} \cdots ,$$

which is equivalently the colimit of the categories \mathbf{G}_n with the evident inclusions. There is similarly a familial monad on globular sets whose algebras are strict ω -categories, with operations outputting an n -cell for all n -dimensional pasting diagrams ([28, Proposition F.2.3]).

Example 1.13. (See also [35, Example 2.14].) Multicategories (symmetric or non-symmetric) are algebras for a familial monad on *multigraphs*¹. Multigraphs are presheaves over the category \mathbf{M} with objects $0, (0, 1), (1, 1), (2, 1), \dots$ and morphisms $s_1, \dots, s_n, t: 0 \rightarrow (n, 1)$ for each $n \geq 0$ with no nontrivial compositions. Cell diagrams

¹We use the word “multigraph” in analogy with “multicategory,” not to mean a graph that can have multiple edges between the same two vertices, which we call simply a “graph.”

X over \mathbf{M} look like graphs with vertices in X_0 and n -to-1 edges as below left in $X_{(n,1)}$ with n sources determined by the functions X_{s_i} and a single target determined by X_t .

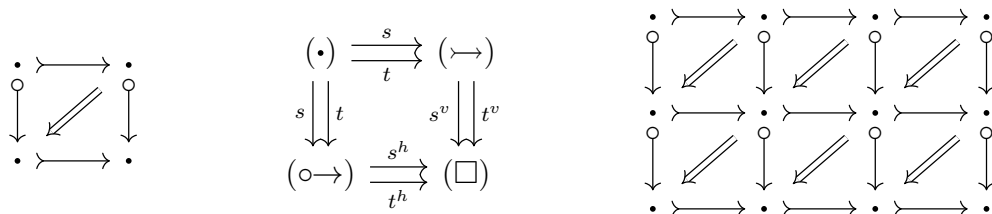


The familial functor of the free non-symmetric multicategory monad has operations and arities corresponding to any “tree” as above center, an arbitrarily large finite composable diagram of these many-to-one edges given by gluing together a source vertex of one such “multi-edge” to the target vertex of another. Each tree provides an operation outputting an n -to-1 edge where n is the number of leaves of the tree (unattached source vertices). These trees describe the possible composites of “multi-arrows” in multicategories, which are precisely the algebras for the monad. As in the free n -category monads, there is only a single operation outputting a vertex with arity the vertex, as there are no non-trivial operations on vertices in a multicategory.

There are also familial monads for variations on the notion of multicategory. The free symmetric multicategory monad on multigraphs has operations given by pairs of a tree as above and a choice of ordering on its leaves, with the underlying tree as the arity. There are also free polycategory and free properad monads on *polygraphs*, whose cells are vertices and n -to- m arrows as above right (see [19, Proposition 2.9]).

Example 1.14. A double graph is a diagram of vertices, two distinct types of edges

drawn as \rhd (“horizontal edges”) and $\circ \rightarrow$ (“vertical edges”), and filled-in squares as below left. Double graphs are precisely presheaves over the category below center with four objects corresponding to vertices, both types of edges, and squares, and relations $s^h s = s^v s$, $s^h t = t^v s$, $t^h s = s^v t$, $t^h t = t^v t$ corresponding to each vertex of the square, or equivalently the cartesian product $\mathbf{G}_1 \times \mathbf{G}_1$.



Double categories (see [15]) are algebras for a familial monad on double graphs whose operations and arities on horizontal and vertical arrows, respectively, look like those for arrows in the free category monad: a single operation for composing a string of n arrows of the same type. For squares, there is for each $n, m \in \mathbb{N}$ a single composition operation with arity the $n \times m$ grid of squares as above right.

An algebra for this monad thus has both its horizontal and vertical arrows form categories, and additionally has horizontal and vertical compositions of squares which satisfy the usual unit, associativity, and interchange equations as any composite of those operations must agree with the unique operation with the relevant grid as its arity. We describe how to formally specify these equations in the data of such a monad in Example 5.5.

2 Grothendieck Constructions

Before discussing polynomials in \mathbf{Cat} built out of Grothendieck fibrations and opfibrations, we review the relevant definitions and provide convenient constructions of pullbacks, composites, and distributivity pullbacks of opfibrations, then establish a basic functoriality result for two sided fibrations.

2.1 Operations on Opfibrations

Grothendieck fibrations and opfibrations are usually defined as functors between categories satisfying certain lifting properties allowing them to be equivalently defined in terms of their fibers over each object and morphism in the codomain. For convenience, we use this “Grothendieck correspondence” between such functors and their fibers, usually considered a theorem, as our definition of (op)fibrations:

Definition 2.1. A functor $p: \mathcal{A} \rightarrow \mathcal{B}$ is an *opfibration* if it is (up to isomorphism of the domain) of the form $p_\Phi: \mathcal{J}\Phi \rightarrow \mathcal{B}$ for some $\Phi: \mathcal{B} \rightarrow \mathbf{Cat}$, where $\mathcal{J}\Phi$ is the following category:

- Objects are pairs (b, x) for $b \in \text{Ob } \mathcal{B}$, $x \in \text{Ob } \Phi b$
- Morphisms are pairs $(i_0, i_1): (b, x) \rightarrow (b', x')$ for $i_0: b \rightarrow b'$ in \mathcal{B} , $i_1: \Phi(i_0)(x) \rightarrow x'$ in $\Phi b'$
- The identity at (b, x) is given by $(\text{id}_b, \text{id}_x)$

- Composites are given by $(i'_0, i'_1) \circ (i_0, i_1) = (i'_0 \circ i_0, i'_1 \circ \Phi(i'_0)(i_1))$
- The functor $p_\Phi: \mathcal{f}\Phi \rightarrow \mathcal{B}$ sends (b, x) to b and (i_0, i_1) to i_0

For an opfibration p , written $p: \mathcal{A} \rightarrow \mathcal{B}$, we write Φ_p for the corresponding functor $\mathcal{B} \rightarrow \mathbf{Cat}$, where $\Phi_p(b)$ is (up to isomorphism) the fiber of p over the object b in \mathcal{B} .

Dually, *fibrations* $\mathcal{A} \rightarrow \mathcal{B}$ are those functors corresponding to an analogous construction for functors $\mathcal{B}^{op} \rightarrow \mathbf{Cat}$, and *discrete fibrations* correspond to functors $\mathcal{B}^{op} \rightarrow \mathbf{Set} \hookrightarrow \mathbf{Cat}$, precisely the categories of elements for functors in $\widehat{\mathcal{B}}$, hence the similar notation f .

(Op)fibrations are often taken to be a more general class of functors corresponding instead to pseudofunctors $\mathcal{B} \rightarrow \mathbf{Cat}$, with those corresponding to strict functors called “split opfibrations.” For our purposes we can restrict to split opfibrations, though it is straightforward to extend the constructions on split opfibrations below to the more general setting. For a more thorough account of fibrations and opfibrations see [20] or [32, Appendix].

Remark 2.2. $\mathcal{f}\Phi$ can be seen as the “lax colimit” of $\Phi: \mathcal{B} \rightarrow \mathbf{Cat}$, in the sense that it is initial among categories with a lax cocone from Φ (see [32, Remark 2.13]). In this case the lax cocone is given by the functors $J_b: \Phi(b) \hookrightarrow \mathcal{f}\Phi: x \mapsto (b, x)$ for each object b in \mathcal{B} and natural transformations $J_i: J_b \Rightarrow J_{b'} \circ \Phi(i)$ for each morphism $i: b \rightarrow b'$ in \mathcal{B} with x component given by

$$(i, \text{id}): (b, x) \rightarrow (b', \Phi(i)(x)).$$

As lax cocones generalize strict cocones, this universal property provides a canonical functor $Q_\Phi: \mathcal{J}\Phi \rightarrow \text{colim}(\Phi)$ from the lax colimit to the strict colimit of Φ in **Cat** (see [32, Example 4.8]).

While the following propositions are usually proven using equivalent definitions of opfibrations in terms of lifting properties ([20, Proposition 3.1]), our main result relies on a more explicit description of pullbacks and composites of opfibrations in terms of **Cat**-valued functors. These constructions are widely understood but do not appear in the literature.

Proposition 2.3. *Opfibrations are closed under pullback.*

Proof. Consider the diagram below left in **Cat**:

$$\begin{array}{ccc} & \mathcal{C} & \mathcal{D} \xrightarrow{\psi} \mathcal{C} \\ & \downarrow f & \phi \downarrow \quad \quad \downarrow u \\ \mathcal{A} \xrightarrow{p} \mathcal{B} & & \mathcal{A} \xrightarrow{p} \mathcal{B} \end{array}$$

We show that the opfibration $p_{(\Phi_p \circ u)}: \mathcal{J}(\Phi_p \circ u) \rightarrow \mathcal{C}$ is the pullback of p along u . For any pair of functors ϕ, ψ commuting as above right, we define $\gamma: \mathcal{D} \rightarrow \mathcal{J}(\Phi_p \circ u)$ by

$$d \mapsto (\psi(d), \phi(d)), \quad (i: d \rightarrow d') \mapsto (\psi(i), \phi(i))$$

where $\psi(d) \in \mathcal{C}$ and $\phi(d) = (b, x)$ for

$$b = p\phi(d) = u\psi(d) \in \text{Ob } \mathcal{B}, \quad x \in \text{Ob } \Phi_p(u\psi(d)).$$

$\psi(i): \psi(d) \rightarrow \psi(d')$ in \mathcal{C} , and $\phi(i) = (i_0, i_1): (b, x) \rightarrow (b', x')$ for

$$i_0 = p\phi(i) = u\psi(i) \text{ in } \mathcal{B}, \quad i_1: \Phi_p(i_0)(x) \rightarrow x' \text{ in } \Phi_p(u\psi(d')).$$

Clearly $p(\Phi_p \circ u) \circ \gamma = \psi$, and likewise ϕ factors as γ followed by the functor

$$\mathfrak{f}u: \mathfrak{f}(\Phi_p \circ u) \rightarrow \mathfrak{f}\Phi_p \cong \mathcal{A}, \quad (c \in \mathcal{C}, x \in \Phi_p(u(c))) \mapsto (u(c), x), \quad (j_0, j_1) \mapsto (u(j_0), j_1).$$

□

Recall the *lax overcategory* \mathbf{Cat}/\mathcal{C} with objects small categories over \mathcal{C} and morphisms diagrams as below in \mathbf{Cat} :

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \rightrightarrows & \\ & \searrow & \swarrow \\ & \mathcal{C} & \end{array}$$

We will also sometimes consider \mathbf{Cat}/\mathcal{C} for \mathcal{C} a large category, the *lax slice* over \mathcal{C} of the inclusion $\mathbf{Cat} \rightarrow \mathbf{CAT}$ of small into large categories, though for our purposes it will not make much difference whether \mathcal{C} is large or small.

Remark 2.4. The assignment $u \mapsto \mathfrak{f}u$ in the proof above extends the assignment \mathfrak{f} into a functor $\mathbf{Cat}/\mathbf{Cat} \rightarrow \mathbf{Cat}$. However, given functors

$$u: \mathcal{C} \rightarrow \mathcal{B}, \quad \Phi: \mathcal{C} \rightarrow \mathbf{Cat}, \quad \Phi': \mathcal{B} \rightarrow \mathbf{Cat},$$

a functor $\mathfrak{f}\Phi \rightarrow \mathfrak{f}\Phi'$ commuting with u corresponds to functors $\phi_c: \Phi(c) \rightarrow \Phi'(u(c))$ which are *lax natural* in c , meaning they are equipped with natural transformations $\phi_i: \Phi'(u(i))\phi_c \Rightarrow \phi_{c'}\Phi(i)$ for each $i: c \rightarrow c'$, functorial in i . Given such data, we can define

$$\mathfrak{f}(u, \phi): \mathfrak{f}\Phi \rightarrow \mathfrak{f}\Phi', \quad (c, x) \mapsto (u(c), \phi_c(x)),$$

sending a map $(i_0: c \rightarrow c', i_1: \Phi(i_0)(x) \rightarrow x')$ to

$$(u(i_0), \Phi'(u(i_0))(\phi_c(x)) \xrightarrow{\phi_{i_0, x}} \phi_{c'}(\Phi(i_0)(x)) \xrightarrow{\phi_{c'}(i_1)} \phi_{c'}(x')),$$

and it is straightforward to check that every such functor arises in this way. When ϕ is strictly natural, this shows that \mathcal{f} extends to a functor $\mathbf{Cat}/\mathbf{Cat} \rightarrow \mathbf{Cat}$

Proposition 2.5. *Opfibrations are closed under composition.*

Proof. Consider two opfibrations $p: \mathcal{A} \twoheadrightarrow \mathcal{B}$, $q: \mathcal{B} \twoheadrightarrow \mathcal{C}$. We define a functor $\Phi: \mathcal{C} \rightarrow \mathbf{Cat}$ and show that the corresponding opfibration agrees with pq . For an object c of \mathcal{C} , let

$$\Phi(c) = \mathcal{f} \left(\Phi_q(c) \xrightarrow{J_c} \mathcal{B} \xrightarrow{\Phi_p} \mathbf{Cat} \right).$$

For a morphism $i: c \rightarrow c'$, we have a morphism

$$\begin{array}{ccc} \Phi_q(c) & \xrightarrow{\Phi_q(i)} & \Phi_q(c') \\ & \begin{array}{c} \xrightarrow{J_i} \\ \searrow J_c \quad \swarrow J_{c'} \end{array} & \\ & \mathcal{B} & \\ & \begin{array}{c} \downarrow \Phi_p \\ \mathbf{Cat} \end{array} & \end{array}$$

in $\mathbf{Cat}/\mathbf{Cat}$. This assignment evidently respects identities and composites, so as \mathcal{f} is functorial over $\mathbf{Cat}/\mathbf{Cat}$, Φ defines a functor $\mathcal{C} \rightarrow \mathbf{Cat}$.

It remains to show that $\mathcal{f}\Phi$ agrees with pq . We define an isomorphism $\mathcal{A} \cong \mathcal{f}\Phi_p \cong \mathcal{f}\Phi$ over \mathcal{C} by, for $i_0: c \rightarrow c'$ in \mathcal{C} , $i_1: \Phi_q(i_0)(x) \rightarrow x'$ in $\Phi_q(c')$, and $i_2: \Phi_p(i_0, i_1)(y) \rightarrow y'$ in $\Phi_p(x')$,

$$\left((c, x), y \right) \mapsto \left(c, (x, y) \right), \quad \left((i_0, i_1), i_2 \right) \mapsto \left(i_0, (i_1, i_2) \right).$$

□

Recall that for a fixed morphism $p: A \rightarrow B$ in a category \mathcal{A} with pullbacks, there are functors $\Sigma_p: \mathcal{A}/A \rightleftarrows \mathcal{A}/B: \Delta_p$, where Δ_p is defined by (choices of) pullback along p and its left adjoint Σ_p by postcomposition with p . $p: A \rightarrow B$ in \mathcal{A} is *exponentiable* if Δ_p also has a right adjoint $\Pi_p: \mathcal{A}/A \rightarrow \mathcal{A}/B$.

Weber shows ([36, Section 2.2]) that p is exponentiable if for all maps $u: X \rightarrow A$, there exists a terminal pullback square among those of the form:

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \Pi_p u \\ X & & B \\ u \downarrow & & \downarrow \\ A & \xrightarrow{p} & B \end{array}$$

This square is called a *distributivity pullback*, and given a choice of distributivity pullbacks $\Pi_p u$ is defined as the map $Y \rightarrow B$ in \mathcal{A}/B .

Proposition 2.6. *Opfibrations are exponentiable in \mathbf{Cat} .*

This was proven in [22, Corollary 6.2] using lifting properties, but as above we construct Π_p explicitly in terms of the functor Φ_p associated to an opfibration p .

Proof. Given $u: \mathcal{X} \rightarrow \mathcal{A}$, $p: \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{Cat} , we construct the following distributivity square:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} \twoheadrightarrow & \mathcal{Y} \\ w \downarrow & & \downarrow v \\ \mathcal{X} & & \mathcal{B} \\ u \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{p} \twoheadrightarrow & \mathcal{B} \end{array}$$

An object in \mathcal{Y} is a pair $(b \in \text{Ob } \mathcal{B}, f: \Phi_p(b) \rightarrow X)$ with $uf = J_b$, and a morphism $(b, f) \rightarrow (b', f')$ is a pair $(i: b \rightarrow b', \sigma: f \Rightarrow f'\Phi(i))$ with $u\sigma = J_i$. The functor v sends (b, f) to b and (i, σ) to i .

q is the pullback of p along v given by Proposition 2.3. Objects of \mathcal{Z} are then triples $(b, f, x \in \text{Ob } \Phi_p(b))$, and morphisms of \mathcal{Z} amount to triples (i, σ, j) where $j: \Phi_p(i)(x) \rightarrow x'$ in $\Phi_p(b')$. The functor $w: \mathcal{Z} \rightarrow \mathcal{X}$ sends (b, f, x) to $f(x)$ and $(i, \sigma, j): (b, f, x) \rightarrow (b', f', x')$ to $f'(j) \circ \sigma_x$. uw agrees with the projection map from Proposition 2.3 as each f is a partial section of u .

To see that this square is terminal, consider a pullback $q': \mathcal{Z}' \rightarrow \mathcal{Y}'$ as below:

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{q'} & \mathcal{Y}' \\ w' \downarrow & & \downarrow v' \\ \mathcal{X} & & \mathcal{B} \\ u \downarrow & \xrightarrow{p} & \downarrow \\ \mathcal{A} & & \mathcal{B} \end{array}$$

There is a functor $k: \mathcal{Y}' \rightarrow \mathcal{Y}$ over \mathcal{B} sending c in \mathcal{Y} to

$$\Phi_p(v'(c)) \cong \Phi_q(c) \xrightarrow{J_c} \mathcal{Z}' \xrightarrow{w'} X,$$

which is a partial section of u as uw' agrees with the projection functor from Proposition 2.3, and defined similarly on morphisms. There is also a functor $\ell: \mathcal{Z}' \rightarrow \mathcal{Z}$ over X sending $(c, x \in \text{Ob } \Phi_p(v'(c)))$ to $(v'(c), k(c), x)$, so that $q\ell = kq'$, and (ℓ, k) are unique with respect to these properties. \square

Remark 2.7. If u is a fibration, then for $i: b \rightarrow b'$ there are functors

$$\text{Fun}_{/\mathcal{A}}(\Phi_p(b'), X) \rightarrow \text{Fun}_{/\mathcal{A}}(\Phi_p(b), X)$$

exhibiting $\Pi_p u$ as the Grothendieck construction for fibrations of the functor

$$\mathcal{B}^{op} \rightarrow \mathbf{Cat}: b \rightarrow \mathit{Fun}/_{\mathcal{A}}(\Phi_p(b), X),$$

and if u is a discrete fibration then each category $\mathit{Fun}/_{\mathcal{A}}(\Phi_p(b), X)$ is discrete, so Π_p preserves (discrete) fibrations. Δ_p also preserves (discrete) fibrations, as does Σ_p when p is a (discrete) fibration.

2.2 Two Sided Fibrations

In the polynomials of the following sections, we consider opfibrations $\mathcal{A} \rightarrow \mathcal{B}$ whose domain is equipped with a functor to another category \mathcal{C} , which is made up of compatible fibrations from each fiber. For $\Phi: \mathcal{B} \rightarrow \mathbf{Cat}$, the data of a functor $\mathcal{A} \rightarrow \mathcal{C}$ is precisely that of an extension of Φ to \mathbf{Cat}/\mathcal{C} . We will be interested in the case when this $\Phi: \mathcal{B} \rightarrow \mathbf{Cat}/\mathcal{C}$ factors through the category $\mathit{Fib}(\mathcal{C})$ (or $\mathit{DFib}(\mathcal{C})$) of categories with a (discrete) fibration to \mathcal{C} and functors which correspond to strict natural transformations in $\mathit{Fun}(\mathcal{C}^{op}, \mathbf{Cat})$. In $\mathit{DFib}(\mathcal{C})$ these are all functors which commute strictly over \mathcal{C} , and in $\mathit{Fib}(\mathcal{C})$ these are the functors over \mathcal{C} which preserve the cartesian morphisms.

Definition 2.8. A *(discrete) two sided fibration* from \mathcal{C} to \mathcal{B} is a diagram of the form

$$\mathcal{C} \xleftarrow{p_1} \mathcal{A} \xrightarrow{p_2} \mathcal{B},$$

where p_2 is an opfibration such that $\Phi_p: \mathcal{B} \rightarrow \mathbf{Cat}$ is equipped with a lift along the

forgetful functor $Fib(\mathcal{C}) \rightarrow \mathbf{Cat}$ (resp. $DFib(\mathcal{C}) \rightarrow \mathbf{Cat}$). Abusing notation slightly, we also write $\Phi_p: \mathcal{B} \rightarrow Fib(\mathcal{C})$ for this lift.

Remark 2.9. This definition agrees with the existing notion of (split) two-sided fibration (see [29, Definition 2.3.4]), as $Fib(\mathcal{C}) \simeq Fun(\mathcal{C}^{op}, \mathbf{Cat})$ and therefore so $Fun(\mathcal{B}, Fib(\mathcal{C})) \simeq Fun(\mathcal{C}^{op} \times \mathcal{B}, \mathbf{Cat})$ as in the standard definition. This shows that two-sided fibrations could be defined dually to the above as a fibration over \mathcal{C} with fibers opfibered over \mathcal{B} , and in particular that p_1 in the above definition is a fibration.

The following will be useful for composing polynomials built from two-sided fibrations.

Proposition 2.10. \mathcal{f} extends to a functor $\mathbf{Cat} // Fib(\mathcal{C}) \rightarrow Fib(\mathcal{C})$.

Proof. As p_1 above is a fibration, it remains only to show that given

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{u} & \mathcal{B}' \\
 \searrow \Phi & \xrightarrow{\phi} & \swarrow \Phi' \\
 & Fib(\mathcal{C}) &
 \end{array}$$

the map $\mathcal{f}(u, \phi): \mathcal{f}\Phi \rightarrow \mathcal{f}\Phi'$ of Remark 2.4 commutes over \mathcal{C} and preserves cartesian morphisms. This follows from the same property of the functors $\phi_b: \Phi(b) \rightarrow \Phi'(u(b))$, as the cocartesian maps on both sides are sent to identities in \mathcal{C} and all of the cartesian morphisms in $\mathcal{f}\Phi$ are contained in some $\Phi(b)$ (by the same property of its objects). \square

We will primarily be interested in obtaining discrete two sided fibrations, which

in addition to naturally arising functors $\mathcal{B} \rightarrow DFib(\mathcal{C}) \simeq \widehat{\mathcal{C}}$ can also be obtained from general two sided fibrations:

Example 2.11. Recall the functor $|-|_{\mathcal{C}}: Fib(\mathcal{C}) \rightarrow DFib(\mathcal{C})$, left adjoint to the inclusion $DFib(\mathcal{C}) \rightarrow Fib(\mathcal{C})$ and sending a fibration u over \mathcal{C} to the discrete fibration whose fibers are the sets of connected components of the fibers of u . $|-|$ is equivalently given by $(\pi_0)_*: Fun(\mathcal{C}^{op}, \mathbf{Cat}) \rightarrow Fun(\mathcal{C}^{op}, \mathbf{Set})$, left adjoint to post-composition with the inclusion $\mathbf{Set} \rightarrow \mathbf{Cat}$.

Given $\Phi: \mathcal{B} \rightarrow Fib(\mathcal{C})$ corresponding to the two sided fibration p , we obtain $|\Phi|: \mathcal{B} \rightarrow Fib(\mathcal{C}) \rightarrow DFib(\mathcal{C})$, and denote the corresponding discrete two sided fibration by $|p|$. The natural unit map $u \rightarrow |u|$ in $Fib(\mathcal{C})$ induces a map $\Phi \rightarrow |\Phi|$ and accordingly $p \rightarrow |p|$, where in the latter the map $\pi: \mathcal{A} \rightarrow |\mathcal{A}|$ sends each element of the (intersection) fiber over (c, b) to its connected component.

3 Fibrous Polynomials in Cat

We now describe a bicategory of “very fibrous polynomials” in \mathbf{Cat} , along with other convenient properties of polynomials which help facilitate the comparison of familial representations and familial functors.

3.1 Special Classes of Polynomials

Recall from [36] that a polynomial p in \mathbf{Cat} from \mathcal{C}' to \mathcal{C} is a diagram as below such that p_2 is exponentiable:

$$\begin{array}{ccc} & \mathcal{A} & \xrightarrow{p_2} & \mathcal{B} & \\ & \swarrow p_1 & & \searrow p_3 & \\ \mathcal{C}' & & & & \mathcal{C} \end{array}$$

Definition 3.1. A polynomial p in \mathbf{Cat} is:

- *fibrous* if p_2 is an opfibration
- *very fibrous* if p_2 is an opfibration and p_3 is a discrete fibration
- *quasi-familial* if (p_1, p_2) is a two sided fibration and p_3 is a discrete fibration
- *familial* if (p_1, p_2) is a discrete two sided fibration and p_3 is a discrete fibration

These properties form a hierarchy: familial \implies quasi-familial \implies very fibrous \implies fibrous. The opfibration p_2 corresponds to a functor $\Phi_p: \mathcal{B} \rightarrow \mathbf{Cat}$, which we also use to denote the functor:

- $\Phi_p: \mathcal{B} \rightarrow \mathbf{Cat} // \mathcal{C}'$ if p is (very) fibrous
- $\Phi_p: \mathcal{B} \rightarrow \mathit{Fib}(\mathcal{C}')$ if p is quasi-familial
- $\Phi_p: \mathcal{B} \rightarrow \mathit{DFib}(\mathcal{C}') \simeq \widehat{\mathcal{C}'}$ if p is familial

Recall ([36, Section 3.2]) that given any polynomial p in \mathbf{Cat} , its associated polynomial functor $P(p)$ is the composite

$$\mathbf{Cat}/\mathcal{C}' \xrightarrow{\Delta_{p_1}} \mathbf{Cat}/\mathcal{A} \xrightarrow{\Pi_{p_2}} \mathbf{Cat}/\mathcal{B} \xrightarrow{\Sigma_{p_3}} \mathbf{Cat}/\mathcal{C}.$$

If p_2 is an opfibration then by Proposition 2.6 we have, for X a category over \mathcal{C}' , $\Pi_{p_2} \Delta_{p_1} X$ is the category with:

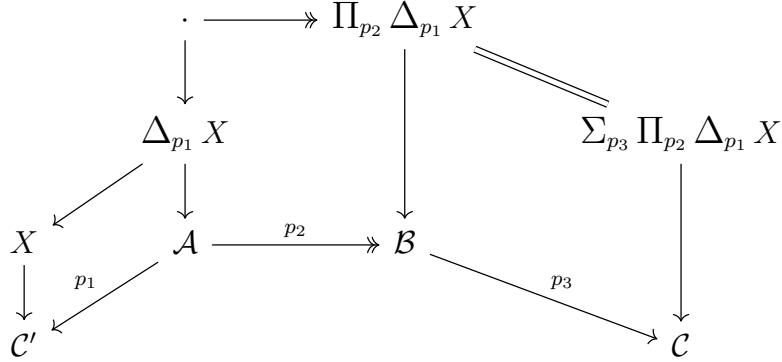
- objects pairs $(b \in \text{Ob } \mathcal{B}, f: \Phi_p(b) \rightarrow X)$ with f commuting over \mathcal{C}
- morphisms pairs $(i: b \rightarrow b', \sigma: f \Rightarrow f' \Phi_p(i)): (b, f) \rightarrow (b', f')$, with σ lying over the canonical transformation from $\Phi_p(b)$ to $\Phi_p(b')$ in \mathcal{C}'

When p is very fibrous, each of these components of $P(p)$ preserves discrete fibrations: discrete fibrations are closed under pullback, exponentiation along an opfibration by Remark 2.7, and composition with a discrete fibration. As discrete fibrations are equivalent to presheaves, such a $P(p)$ therefore restricts to a functor between presheaf categories:

Definition 3.2. For p a fibrous polynomial as above, we associate to it the functor $P_d(p)$, defined as the composite

$$\widehat{\mathcal{C}} \xrightarrow{\Delta_{p_1}} \widehat{\mathcal{A}} \xrightarrow{\Pi_{p_2}} \widehat{\mathcal{B}} \xrightarrow{\Sigma_{p_3}} \widehat{\mathcal{C}}.$$

Consider a diagram X in $\widehat{\mathcal{C}'}$, represented below as a discrete fibration over \mathcal{C}' :



Unwinding the definitions, $P_d(p)(X) = \Sigma_{p_3} \Pi_{p_2} \Delta_{p_1} X$ has c -cells

$$\coprod_{b \in \mathcal{B}_c} \text{Fun}_{/\mathcal{C}'}(\Phi_p(b), X).$$

If p is familial, Φ_p lands in $DFib(\mathcal{C}') \simeq \widehat{\mathcal{C}'}$ and we have

$$P_d(p)(X)_c \cong \coprod_{b \in \mathcal{B}_c} \text{Hom}_{\widehat{\mathcal{C}'}}(\Phi_p(b), X),$$

hence $P_d(p)$ is a familial functor.

Proposition 3.3. *For any very fibrous polynomial p , $P_d(p)$ is a familial functor.*

Proof. Let p be quasi-familial; we can form the familial polynomial $|p|$, called its *familial replacement*, by replacing $\Phi_p: \mathcal{B} \rightarrow Fib(\mathcal{C}')$ with $|\Phi_p|: \mathcal{B} \rightarrow DFib/\mathcal{C}'$. By the adjunction discussed in Example 2.11, we have for X discretely fibered over \mathcal{C}'

$$\text{Fun}_{/\mathcal{C}'}(\Phi_p(b), X) \cong \text{Hom}_{Fib(\mathcal{C}')}(\Phi_p(b), X) \cong \text{Hom}_{\widehat{\mathcal{C}'}}(|\Phi_p|(b), X),$$

natural in b , which establishes an isomorphism

$$\Pi_{p_2} \Delta_{p_1} \cong \Pi_{|p|_2} \Delta_{|p|_1}.$$

As $|p|_3 = p_3$, this shows that $P_d(p) \cong P_d(|p|)$, so $P_d(p)$ is a familial functor.

The general proof for a very fibrous polynomial proceeds similarly by noting that the left adjoint $\mathbf{Cat}/\mathcal{C}' \rightarrow \mathit{Fib}(\mathcal{C}')$ to the inclusion functor extends to $\mathbf{Cat}/\mathcal{C}'$, but the only example we will need at this level of generality is the identity polynomial discussed below, so we do not discuss this further. \square

P_d is full in the sense that it can recover any familial functor by a familial polynomial.

Definition 3.4. For any familial functor $\widehat{\mathcal{C}}' \rightarrow \widehat{\mathcal{C}}$ with representation (S, E) , we can form the familial polynomial $\mathbf{gr}(S, E)$ given by

$$\begin{array}{ccc} & \oint E & \xrightarrow{p_2} \twoheadrightarrow \int S \\ p_1 \swarrow & & \searrow p_3 \\ \mathcal{C}' & & \mathcal{C} \end{array}$$

where $\oint E$ is the Grothendieck construction of the functor $\int S \xrightarrow{E} \widehat{\mathcal{C}}' \cong \mathit{DFib}/\mathcal{C}'$. $P_d(\mathbf{gr}(S, E))$ then agrees with the familial functor associated to (S, E) by Definition 3.2.

Example 3.5. The representation of the identity functor on $\widehat{\mathcal{C}}$ is given by (S^0, E^0) , where S^0 is the terminal functor $\mathcal{C}^{op} \rightarrow \mathbf{Set}$ and $E^0: \int S^0 \cong \mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is the Yoneda embedding (Definition 1.9). The Grothendieck construction of E^0 is then the discrete two sided fibration with $\mathit{Hom}_{\mathcal{C}}(c', c)$ as the (intersection) fiber over (c', c) . Morphisms in the fiber over c' are given by commuting triangles under c' , and morphisms from in the fiber over c are given by commuting triangles over c , with general morphisms

given by composites of these which form commutative squares (below left). $\mathbf{gr}(S^0, E^0)$ is then isomorphic to the polynomial below right:

$$\begin{array}{ccc}
 c' & \longrightarrow & \cdot \\
 \downarrow & \searrow & \downarrow \\
 \cdot & \longrightarrow & c
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathcal{C} & \xrightarrow{\text{cod}} \mathcal{C} \\
 \swarrow \text{dom} & & \parallel \\
 \mathcal{C} & & \mathcal{C}
 \end{array}$$

3.2 Cartesian Morphisms

To extend \mathbf{gr} to a functor from familial representations to polynomials, we recall the definition of morphisms between polynomials.

In [36], morphisms f between polynomials p and q are commuting diagrams of the following form:

$$\begin{array}{ccccc}
 & \mathcal{A} & \xrightarrow{p_2} & \mathcal{B} & \\
 & \swarrow p_1 & \lrcorner & \searrow p_3 & \\
 \mathcal{C}' & & & & \mathcal{C} \\
 & \swarrow q_1 & & \searrow q_3 & \\
 & \mathcal{A}' & \xrightarrow{q_2} & \mathcal{B}' & \\
 & \downarrow u_0 & & \downarrow u_1 & \\
 & & & &
 \end{array}$$

which for fibrous polynomials amounts to a (pseudo) natural isomorphism $\Phi_p \cong \Phi_q \circ u_1$ in $\mathbf{Cat} // \mathcal{C}$ with components in the subcategory $\mathbf{Cat} / \mathcal{C}$.

These *cartesian* morphisms between polynomials induce natural transformations between polynomial functors as follows:

$$\sum_{p_3} \prod_{p_2} \Delta_{p_1} \cong \sum_{q_3} \sum_{u_1} \prod_{p_2} \Delta_{u_0} \Delta_{q_1} \cong \sum_{q_3} \sum_{u_1} \Delta_{u_1} \prod_{q_2} \Delta_{q_1} \xrightarrow{\epsilon} \sum_{q_3} \prod_{q_2} \Delta_{q_1}$$

The first isomorphism comes from pseudofunctoriality of Σ, Δ , the second is the Beck-Chevalley isomorphism for the pullback square, and the final map is the counit

of the adjunction $\Sigma_u \dashv \Delta_u$. ϵ is cartesian, hence so is the induced natural transformation ([17, 2.1]).

Lemma 3.6. $P: \mathbf{Poly}(\mathcal{C}', \mathcal{C})_\times \rightarrow \mathbf{Poly}(\mathbf{Cat}/\mathcal{C}', \mathbf{Cat}/\mathcal{C})_\times$, where the codomain is the category of polynomial functors and cartesian transformations, is fully faithful.

Proof. While \mathbf{Cat} is not locally cartesian closed, as every functor $\mathcal{C} \rightarrow 1$ is exponentiable, \mathbf{Cat}/\mathcal{C} has the same tensoring and enrichment as described in [17, 1.3], which suffices to replicate the proof of [17, Proposition 2.9] in this setting. It then remains only to observe that any strict natural transformation between \mathbf{Cat} -enriched functors is strong, so any cartesian transformation between polynomial functors is uniquely represented by a cartesian morphism of polynomials. \square

As $\widehat{\mathcal{C}} \simeq DFib(\mathcal{C})$ forms a full subcategory of \mathbf{Cat}/\mathcal{C} , P_d also sends cartesian morphisms to cartesian natural transformations, which for (u_0, u_1) as above unwinds to the map

$$P_d(p)(X)_c \cong \prod_{b \in \mathcal{B}_c} Fun_{/\mathcal{C}'}(\Phi_p(b), X) \rightarrow \prod_{b' \in \mathcal{B}'_c} Fun_{/\mathcal{C}'}(\Phi_q(b'), X) \cong P_d(q)(X)$$

sending $(b, f: \Phi_p(b) \rightarrow X)$ to $(u_1(b), \Phi_q(u_1(b)) \xrightarrow{(u_0)_b^{-1}} \Phi_p(b) \xrightarrow{f} X)$.

Definition 3.7. Given a morphism $(\phi^S, \phi^E): (S, E) \rightarrow (S', E')$ of familial representations from \mathcal{C}' to \mathcal{C} , we have the following cartesian morphism of familial polynomi-

als:

$$\begin{array}{ccccc}
 & \mathcal{F}E & \xrightarrow{p_2} & \int S & \\
 & \swarrow p_1 & & \downarrow & \searrow p_2 \\
 \mathcal{C}' & \mathcal{F}\phi^E & \lrcorner & \int \phi^S & \mathcal{C} \\
 & \swarrow q_1 & & \downarrow & \searrow q_3 \\
 & \mathcal{F}E' & \xrightarrow{q_2} & \int S' &
 \end{array}$$

where the square is a pullback as ϕ^E restricts to an isomorphism $E(t) \cong E'(\phi^S(t))$ on the fibers of p_2, q_2 .

Every cartesian morphism between parallel familial polynomials arises uniquely in this manner, as functors $u_0: \int S \rightarrow \int S'$ over \mathcal{C} are in bijective correspondence with maps $u_0: S \rightarrow S'$ in $\widehat{\mathcal{C}}$, and a functor $u_1: \mathcal{F}E \rightarrow \mathcal{F}E'$ commuting with the rest of such a diagram restricts to natural isomorphisms $\int Et \rightarrow \int E'u_0(t)$ over \mathcal{C}' , corresponding to isomorphisms $Et \rightarrow E'u_0(t)$ in $\widehat{\mathcal{C}'}$.

\mathbf{gr} therefore extends to a fully faithful functor from $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$. Denoting by $\mathbf{Poly}^{vf}(\mathcal{C}', \mathcal{C})_{\times}$ the category of very fibrous polynomials from \mathcal{C}' to \mathcal{C} and cartesian morphisms between them, we have now established the following.

Proposition 3.8. $H: \mathbf{Rep}(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{Fam}(\mathcal{C}', \mathcal{C})$ factors as

$$\mathbf{Rep}(\mathcal{C}', \mathcal{C}) \xrightarrow{\mathbf{gr}} \mathbf{Poly}^{vf}(\mathcal{C}', \mathcal{C})_{\times} \xrightarrow{P_d} \mathbf{Fam}(\mathcal{C}', \mathcal{C}),$$

where \mathbf{gr} is fully faithful and P_d is surjective on objects.

Remark 3.9. As \mathbf{gr} is fully faithful and essentially surjective onto familial polynomials, the full subcategory $\mathbf{Poly}^{fam}(\mathcal{C}', \mathcal{C})_{\times}$ of familial polynomials and cartesian morphisms is equivalent to $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$ and therefore $\mathbf{Fam}(\widehat{\mathcal{C}'}, \widehat{\mathcal{C}'})$.

3.3 Vertical Morphisms

While they are not included in the bicategory of polynomials in \mathbf{Cat} defined in [36], [17] describe a larger bicategory of polynomials and morphisms between them, in the more restrictive setting of a locally cartesian closed category. The additional morphisms are sent by an extension of P to non-cartesian transformations of polynomial functors, and morphisms from p to q admit a factorization system with right class the cartesian morphisms and left class the *vertical morphisms*, given by diagrams of the following form:

$$\begin{array}{ccccc}
 & & \mathcal{A} & \xrightarrow{p_2} & \mathcal{B} \\
 & p_1 \swarrow & \uparrow v & & \searrow p_3 \\
 \mathcal{C}' & & & & \mathcal{C} \\
 & q_1 \swarrow & & & \searrow p_3 \\
 & & \mathcal{A}' & \xrightarrow{q_2} & \mathcal{B}
 \end{array}$$

which for fibrous p, q amounts to (by Remark 2.4) a lax natural transformation $\Phi_q \rightarrow \Phi_p$ in $\mathbf{Cat}/\mathcal{C}'$ with components in \mathbf{Cat}/\mathcal{C} and lax structure lying over identities in \mathcal{C} .

If v is exponentiable there is a transformation $P(p) \rightarrow P(q)$ given by

$$\sum_{p_3} \prod_{p_2} \Delta_{p_1} \xrightarrow{\eta} \sum_{p_3} \prod_{p_2} \prod_v \Delta_v \Delta_{p_1} \cong \sum_{p_3} \prod_{q_2} \Delta_{q_1},$$

where η is the unit of the adjunction $\Delta_v \dashv \prod_v$ and the second isomorphism comes from pseudofunctoriality. However, when p_2, q_2 are opfibrations, the desired transformation can be defined for any v .

Lemma 3.10. (Analogue of [17, Proposition 2.8]) For fibrous polynomials p, q as

above, natural transformations $P(p) \Rightarrow P(q): \mathbf{Cat}/\mathcal{C}' \rightarrow \mathbf{Cat}/\mathcal{C}$ that restrict to the identity on $\text{id}_{\mathcal{C}'}$ correspond bijectively with maps v as above.

Proof. As $\text{id}_{\mathcal{C}'}$ is terminal in $\mathbf{Cat}/\mathcal{C}'$ and $P(p)(\text{id}_{\mathcal{C}'}) = P(q)(\text{id}_{\mathcal{C}'}) \cong \mathcal{B}$, any such transformation lifts uniquely to a natural transformation

$$\prod_{p_2} \Delta_{p_1} \rightarrow \prod_{q_2} \Delta_{q_1}: \mathbf{Cat}/\mathcal{C}' \rightarrow \mathbf{Cat}/\mathcal{B},$$

so it suffices to show that such transformations β correspond bijectively with lax natural transformations $\phi: \Phi_q \rightarrow \Phi_p$ strict over \mathcal{C} .

$\beta: \prod_{p_2} \Delta_{p_1} \rightarrow \prod_{q_2} \Delta_{q_1}$, restricted to the fiber over $b \in \text{Ob } \mathcal{B}$, is a map

$$\text{Fun}_{/\mathcal{C}}(\Phi_p(b), X) \rightarrow \text{Fun}_{/\mathcal{C}}(\Phi_q(b), X)$$

natural in X , which by Yoneda is uniquely determined by a functor $\phi_b: \Phi_q(b) \rightarrow \Phi_p(b)$ over \mathcal{C} (this is essentially the argument in [17, Proposition 2.8]). To extend this correspondence to morphisms, for each $i: b \rightarrow b'$ in \mathcal{B} β requires a mapping, natural in X , from transformations as pictured below (right side) to natural transformations filling in the outer diagram (all over \mathcal{C}):

$$\begin{array}{ccc}
 \Phi_q(b) & \xrightarrow{\phi_b} & \Phi_p(b) \\
 \downarrow \Phi_q(i) & & \downarrow \Phi_p(i) \\
 \Phi_q(b') & \xrightarrow{\phi_{b'}} & \Phi_p(b')
 \end{array}
 \begin{array}{c}
 \searrow u \\
 \Downarrow \\
 \nearrow u'
 \end{array}
 \rightarrow X$$

Such a mapping could arise from precomposition with a natural transformation

$\phi_i: \Phi_p(i)\phi_b \Rightarrow \phi_{b'}\Phi_q(i)$ (satisfying coherence conditions corresponding to functoriality in i). In a Yoneda style argument, we can set

$$X = \Phi_p(b'), \quad u = \Phi_p(i), \quad u' = \text{id}_{\Phi_p(b')}$$

in the diagram above and apply such a mapping to the identity transformation on $\Phi_p(i)$ to recover a transformation $\Phi_p(i)\phi_b \Rightarrow \phi_{b'}\Phi_q(i)$, and it is straightforward to check that these transformations satisfy the desired coherence conditions and provide a bijective correspondence between such β 's and ϕ 's. \square

Remark 3.11. The analogous extension of P_d sends v to the natural transformation

$$P_d(p)(X)_c \cong \coprod_{b \in \mathcal{B}_c} \text{Fun}_{/\mathcal{C}'}(\Phi_p(b), X) \rightarrow \coprod_{b' \in \mathcal{B}_c} \text{Fun}_{/\mathcal{C}'}(\Phi_q(b'), X) \cong P_d(q)(X)$$

mapping $(b, f: \Phi_p(b) \rightarrow X)$ to $(b, \Phi_q(b) \xrightarrow{v_b} \Phi_p(b) \xrightarrow{f} X)$.

Just as in Lemma 3.6, the tensoring of \mathbf{Cat}/\mathcal{C} over \mathbf{Cat} lets us replicate the proof of [17, Proposition 2.4] in this setting. Therefore, as all strict natural transformations in a \mathbf{Cat} -enriched category are strong, the functor $\text{Poly}(\mathbf{Cat}/\mathcal{C}', \mathbf{Cat}/\mathcal{C}) \rightarrow \mathbf{Cat}/\mathcal{C}$ given by evaluating a polynomial functor or cartesian transformation at the terminal object $\text{id}_{\mathcal{C}'}$ is a fibration. The cartesian maps with respect to this fibration are the cartesian transformations, and vertical maps are those whose component at $\text{id}_{\mathcal{C}'}$ is the identity.

The factorization system on cartesian and vertical maps with respect to this fibration provides a canonical means of commuting past each other cartesian and vertical morphisms of polynomials from \mathcal{C}' to \mathcal{C} , which uniquely represent cartesian

and vertical transformations, respectively, of the corresponding functors (Lemma 3.6, Lemma 3.10). Any composite of cartesian and vertical morphisms then has a unique factorization as a vertical morphism followed by a cartesian morphism, which suffices to define the category $\mathbf{Poly}^f(\mathcal{C}', \mathcal{C})$ of fibrous polynomials from \mathcal{C}' to \mathcal{C} and general morphisms between them of this form, with full subcategory $\mathbf{Poly}^{vf}(\mathcal{C}', \mathcal{C})$ of very fibrous polynomials.

Remark 3.12. The transformation induced by a vertical morphism is only cartesian when the map v is an isomorphism, in which case it is identified with the cartesian morphism with (u_0, u_1) given by (v^{-1}, id) .

We have now extended P to a functor $\mathbf{Poly}^f(\mathcal{C}', \mathcal{C}) \rightarrow Poly(\mathbf{Cat}/\mathcal{C}', \mathbf{Cat}/\mathcal{C})$, and P_d to a functor $\mathbf{Poly}^{vf} \rightarrow Fam(\widehat{\mathcal{C}'}, \widehat{\mathcal{C}})$.

Example 3.13. Consider the familial polynomial $\mathbf{gr}(S^0, E^0)$ of Example 3.5. We have the following vertical morphism ϵ from $\mathbf{gr}(S^0, E^0)$ to the identity polynomial on \mathcal{C}

$$\begin{array}{ccccc}
 & & \mathcal{C}^{\rightarrow} & \xrightarrow{\text{cod}} & \mathcal{C} \\
 & \swarrow \text{dom} & \uparrow & & \parallel \\
 \mathcal{C} & & \mathcal{C} & & \mathcal{C} \\
 & \searrow \text{id}_{(-)} & \parallel & & \parallel \\
 & & \mathcal{C} & \xlongequal{\quad} & \mathcal{C}
 \end{array}$$

where the vertical map (which is not exponentiable) sends an object c in \mathcal{C} to its identity morphism id_c in $\mathcal{C}^{\rightarrow}$. The transformation

$$P_d(\epsilon): P_d\left(\mathbf{gr}(S^0, E^0)\right)(X)_c \cong Fun_{/\mathcal{C}}\left(\int y(c), X\right) \rightarrow Fun_{/\mathcal{C}}(*, X) \cong P_d(1_{\mathcal{C}})(X)_c$$

is induced by the map $* \rightarrow \int y(c) \cong \mathcal{C}/c$ picking out id_c , and by Yoneda this is an

isomorphism. $\mathbf{gr}(S^0, E^0)$ is the familial replacement of $1_{\mathcal{C}}$, in the sense of Proposition 3.3.

Example 3.14. For a quasi-familial polynomial p , the map $\pi: \mathcal{A} \rightarrow |\mathcal{A}|$ forms a vertical morphism of polynomials from $|p|$ to p , which P_d sends to an isomorphism by Proposition 3.3, π being the unit of the adjunction between two sided fibrations and discrete two sided fibrations. This will be the key to comparing the compositions of familial representations and familial polynomials.

3.4 Bicategory Structure

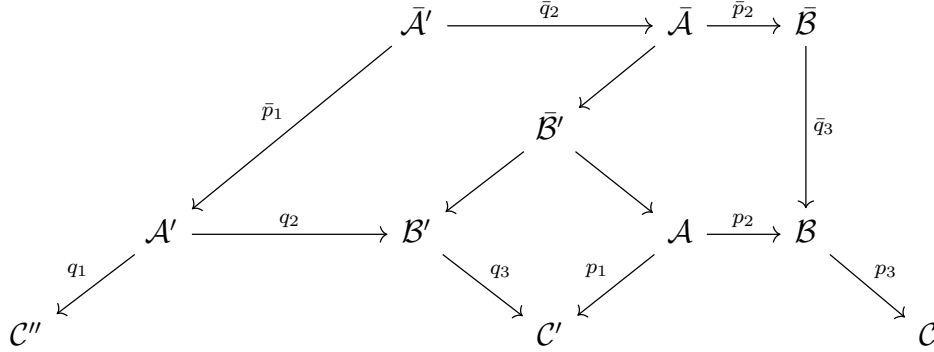
[36] constructs a bicategory whose objects are small categories with morphism categories $\mathbf{Poly}(\mathcal{C}', \mathcal{C})_{\times}$, and a bifunctor $\mathbf{Poly}_{\times} \rightarrow \mathbf{CAT}$ sending \mathcal{C} to \mathbf{Cat}/\mathcal{C} and each polynomial to the corresponding polynomial functor, likewise for cartesian transformations. The bicategory structure of polynomials in [17] which also includes vertical morphisms, while only claimed for locally cartesian closed categories, agrees on cartesian morphisms with that of [36].

Extending Weber's bifunctor to a bicategory of polynomials on \mathbf{Cat} with vertical morphisms would require, at a minimum, restricting vertical morphisms to those whose map $\mathcal{A}' \rightarrow \mathcal{A}$ is exponentiable. To avoid this restriction, we instead work with the sub-bicategory of (very) fibrous polynomials, between which all vertical morphisms induce unique transformations of the corresponding polynomial functors.

Lemma 3.15. *Fibrous, very fibrous, and quasi-familial polynomials are each closed*

under polynomial composition.

Proof. Recall ([17, 1.11], [36, Definition 3.1.7]) that for composable polynomials p, q their composite is defined as the outer diagram below, where all squares are pullbacks and the pentagon on the right is a distributivity pullback:



This definition requires a choice of pullbacks and distributivity pullbacks in \mathbf{Cat} , which is provided by the constructions in Proposition 2.3 and Proposition 2.6.

For p, q fibrous, $\bar{p}_2\bar{q}_2$ is an opfibration by Proposition 2.3 and Proposition 2.5. If they are very fibrous, \bar{q}_3 is a discrete fibration by Remark 2.7, hence so is $p_3\bar{q}_3$.

Now assume p, q are quasi-familial. $\bar{\mathcal{B}}$ is precisely $P_d(p)(\mathcal{B}')$, so its objects are of the form $(b, f: \Phi_p(b) \rightarrow \mathcal{B}')$, where f commutes over \mathcal{C}' . Using, Proposition 2.3 and Proposition 2.5, $\bar{p}_2\bar{q}_2$ corresponds to the functor $\Phi: \bar{\mathcal{B}} \rightarrow \mathbf{Cat}$ sending (b, f) to

$$\oint \left(\Phi_p(b) \xrightarrow{f} \mathcal{B}' \xrightarrow{\Phi_q} \mathit{Fib}(\mathcal{C}'') \right),$$

where Φ factors through $\mathit{Fib}(\mathcal{C}'')$ by Proposition 2.10 as a morphism in $\bar{\mathcal{B}}$ from (b, f) to (b', f') is a natural transformation $f \Rightarrow f' \circ \Phi_p(i)$ over \mathcal{C}' for $i: b \rightarrow b'$ (which in

fact must be the identity by discreteness of q_3). Therefore $(q_1\bar{p}_1, \bar{p}_2\bar{q}_2)$ forms a two sided fibration. \square

Remark 3.16. As discussed above, in the composite fibrous polynomial $\bar{\mathcal{B}}$ consists of diagrams in \mathcal{B}' indexed by a fiber of p_2 , and the fiber in $\bar{\mathcal{A}}'$ over such a diagram is the lax colimit of the corresponding fibers of q_2 . In the next section we use the comparison map from each such lax colimit to a strict colimit to analyze composites of familial polynomials, which are not closed under composition.

Theorem 3.17. *Small categories, fibrous polynomials between them, and general morphisms of polynomials form a bicategory \mathbf{Poly}^f under polynomial identities and composition, with a bifunctor $P: \mathbf{Poly}^f \rightarrow \mathbf{CAT}$ sending \mathcal{C} to \mathbf{Cat}/\mathcal{C} and acting on morphism categories by $P: \mathbf{Poly}^f(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{CAT}(\mathbf{Cat}/\mathcal{C}', \mathbf{Cat}/\mathcal{C})$. Very fibrous polynomials form a sub-bicategory \mathbf{Poly}^{vf} .*

Proof. As fibrous polynomials are closed under polynomial identities and composition which agree with identities and composition of polynomial functors, it suffices to define horizontal composites of morphisms of polynomials and show that it agrees in the appropriate sense with horizontal composition in \mathbf{CAT} . These constructions can proceed exactly as in [17] by a transport argument, as by Lemma 3.6, Lemma 3.10, and the analogue of [17, Proposition 2.4] we have the analogue of [17, Lemma 2.15], upon which these constructions rely.

As the inclusion $\mathbf{Poly}^{vf}(\mathcal{C}', \mathcal{C}) \hookrightarrow \mathbf{Poly}^f(\mathcal{C}', \mathcal{C})$ is full, to show that \mathbf{Poly}^{vf} is a subcategory is suffices to note that very fibrous polynomials include the identities

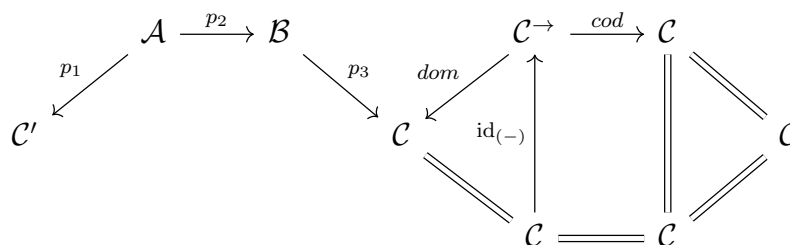
and are closed under horizontal composition. □

Corollary 3.18. *There is a bifunctor $P_d: \mathbf{Poly}^{vf} \rightarrow \mathbf{CAT}$ sending a small category \mathcal{C} to $\widehat{\mathcal{C}}$ and all polynomials to familial functors.*

Proof. P_d is constructed as a sub-bifunctor of P given by restricting the categories \mathbf{Cat}/\mathcal{C} to discrete fibrations over \mathcal{C} , the category of which is equivalent to $\widehat{\mathcal{C}}$. That P_d sends all polynomials to familial functors follows from Proposition 3.3. □

Note that P_d does not land in **Fam** as vertical morphisms are not necessarily sent to cartesian transformations, though as discussed below the vertical morphisms we are interested will be sent to isomorphisms.

Example 3.19. Consider the composition of a fibrous polynomial p with the familial polynomial $|1_{\mathcal{C}}| \cong \mathbf{gr}(S^0, E^0)$:



The resulting composite opfibration $\bar{\mathcal{A}} \rightarrow \bar{\mathcal{B}}$ with corresponding functor $\Phi: \bar{\mathcal{B}} \rightarrow \mathbf{Cat}/\mathcal{C}'$ has $\bar{\mathcal{B}} = P_d(|1_{\mathcal{C}}|)(\mathcal{B}) \cong \mathcal{B}$ by Example 3.13, and for b a c -cell of \mathcal{B} ,

$$\Phi(p) = \oint \left(\mathcal{C}/c \cong \int y(c) \xrightarrow{b} \mathcal{B} \xrightarrow{\Phi_p} \mathbf{Cat}/\mathcal{C} \right).$$

Up to the unitor of \mathbf{Poly}^f , $\epsilon \cdot \text{id}_p$ can be factored as the vertical followed by cartesian morphism:

$$\begin{array}{ccccc}
 & \bar{\mathcal{A}} & \longrightarrow & \bar{\mathcal{B}} & \\
 & \swarrow & & \searrow & \\
 \mathcal{C}' & \longleftarrow & \mathcal{A} & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{C} \\
 & \swarrow & & \searrow & & & \\
 & p_1 & & p_3 & & & \\
 & & \mathcal{A} & \xrightarrow{p_2} & \mathcal{B} & &
 \end{array}$$

$\uparrow (\text{id}_{(-)}, -)$ \parallel $\downarrow \wr$
 \parallel \parallel \parallel

where the natural transformation $\Phi_p \Rightarrow \Phi$ corresponding to the vertical map sends x in $\Phi_p(b)$ to (id_c, x) , with the evident action on morphisms.

The composite of $|1_{\mathcal{C}'}|$ and p is then the opfibration $\bar{\mathcal{A}} \rightarrow \bar{\mathcal{B}}$ with corresponding functor $\Phi: \bar{\mathcal{B}} \rightarrow \mathbf{Cat}/\mathcal{C}'$, where $\bar{\mathcal{B}} = P_d(p)(\text{id}_{\mathcal{C}'}) \cong \mathcal{B}$ as $\Pi_{p_2} \Delta_{p_1}$ as a right adjoint preserves terminal objects, and

$$\Phi(p) = \oint \left(\Phi_p(b) \rightarrow \mathcal{C}' \xrightarrow{\text{id}_{\mathcal{C}'}} \mathbf{Cat}/\mathcal{C}' \right).$$

Up to the unitor of \mathbf{Poly}^f and a cartesian morphism like above, $\text{id}_p \cdot \epsilon: |1_{\mathcal{C}'}|p \rightarrow p$ is the vertical morphism given by the transformation $\Phi_p \Rightarrow \Phi$ sending x in $\Phi_p(b)$ lying over c in \mathcal{C}' to (x, id_c) , with the evident action on morphisms.

4 The Bicategory Rep

The equivalence of categories $H: \mathbf{Rep}(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{Fam}(\widehat{\mathcal{C}'}, \widehat{\mathcal{C}})$ in Proposition 1.7, ranging over any small categories $\mathcal{C}', \mathcal{C}$ and landing in the morphism categories of the 2-category \mathbf{Fam} , resembles the functors on morphism categories making up a bifunctor from a bicategory \mathbf{Rep} with objects small categories and morphism categories

given by $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$. The main technical result of this chapter is that this is in fact the case.

Theorem 4.1. *There is a bicategory \mathbf{Rep} with objects small categories, 1-cells familial representations, and 2-cells given by morphisms of representations, such that $H: \mathbf{Rep} \rightarrow \mathbf{Fam}$ sending a small category \mathcal{C} to its presheaf category $\widehat{\mathcal{C}}$ and acting on morphisms as described above is a biequivalence.*

Proof. It suffices to show that \mathbf{Rep} is a bicategory and H is a bifunctor, as H is bijective on objects and an equivalence on morphism categories by Proposition 1.7. To do so, we consider the diagram

$$\mathbf{Rep} \xrightarrow{\mathbf{gr}} \mathbf{Poly}^{vf} \xrightarrow{P_d} \mathbf{CAT},$$

where P_d is the bifunctor from Corollary 3.18 landing in familial functors between presheaf categories (but not necessarily cartesian transformations), \mathbf{gr} has the elements of a *colax* bifunctor (Theorem 4.3), and \mathbf{Poly}^{vf} is the bicategory of *very fibrous* categorical polynomials in \mathbf{Cat} .

We show (Remark 4.4) that P_d sends the colax structure maps of \mathbf{gr} to natural isomorphisms in \mathbf{Fam} , so that the composite $P_d \circ \mathbf{gr}$ has the elements of a bifunctor. Furthermore, the composite (unlike P_d) lands in cartesian natural transformations (Section 3.2), so it factors through \mathbf{Fam} . The assignment $\mathbf{Rep} \rightarrow \mathbf{Fam}$ then sends \mathcal{C} to $\widehat{\mathcal{C}}$ and agrees with H on morphism categories (Proposition 3.8), so H has the elements of a bifunctor.

By “elements of a (colax) bifunctor”, we mean that after describing the identities, composites, unitors, and associators of \mathbf{Rep} , but without directly proving that the triangle and pentagon laws hold, \mathbf{gr} and its colax structure maps are shown to satisfy the unitality and associativity equations of a colax bifunctor. The composite H then has all of the elements of a bifunctor except for a complete proof that its domain \mathbf{Rep} is a bicategory. To complete the proof, we recall (Proposition 4.5) that given such data with H consisting of faithful functors on morphism categories, the triangle and pentagon equations for \mathbf{Rep} can be deduced from those for \mathbf{Fam} and the unitality and associativity equations for H .

□

We now proceed to concretely define the bicategory \mathbf{Rep} and prove the outstanding claims in the proof above.

The bicategory \mathbf{Rep} will have small categories as objects and $\mathbf{Rep}(\mathcal{C}', \mathcal{C})$ as morphism categories, while simultaneously assembling the functors $\mathbf{gr}: \mathbf{Rep}(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{Poly}^f(\mathcal{C}', \mathcal{C})$ into an identity-on-objects colax bifunctor $\mathbf{gr}: \mathbf{Rep} \rightarrow \mathbf{Poly}^{vf}$. The colax coherence maps for \mathbf{gr} are sent to isomorphisms by $P_d: \mathbf{Poly}^f \rightarrow \mathbf{CAT}$, endowing the composite $P_d \mathbf{gr}: \mathbf{Rep} \rightarrow \mathbf{CAT}$, sending a representation to its associated familial functor, with the structure of a bifunctor.

4.1 Identity

The identity representation in $\mathbf{Rep}(\mathcal{C}, \mathcal{C})$ is given by (S^0, E^0) . The identitor of \mathbf{gr} at \mathcal{C} is the (vertical) transformation in \mathbf{Poly}^f from Example 3.13:

$$\begin{array}{ccccc}
 & & \mathcal{C} & \xrightarrow{\text{cod}} & \mathcal{C} \\
 & \swarrow \text{dom} & \uparrow \text{id}_{(-)} & & \parallel \\
 \mathcal{C} & & \mathcal{C} & & \mathcal{C} \\
 & \searrow & \parallel & & \parallel \\
 & & \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \\
 & & & & \parallel \\
 & & & & \mathcal{C}
 \end{array}$$

The identitor ϵ goes from $\mathbf{gr}(S^0, E^0)$ to $1_{\mathcal{C}}$ and is not invertible in $\mathbf{Poly}^f(\mathcal{C}, \mathcal{C})$, though by Example 3.13 it is sent to an isomorphism by P_d . The direction of ϵ is opposite that of a *lax* bifunctor, so as the same holds for the productor below, \mathbf{gr} will be a *colax* bifunctor, which we prove in the following subsections.

4.2 Composition

In Definition 1.10, we showed that for familial functors $F: \widehat{\mathcal{C}}' \rightarrow \widehat{\mathcal{C}}$ and $G: \widehat{\mathcal{C}}'' \rightarrow \widehat{\mathcal{C}}'$ with representations (S, E) and (S', E') , their composite has representation (SS', EE') , where

$$SS'_c = \coprod_{t \in S_c} \text{Hom}(Et, S'), \quad EE'(t, f) = \text{colim}_{x: y(c') \rightarrow Et} E'f(x).$$

Let this define the horizontal composition in \mathbf{Rep} , where both of these formulas are functorial in S, E, S', E' .

From the proof of Lemma 3.15, the polynomial composite $\mathbf{gr}(S, E)\mathbf{gr}(S', E')$ is of the form

$$\begin{array}{ccc} & \mathcal{A} & \xrightarrow{p_2} \int SS' \\ p_1 \swarrow & & \searrow p_3 \\ \mathcal{C}'' & & \mathcal{C} \end{array}$$

where $\Phi_p(t, f: Et \rightarrow S')$ is given by

$$\oint \left(\int Et \xrightarrow{f} \int S' \xrightarrow{E'} \widehat{\mathcal{C}}'' \cong DFib(\mathcal{C}'') \right).$$

$\mathbf{gr}(S, E)\mathbf{gr}(S', E')$ is quasi-familial by Lemma 3.15 but not familial, as $\Phi_p(t, f) \rightarrow \mathcal{C}''$ is not a discrete fibration: the fiber over c'' includes nontrivial morphisms of the form

$$(i_x, \text{id}): (Ei_t(x), y) \rightarrow (x, E'f(i_x)(y))$$

for $i: d \rightarrow c$ in \mathcal{C} , $x \in Et_c$, and $y \in E'f(Ei_t(x))_{c''}$.

Proposition 4.2. *The familial replacement $|\mathbf{gr}(S, E)\mathbf{gr}(S', E')|$ is isomorphic to $\mathbf{gr}(SS', EE')$.*

Proof. We have the following chain of isomorphisms

$$P_d(\mathbf{gr}(SS', EE')) \cong P_d(\mathbf{gr}(S, E))P_d(\mathbf{gr}(S', E')) = P_d(\mathbf{gr}(S, E)\mathbf{gr}(S', E')) \cong P_d(|\mathbf{gr}(S, E)\mathbf{gr}(S', E')|)$$

by Definition 1.10, bifunctionality of P_d , and Proposition 3.3 respectively.

As P_d restricted to familial polynomials reflects isomorphisms (Remark 3.9),

$$\mathbf{gr}(SS', EE') \cong |\mathbf{gr}(S, E)\mathbf{gr}(S', E')|. \quad \square$$

More conceptually, the corresponding $\Phi_{|p_2|}(t, f)_c$ is the set of connected components of $\oint(E' \circ f)$ over c , which are precisely the c -cells of $EE'(t, f) =$

$\operatorname{colim}_{x: y(c') \rightarrow Et} E'f(x)$, and $(\pi_0)_*: \oint(E' \circ f) \rightarrow \operatorname{colim}(E' \circ f)$ is the canonical map from the lax colimit of $E' \circ f$ to the strict colimit. The corresponding vertical map π , pictures below, is the productor $\mathbf{gr}(SS', EE') \rightarrow \mathbf{gr}(S, E)\mathbf{gr}(S', E')$.

$$\begin{array}{ccccc}
 & & \oint EE' & \longrightarrow & \int SS' \\
 & \swarrow & \uparrow \pi & & \parallel \\
 \mathcal{C}'' & & \mathcal{A} & \longrightarrow & \int SS' \\
 & \swarrow & & & \parallel \\
 & & & & \int SS' \\
 & \searrow & & & \searrow \\
 & & & & \mathcal{C}
 \end{array}$$

By Example 3.14, P_d sends π to an isomorphism.

4.3 Unitors

The left unitor λ in **Rep** sends $(*_c, t: y(c) \rightarrow S) \in S^0Sc$ to $t \in Sc$ and has as its E -part the canonical $\operatorname{colim}_{j: y(c') \rightarrow y(c)} E(tj) \cong Et$. The right unitor ρ sends $(t, !: Et \rightarrow S^0) \in SS^0c$ to $t \in Sc$ and has as its E -part the canonical $\operatorname{colim}_{x: y(c') \rightarrow Et} y(c') \cong Et$.

The left unitality law making \mathbf{gr} a colax bifunctor amounts to the following for (S, E) a representation from \mathcal{C}' to \mathcal{C} :

$$\begin{array}{ccc}
 1_{\mathcal{C}}\mathbf{gr}(S, E) & \xleftarrow{\epsilon \cdot \operatorname{id}} & \mathbf{gr}(S^0, E^0)\mathbf{gr}(S, E) \\
 \parallel & & \uparrow \pi \\
 \mathbf{gr}(S, E) & \xleftarrow{\mathbf{gr}(\lambda)} & \mathbf{gr}(S^0S, E^0E)
 \end{array}$$

By Example 3.13 and Example 3.19, for $\Phi: \int S^0S \rightarrow \operatorname{Fib}(\mathcal{C}')$ corresponding to $\mathbf{gr}(S^0, E^0)\mathbf{gr}(S, E)$ and $t \in Sc$, $\epsilon \cdot \operatorname{id}$ is the composite of a cartesian morphism

containing λ^S and the vertical morphism given by the natural transformation

$$\int Et \xrightarrow{(\text{id}_c, -)} \oint \left(\mathcal{C}/c \xrightarrow{t} \int S \xrightarrow{E} DFib(\mathcal{C}') \right) = \Phi(*_c, t),$$

which composed with the transformation $(\pi_0)_*$ contracting to identities the morphisms of the form $(i: c'' \rightarrow c', \text{id}_t)$ in $\Phi(*, t)$ yields the canonical isomorphism

$E(t) \cong \text{colim}_{j: y(c') \rightarrow y(c)} E(tj)$, inverse to that of λ . The top composite then amounts to

$$\begin{array}{ccccc} & \oint E^0 E & \longrightarrow & \int S^0 S & \\ & \swarrow & & \searrow & \\ \mathcal{C}' & \longleftarrow \oint E & \longrightarrow & \int S^0 S & \longrightarrow \mathcal{C} \\ & \swarrow & & \searrow & \\ & \oint E & \longrightarrow & \int S & \end{array}$$

$\uparrow \int (\lambda^E)^{-1}$ \parallel
 \parallel $\int \lambda^S \downarrow$

which by Remark 3.12 is precisely $\mathbf{gr}(\lambda)$, so the diagram commutes.

The right unitality law is the square:

$$\begin{array}{ccc} \mathbf{gr}(S, E)1_{\mathcal{C}'} & \xleftarrow{\text{id} \cdot \epsilon} & \mathbf{gr}(S, E)\mathbf{gr}(S^0, E^0) \\ \parallel & & \uparrow \phi \\ \mathbf{gr}(S, E) & \xleftarrow{\mathbf{gr}(\rho)} & \mathbf{gr}(SS^0, EE^0) \end{array}$$

Similarly, for $\Phi: \int SS^0 \rightarrow Fib(\mathcal{C}')$ corresponding to $\mathbf{gr}(S, E)\mathbf{gr}(S^0, E^0)$ and $t \in Sc$, $\text{id} \cdot \epsilon$ is the vertical morphism given by the natural transformation

$$\int Et \xrightarrow{(-, \text{id})} \oint \left(\int Et \rightarrow \mathcal{C}' \xrightarrow{c'/-} DFib(\mathcal{C}') \right) = \Phi(t, !),$$

which composed with the transformation $(\pi_0)_*$ contracting morphisms of the form

$$(i_t, \text{id}_{c'}) : (Et_i(x), ji: c'' \rightarrow c) \rightarrow (x, j: c' \rightarrow c)$$

in $\Phi(t, !)$ for $i: c'' \rightarrow c'$ yields the canonical isomorphism $E(t) \cong \operatorname{colim}_{x: y(c') \rightarrow Et} y(c')$, inverse to that of ρ . The top composite then agrees with $\int \rho$ as in the left unitality square.

4.4 Associator

In this section, we fix the following familial polynomials

$$\mathcal{C}''' \xrightarrow{r=\mathbf{gr}(S'', E'')} \mathcal{C}'' \xrightarrow{q=\mathbf{gr}(S', E')} \mathcal{C}' \xrightarrow{p=\mathbf{gr}(S, E)} \mathcal{C},$$

writing $\mathcal{A}_{pq} \rightarrow \mathcal{B}_{pq}$ to denote the opfibration in the polynomial pq corresponding to Φ_{pq} , and likewise for the other composites. We will show that the following diagram commutes in $\mathbf{Poly}^f(\mathcal{C}''', \mathcal{C})$, which using the shorthand $|pq| = \mathbf{gr}(SS', EE')$ suggested by Proposition 4.2 expresses the associativity law for the colax bifunctor \mathbf{gr} :

$$\begin{array}{ccc} (pq)r & \xrightarrow{\alpha^{\mathbf{Poly}}} & p(qr) \\ \pi \cdot \text{id} \uparrow & & \uparrow \text{id} \cdot \pi \\ |pq|r & & p|qr| \\ \pi \uparrow & & \uparrow \pi \\ |(pq)r| & \xrightarrow{\mathbf{gr}(\alpha^{\mathbf{Rep}})} & |p(qr)| \end{array}$$

For $t \in Sc$ and $f: Et \rightarrow S'$, we have

$$\mathcal{B}_{|(pq)r|c} = \mathcal{B}_{|pq|r}c = (SS')S''c = \coprod_{(t,f) \in SS'c} \operatorname{Hom}_{\widehat{\mathcal{C}}''}(\operatorname{colim}(E' \circ f f), S'')$$

$$\mathcal{B}_{(pq)r}c = P_d(pq)(S'')c = \coprod_{(t,f)} \operatorname{Hom}_{\operatorname{Fib}(\mathcal{C}'')}(\oint(E' \circ f f), f S'')$$

where $\int Et \xrightarrow{f} \int S' \xrightarrow{E'} \widehat{\mathcal{C}}''$. The \mathcal{B} -component of $\pi: |(pq)r| \rightarrow |pq|r$ is the identity and that of $\pi \cdot \text{id}$ is the map $\mathcal{B}_{|pq|r} \rightarrow \mathcal{B}_{(pq)r}$ induced by the maps $Q: \oint(E' \circ f f) \rightarrow \int \text{colim}(E' \circ f f)$, which is an isomorphism by Example 3.14 as $\int S''$ is a discrete fibration. Meanwhile, as $P_d(q)(S'') = S'S''$, we have

$$\mathcal{B}_{|p(qr)|c} = \mathcal{B}_{p|qr|c} = \mathcal{B}_{p(qr)c} = P_d(p)(S'S'') = S(S'S'')_c = \coprod_{t \in Sc} \text{Hom}_{\widehat{\mathcal{C}}''}(Et, S'S'')$$

with the \mathcal{B} component of $|p(qr)| \xrightarrow{\pi} p|qr| \xrightarrow{\text{id} \cdot \pi} p(qr)$ the identity.

The associator $\alpha^{\mathbf{Rep}}$ has S -component given by

$$\left(t \in Sc, f: Et \rightarrow S', F: \text{colim}(E' \circ f f) \rightarrow S'' \right) \in (SS')S''c \quad \mapsto \quad (t, G) \in S(S'S'')c$$

where G sends $x \in Et_{c'}$ to

$$\left(f(x) \in S'c', F_x: E'f(x) \rightarrow \text{colim}(E' \circ f f) \xrightarrow{F} S'' \right) \in S'S''c'.$$

This is an isomorphism as every such G uniquely arises in this way: if G sends x to $\left(g(x) \in S'c', G_x: E'g(x) \rightarrow S'' \right)$ we can set $f(x) = g(x)$ and use (G_x) to induce F from the colimit.

The associator $\alpha^{\mathbf{Poly}}$ has \mathcal{B} -component

$$\coprod_{(t,f)} \text{Hom}_{\text{Fib}(\mathcal{C}'')} \left(\oint(E' \circ f f), \int S'' \right) \rightarrow \coprod_{t \in Sc} \text{Hom}_{\widehat{\mathcal{C}}''}(Et, S'S'')$$

sending (t, f, \bar{F}) in the domain, where by Example 3.14 $\bar{F}: \oint(E' \circ f f) \rightarrow \int S''$ must be of the form $Q \circ \int F$ for some $F: \text{colim}(E' \circ f f) \rightarrow S''$, to the same G constructed for F above. This shows that the \mathcal{B} -parts of the associativity diagram commute.

We now fix (t, f, F) as above along with the corresponding \bar{F} and G that they determine. The fiber of $\mathcal{A}_{(pq)r}$ over (t, f, \bar{F}) is

$$\oint \left(\oint (E' \circ f) \xrightarrow{Q} \int \text{colim}(E' \circ f) \xrightarrow{fF} \int S'' \xrightarrow{E''} \widehat{\mathcal{C}}''' \right),$$

while the fiber of $\mathcal{A}_{|(pq)r|}$ over (t, f, \bar{F}) is

$$\int \text{colim} \left(\int \text{colim}(E' \circ f) \xrightarrow{fF} \int S'' \xrightarrow{E''} \widehat{\mathcal{C}}''' \right).$$

The \mathcal{A} -part of the map $(\pi \cdot \text{id}) \circ \pi$ between them contracts first the inner lax colimit to a strict colimit, then contracts the outer lax colimit to a strict colimit (which Q no longer affects, see [32, Example 4.8]). The fiber of $\mathcal{A}_{p(qr)}$ over (t, G) is

$$\oint \left(x \mapsto \oint \left(\int E' f(x) \xrightarrow{fF_x} \int S'' \xrightarrow{E''} \widehat{\mathcal{C}}''' \right) \right)$$

for x in Et , while the fiber of $\mathcal{A}_{|p(qr)|}$ over (t, G) is

$$\int \text{colim} \left(x \mapsto \text{colim}(E'' \circ f F_x) \right).$$

Similarly, the \mathcal{A} -part of the map $(\text{id} \cdot \pi) \circ \pi$ between them contracts first the inner and then the outer lax colimits to strict ones.

The E component of $\alpha^{\mathbf{Rep}}$ is given by the colimit decomposition isomorphism (see [4, Lemma 7.13], [32, Theorem 5.4]),

$$\text{colim}_{\substack{x' \in \text{colim} \\ x \in Et}} E' F(x') \cong \text{colim}_{x \in Et} \text{colim}_{x' \in E' f(x)} E'' F_x(x'),$$

while the \mathcal{A} component of $\alpha^{\mathbf{Poly}}$ is the analogous isomorphism for lax colimits, sending $\left((x, x'), x'' \right)$ to $\left(x, (x', x'') \right)$ as in Proposition 2.5. Both sides of the \mathcal{A} -part of the

associativity equation, where the cartesian associator maps act in the direction indicated and the vertical productor maps act in the opposite direction, send $\left((x, x'), x''\right)$ to the image of $\left(x, (x', x'')\right)$ in the strict double colimit, either by first reindexing with $\alpha^{\mathbf{Poly}}$ then quotienting twice from lax to strict colimits or first quotienting then reindexing with $\alpha^{\mathbf{Rep}}$.

In conclusion, the diagram commutes, showing \mathbf{gr} satisfies the associativity law for colax bifunctors. In somewhat counterintuitive fashion, we have now proven that \mathbf{gr} is by all accounts a colax bifunctor before proving that the identity, unitor, product, and associator in \mathbf{Rep} satisfy the laws of a bicategory. This is resolved below, but for now we summarize our results on \mathbf{gr} in the following:

Theorem 4.3. $\mathbf{gr}: \mathbf{Rep} \rightarrow \mathbf{Poly}^{vf}$ has the data, structure, and properties of a colax bifunctor.

Remark 4.4. $P_d: \mathbf{Poly}^{vf} \rightarrow \mathbf{CAT}$ is a bifunctor which sends the colax structure maps ϵ, π of \mathbf{gr} to isomorphisms, so the composite $P_d \mathbf{gr}$ has the structure and properties of not just a colax bifunctor, but an actual bifunctor, as the structure maps of the composite are build out of those of P_d along with P_d applied to those of \mathbf{gr} , and all of these maps are isomorphisms.

4.5 Pentagon and Triangle Laws

To demonstrate that the pentagon and triangle laws hold in \mathbf{Rep} , we recall the following more general fact about bicategories:

Proposition 4.5. *Assume \mathbf{A} denotes all of the data and structure of a bicategory (objects, categories of morphisms, identities, composition functors, unitors $\lambda_{\mathbf{A}}, \rho_{\mathbf{A}}$, associator $\alpha_{\mathbf{A}}$), \mathbf{B} is a bicategory, and H contains the data, structure, and properties of a bifunctor $\mathbf{A} \rightarrow \mathbf{B}$ (mapping on objects, functors between morphism categories, productor π , identitor ϵ , unitality and associativity equations) such that the functor $H_{\mathcal{C}', \mathcal{C}}: \mathbf{A}(\mathcal{C}', \mathcal{C}) \rightarrow \mathbf{B}(\mathcal{C}', \mathcal{C})$ is faithful for all objects $\mathcal{C}, \mathcal{C}'$ in \mathbf{A} . Then \mathbf{A} is a bicategory and H is a bifunctor.*

Proof. By definition of \mathbf{A} and H , it only remains to show that the pentagon and triangle laws hold in \mathbf{A} , and as H is locally injective on 2-cells, it suffices to show that the images of the pentagon and triangle diagrams commute in \mathbf{B} . We show this for the triangle law; the (much larger) diagram for the pentagon is constructed similarly from the pentagon diagram in \mathbf{B} . For composable 1-cells p, q in \mathbf{A} , we have the following diagram:

$$\begin{array}{ccccc}
H((p1)q) & \xrightarrow{H(\alpha_{\mathbf{A}})} & & & H(p(1q)) \\
\pi^{-1} \downarrow & & & & \downarrow \pi^{-1} \\
H(p1)H(q) \xrightarrow{\pi^{-1} \cdot \text{id}} (H(p)H(1))H(q) & \xrightarrow{\alpha_{\mathbf{B}}} & H(p)(H(1)H(q)) \xrightarrow{\text{id} \cdot \pi^{-1}} & H(p)H(1q) & \\
H(\rho_{\mathbf{A}}) \cdot \text{id} \downarrow & \text{id} \cdot \epsilon^{-1} \cdot \text{id} \downarrow & & \downarrow \text{id} \cdot \epsilon^{-1} \cdot \text{id} & \downarrow \text{id} \cdot H(\lambda_{\mathbf{A}}) \\
H(p)H(q) \xleftarrow{\rho_{\mathbf{B}} \cdot \text{id}} (H(p)1)H(q) & \xrightarrow{\alpha_{\mathbf{B}}} & H(p)(1H(q)) \xrightarrow{\text{id} \cdot \lambda_{\mathbf{B}}} & H(p)H(q) & \\
& \searrow \rho_{\mathbf{B}} \cdot \text{id} & & \swarrow \text{id} \cdot \lambda_{\mathbf{B}} & \\
& & H(p)H(q) & & \\
& \searrow \pi & \downarrow \pi & \swarrow \pi & \\
& & H(pq) & &
\end{array}$$

The diagram commutes by the associativity and unitality of H , naturality of $\alpha_{\mathbf{B}}$, and the triangle law for \mathbf{B} . The outer left and right composites equal $H(\rho_{\mathbf{A}} \cdot \text{id})$ and $H(\text{id} \cdot \lambda_{\mathbf{A}})$, respectively, by naturality of π , so this is precisely the image under H of the triangle diagram for p, q in \mathbf{A} . \square

CHAPTER 2
HIGHER CATEGORY THEORIES

5 Higher Category Schema

Our motivation for the preceding technical results is to describe familial monads on $\widehat{\mathcal{C}}$ exclusively in terms of their operations and arities, for which we use the following immediate consequence of Theorem 4.1.

Theorem 5.1. *For any small category \mathcal{C} , H restricts to an equivalence between monoids in $\mathbf{Fam}(\widehat{\mathcal{C}}, \widehat{\mathcal{C}})$ and monoids in the monoidal category $\mathbf{Rep}(\mathcal{C}, \mathcal{C})$.*

In this section we begin to explore how this result provides a unifying language for different higher category theories. We first give an explicit description of what constitutes a monoid in $\mathbf{Rep}(\mathcal{C}, \mathcal{C})$ (sometimes denoted simply $\mathbf{Rep}_{\mathcal{C}}$) and show how this information encodes equations between composition operations in a higher category theory. We then discuss how these “schema” or behave when there are top-dimensional cell shapes or the cell shapes are a restriction of those of another higher category theory.

5.1 Monad Representations

By Theorem 5.1, a familial monad is uniquely determined by a monoid in $\mathbf{Rep}(\mathcal{C}, \mathcal{C})$, which by the definitions in Section 4 consists of the data below. We call such a familial

representation over \mathcal{C} a *monad representation* or *higher category schema* and say that such a monad T is *represented by* the pair (S, E) , which implicitly includes the data of η, μ described below.

- A functor $S: \mathcal{C}^{op} \rightarrow \mathbf{Set}$
- A functor $E: \int S \rightarrow \widehat{\mathcal{C}}$

These alone define the functor $T = H_{(S,E)}: \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ by

$$TX_c = \coprod_{t \in Sc} \text{Hom}(Et, X).$$

- A map η^S from $S^0: c \mapsto \{*_c\}$ to S , for which we will often write simply $\eta(c) \in Sc$, along with an isomorphism $\eta^E: E \int \eta^S \rightarrow E^0$, which amounts to natural isomorphisms

$$E\eta(c) \cong y(c)$$

This provides the unit map

$$\eta_X: X_c \cong \text{Hom}(y(c), X) \cong \text{Hom}(E\eta(c), X) \hookrightarrow \coprod_{t \in Sc} \text{Hom}(Et, X) = TX_c.$$

- A map μ^S from $SS: c \mapsto \coprod_{t \in Sc} \text{Hom}(Et, S)$ to S , for which we will often write simply $\mu(t, f) \in Sc$, along with an isomorphism $\mu^E: E \int \mu^S \rightarrow EE$, which amounts to natural isomorphisms

$$E\mu(t, f) \cong \text{colim}_{x \in Et} Ef(x)$$

This provides the multiplication map

$$TTX_c \cong \coprod_{(t \in Sc, f: Et \rightarrow S)} \text{Hom} \left(\text{colim}_{x \in Et} Ef(x), X \right) \cong \coprod_{(t, f)} \text{Hom} \left(E\mu(t, f), X \right) \rightarrow \coprod_{t' \in Sc} \text{Hom}(Et', X) = TX_c.$$

- Commutativity of the following left unitality diagrams:

$$\begin{array}{ccc}
 S^0 S & \xrightarrow{\eta^S; id} & SS \\
 & \searrow \lambda^S & \downarrow \mu^S \\
 & & S
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{colim}_{\iota: y(c') \rightarrow y(c)} E(t\iota) & \xleftarrow{\text{colim } id} & \text{colim}_{x: y(c') \rightarrow E\eta(c)} E(t\eta^E(x)) \\
 & \nwarrow \lambda^E & \uparrow \mu^E \\
 & & Et = E\mu(\eta(c), t: y(c) \rightarrow S)
 \end{array}$$

- Commutativity of the following right unitality diagrams:

$$\begin{array}{ccc}
 SS & \xleftarrow{id; \eta^S} & SS^0 \\
 \mu^S \downarrow & \swarrow \rho^S & \\
 S & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{colim}_{x: y(c') \rightarrow Et} E\eta(c') & \xrightarrow{\text{colim } id} & \text{colim}_{x: y(c') \rightarrow Et} y(c') \\
 \mu^E \uparrow & \searrow \rho^E & \\
 E\mu(t, \eta^S \circ !: Et \rightarrow S^0 \rightarrow S) = Et & &
 \end{array}$$

- Commutativity of the following associativity diagrams, where for $t \in Sc$, $f: Et \rightarrow S$, $F: \text{colim}_{x: y(c') \rightarrow Et} Ef(x) \rightarrow S$, $\alpha^S((t, f), F)$ is given by (t, G) where $G: Et \rightarrow SS$ with $G(x) = (f(x), F|_{Ef(x)})$:

$$\begin{array}{ccc}
 (SS)S & \xrightarrow{\alpha^S} & S(SS) \\
 \mu^S; id \downarrow & & \downarrow id; \mu^S \\
 SS & & SS \\
 \mu^S \searrow & & \swarrow \mu^S \\
 & S &
 \end{array}$$

$$\begin{array}{ccc}
x' : y(c'') \rightarrow \operatorname{colim}_{x : y(c') \rightarrow Et} Ef(x) & EF(x') \xrightarrow{\rho^E} & \operatorname{colim}_{x : y(c') \rightarrow Et} \operatorname{colim}_{x' : y(c'') \rightarrow Ef(x)} EG(x)(x') \\
\operatorname{colim}_{\mu^E} \operatorname{id} \uparrow & & \uparrow \operatorname{colim}_{\operatorname{id}} \mu^E \\
\operatorname{colim}_{x' : y(c'') \rightarrow E\mu(t, f)} EF(\mu^E x') & & \operatorname{colim}_{x : y(c') \rightarrow Et} E\mu(f(x), G(x)) \\
\mu^E \uparrow & & \uparrow \mu^E \\
E\mu(\mu(t, f), F\mu^E) & \xlongequal{\quad} & E\mu(t, x \mapsto \mu(f(x), G(x)))
\end{array}$$

These equations ensure that (T, η, μ) satisfy the unit and associativity laws. Recall that $\lambda^S, \rho^S, \alpha^S$ are defined (in Section 4) by

$$\lambda^S(*_c, t : y(c) \rightarrow S) = t \in Sc, \quad \rho^S(t, ! : Et \rightarrow S^0) = t \in Sc,$$

$$\alpha^S\left((t, f : Et \rightarrow S), F : \operatorname{colim}_{x : y(c') \rightarrow Et} Ef(x) \rightarrow S\right) = \left(t, G : Et \rightarrow SS : x \mapsto (f(x), F_x)\right),$$

where F_x is the composite $Ef(x) \rightarrow \operatorname{colim}_{x' : y(c') \rightarrow Et} Ef(x') \xrightarrow{F} S$.

These isomorphisms and equations may appear tedious to check, but in practice many interesting higher category theories have properties that ensure they arise automatically.

Many examples have E land in *rigid* diagrams, objects in $\widehat{\mathcal{C}}$ with no nontrivial automorphisms, in which case all isomorphisms in the class of the representing diagrams Et are then unique.

Definition 5.2. A familial representation is *rigid* if all of the arities Et are rigid.

In this case it suffices to define $\eta : S^0 \rightarrow S, \mu : SS \rightarrow S$ such that $Ej\eta \cong E^0$ and $Ej\mu \cong EE$ hold as properties rather than structure, then check only the diagrams for the S -parts above. This is the case for all of our recurring examples.

Another common property among our examples is that the set Sc of operations outputting a c -cell contains at most one operation with each possible (isomorphism class of) arity. This and the isomorphisms $E f \eta \cong E^0$ and $E f \mu \cong EE$ ensure that the S -parts of the unit and associativity equations hold automatically, so it suffices to show that the arities present are closed under the appropriate colimits (and if they are not rigid check that the E -parts of the monoid equations hold).

This is the case for n -categories, non-symmetric multicategories, and double categories. Symmetric multicategories by contrast have $n!$ different operations with arity given by each tree with n leaves. However, a slightly more permissive condition with the same effect does apply to symmetric multicategories.

Definition 5.3. A familial representation is *shapely* ([19]) if among the functors

$$\mathcal{C}/c \cong \int y(c) \xrightarrow{t} \int S \xrightarrow{E} \widehat{\mathcal{C}}$$

for $t \in Sc$, each isomorphism class of functor $\mathcal{C}/c \rightarrow \widehat{\mathcal{C}}$ appears at most once.

By naturality of η, μ , this also ensures that the S -parts of the monoid equations automatically hold. Shapeliness can be regarded as a general form of strict associativity: any pair of composites $\mu(t, f), \mu(t', f')$ with $t, t' \in Sc$ that have the same arity (and restrictions of that arity) must be equal. Counterexamples can then only be found in higher categories with multiple different ways of composing the same diagram that don't merely rearrange its faces: in bicategories for instance, there are different operations for composing n successive arrows for each parenthesization.

Example 5.4. The monad structure on the familial representation (S, E) for categories in Example 1.2 (with $S0 = \{0\}$, $S1 = \mathbb{N}$, $En = \xrightarrow{n}$) is given by

$$\eta(0) = 0 \in S0, \quad \eta(1) = 1 \in S1, \quad \mu(0, 0) = 0 \in S0, \quad \mu(n, (m_1, \dots, m_n)) = \sum_i m_i \in S1.$$

It suffices to define μ in terms of $\mu(0, 0)$ and $\mu(n, (m_1, \dots, m_n))$ as a map $\xrightarrow{0} \cong y(0) \rightarrow S$ is determined by an element of $S0$ and a map $\xrightarrow{n} \rightarrow S$ is determined by n different elements of $S1$, since all of the vertices in \xrightarrow{n} must be sent to the unique element of $S0$.

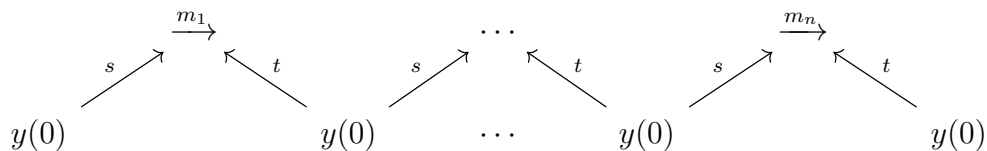
This definition of μ encodes both the classical unit and associativity equations for categories in an *unbiased* way: $\mu(2, (0, 1))$ and $\mu(2, (1, 0))$ describe the binary composite on either side of a single edge with an identity (the edge produced by a path of length 0 in a graph), and both equal $1 \in S1$, the unit operation on edges, so both composites must agree with the original edge. Similarly $\mu(2, (2, 1)) = 3 = \mu(2, (1, 2))$ imposes associativity.

As this representation is rigid, to give the isomorphisms η^E, μ^E it suffices to check that there merely exist isomorphisms

$$E\eta(c) \cong y(c) \quad E\mu(t, f) \cong \operatorname{colim}_{x \in Et} Ef(x) = \operatorname{colim} \left(\int Et \xrightarrow{ff} \int S \xrightarrow{E} \widehat{\mathcal{C}} \right).$$

For η this follows from the graphs $\xrightarrow{0}$ and $\xrightarrow{1}$ being the single vertex and single edge, respectively. For μ , $\int En$ is the category with objects $(0, 0), \dots, (0, n), (1, 1), \dots, (1, n)$ and morphisms $s : (0, i) \rightarrow (1, i + 1)$ and $t : (0, i) \rightarrow (1, i)$ with no nontrivial compos-

ites. The the colimit on the right hand side above is of the diagram



whose colimit is indeed $\xrightarrow{\sum_i m_i}$. This colimit can be envisioned as “plugging $\xrightarrow{m_i}$ into the i th arrow of \xrightarrow{n} ” which results in a (usually) longer string of arrows.

As this representation is shapely, we could have merely checked that there existed *some* operation in $S1$ with this colimit as its arity rather than picking out μ in advance, but in practice there isn’t much difference between the two.

In an algebra A for this monad (a category), this definition of η ensures that the operation 1 sending an arrow in A to an arrow in A is the identity (as is the operation 0 on vertices), while μ ensures that the n -ary composite of adjacent m_i -ary composites agrees with the $(\sum_i m_i)$ -ary composite of the underlying string of arrows, which encodes the unit and associativity equations.

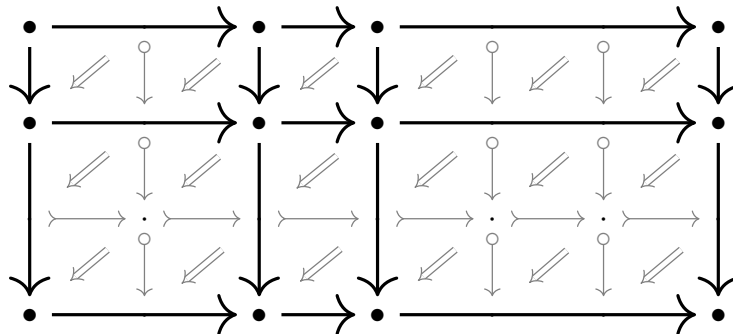
n -categories, double categories, and multicategories also have their monad structure determined from their operations and arities, as they are all rigid and shapely. This means that to show they fit the definition of a higher category schema it suffices to show that their arities are closed under the appropriate colimits.

Example 5.5. Recall that the representation (S, E) of the familial monad on $\mathbf{G}_1 \times \mathbf{G}_1$ whose algebras are double categories has

$$S(\bullet) = \{0\}, \quad S(\dashv) = \mathbb{N}, \quad S(\circ \dashv) = \mathbb{N}, \quad S(\square) = \mathbb{N} \times \mathbb{N},$$

where for both $Ss^v, St^v : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ are the first projection while Ss^h, St^h are both the second projection. This is because the vertical source and target of an $n \times m$ grid are both the string of n horizontal arrows while the horizontal source and target of such a grid are both the string of m vertical arrows. $E0$ is the single vertex, En for $n \in S(\rightarrow)$ is the string of n horizontal arrows, Em for $m \in S(\circ\rightarrow)$ is the string of m vertical arrows, and $E(n, m)$ is the $n \times m$ grid of squares shown in Example 1.14, which we denote by $\xrightarrow{n} \otimes \xrightarrow{m}$.

This familial representation is rigid and shapely, so it suffices to show that these arities are closed under the appropriate colimits. This is clearly the case for operations outputting a vertex or either arrow type as in Example 5.4. A map $f : \xrightarrow{n} \otimes \xrightarrow{m} \rightarrow S$, as $Ss^h = St^h = \pi_2$, must send all squares in the same row of the grid to pairs (i, j) with the same j ; similarly, as $Ss^v = St^v = \pi_1$, all squares in the same column must be sent to pairs (i, j) with the same i . Such a map is then entirely determined by n choices of $i \in S(\rightarrow)$ and m choices of $j \in S(\circ\rightarrow)$. The corresponding colimit defining $\mu((n, m), f)$ then arises from plugging new grids into the squares of $\xrightarrow{n} \otimes \xrightarrow{m}$ which *agree on their boundaries*, and this ensures that the resulting double graph is again a rectangular grid of squares, as below.



Composite operations in 2-categories behave similarly, where the operations plugged into the 2-cells of a pasting diagram must agree on their vertical boundaries (as horizontal boundaries are always just a single vertex on either side).

Example 5.6. (See also [35, Example 2.14].) The definition of η and μ for the free nonsymmetric multicategory monad on multigraphs (Example 1.13) is largely analogous to Example 5.4 where n -leaf trees are plugged into the n -to-1 edges of another tree to form a composite tree, but for symmetric multicategories as alluded to above the way the shapeliness determines how η, μ are defined is more complicated, as it determines not only the underlying tree of the composite but also the permutation of the leaves.

Note that for each operation $t \in S(n, 1)$ for the free symmetric multicategory monad, determined by a tree Et and an ordering on its n leaves, the map $E(s_i)_t : E\eta(0) \cong y(0) \rightarrow Et$ are given by the inclusion of the vertex as the i th leaf in the tree Et according to that ordering. For nonsymmetric multicategories the *planar ordering* on the leaves is always used, meaning $a \leq b$ when there is a multi-edge in the tree such that the leaf a descends from its i th source and the leaf b descends from its j th source with $i \leq j$.

Consider how to choose $\eta(n, 1) \in S(n, 1)$: the operation must have as its arity the height 1 tree with n leaves, but there are $n!$ such operations corresponding to the possible orders on the leaves. Naturality of the isomorphism $\eta^E : E\eta(n, 1) \cong y(n, 1)$ means that $Es_i : E\eta(0) \rightarrow E\eta(n, 1)$ agrees (up to the isomorphisms η^E) with the inclusion $y(0) \rightarrow y(n, 1)$ of the vertex as the i th source of the multi-edge $y(n, 1)$.

This condition, for all i , imposes that $\eta(n, 1)$ must be the height 1 tree with n leaves with the planar order on the leaves. While there are many operations with arity a single n -to-1 edge, when accounting for the faces (inclusions of $y(0)$ as leaves) it becomes unique.

μ is similarly determined. When the composite tree $E\mu(t, f)$ is constructed by plugging m -leaf trees with ordered leaves $Ef(x)$ into each m -to-1 edge x in the tree Et with an order on its n leaves, naturality of μ^E requires that the i th ($1 \leq i \leq n$) inclusion of a vertex into the composite tree $E\mu(t, f)$ corresponds to the i th leaf of Et . As μ^E mediates between $E\mu(t, f)$ and the colimit

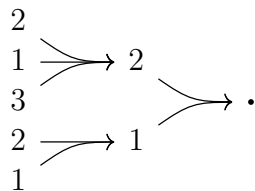
$$\text{colim}(f Et \xrightarrow{f} f S \xrightarrow{E} \widehat{\mathcal{C}})$$

where the i th leaf of Et is a vertex including into $Ef(x)$ for the multi-edge x in Et to which the leaf belongs, in the colimit this is the j th leaf in $Ef(x)$, where the i th leaf of Et is the j th source of x . Therefore, the order on $E\mu(t, f)$ is constructed from the planar order by reordering according to the order on each $Ef(x)$ for all multi-edges x in Et , then again reordering by the order on Et .¹

For example, consider the tree Et below with its leaves ordered as $(3, 2, 5, 1, 4)$ (with respect to the planar order) and the map $f : Et \rightarrow S$ sending each n -to-1 edge to the height 1 tree consisting of a single n -to-1 edge with the orders on leaves as

¹For this description it is perhaps easier to use permutations of leaves in place of orders.

written below.



The composite $E\mu(t, f)$ has the same underlying tree as Et . To construct the order on the leaves, start with the planar order $(1, 2, 3, 4, 5)$. Applying the orders on the leftmost multi-edges yields $(2, 1, 3, 5, 4)$, then applying the order on the rightmost 2-to-1 edge yields $(5, 4, 2, 1, 3)$. Finally, applying the order on Et (essentially, composing the two permutations of the planar order) turns this into $(4, 1, 2, 3, 5)$. This construction agrees with composing multimorphisms and their symmetric group actions in a symmetric multicategory.

5.2 Endpoint Objects

Endpoint objects in a category of cell shapes are precisely those which are “top dimensional” in the sense of not being a face of any other cell shape. These objects are the cell shapes that can be affected by enrichment, just as enriched categories only modify the structure of the arrows in a category, not the objects.

Definition 5.7. An object e in a category \mathcal{C} is an endpoint if it has no non-identity outgoing morphisms.

Example 5.8. In the category $\mathbf{G}_1 = 0 \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} 1$, the object 1 is an endpoint. This is the

sense in which edges are the top-dimensional cells in a graph. More generally

$$\mathbf{G}_n = 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} n$$

has endpoint object n , as n -cells are top-dimensional in n -globular sets, and $\mathbf{G}_1 \times \mathbf{G}_1$ has the square as an endpoint object.

A choice of endpoint object e in a category \mathcal{C} gives rise to a canonical functor $\mathcal{C} \rightarrow \mathbf{\Delta}^1$ sending e to 1 and all other objects to 0.

We will often write $e = \{e_k\}$ for a set of endpoint objects indexed by elements k of some implicit indexing set, and when X is in $\widehat{\mathcal{C}}$ we write X_e for $\coprod_k X_{e_k}$.

Example 5.9. In \mathbf{M} , the indexing category for multigraphs, all n -to-1 edges are top-dimensional.

Proposition 5.10. *For e a set of endpoint objects in \mathcal{C} , X in $\widehat{\mathcal{C}}$, and $Y : \int X \rightarrow \widehat{\mathcal{C}}$ with $Y(x)_e$ empty whenever $x \notin X_e$,*

$$(\operatorname{colim}_{x \in X} Y(x))_e \cong \coprod_{x \in X_e} Y(x)_e.$$

Proof. $Y(x)_e$ is assumed empty whenever $x \notin X_e$, and as e is an endpoint object no two e -cells in X are related by maps in $\int X$. Therefore the e -cells in the colimit, generally a quotient of the disjoint union of those in $Y(x)$ for all x , are simply the disjoint union of $Y(x)_e$ for $x \in X_e$ as none of these e -cells will be related by maps in the diagram Y .. □

For e a set of endpoint objects, we let $\mathcal{C}\setminus e$ denote the full subcategory of objects in \mathcal{C} which are not in e .

Definition 5.11. A famillal endofunctor on $\widehat{\mathcal{C}}$ represented by (S, E) is e -graded for a set of endpoint objects e if Et_e is empty for all $t \in Sc$ with c in $\mathcal{C}\setminus e$.

Example 5.12. For the choices of endpoint(s) e described above, the free n -category, double category, and multicategory monads are all e -graded, as only the arities of operations outputting e -cells contain any e -cells.

Corollary 5.13. For familial endofunctors on $\widehat{\mathcal{C}}$ represented by (S, E) and e -graded (S', E') , and $(t \in Sc, f : Et \rightarrow S') \in SS'c$,

$$EE'(t, f)_e \cong \coprod_{x \in Et_e} E'f(x)_e.$$

Proof. This follows immediately from the previous proposition and the definition of $EE'(t, f)$ as a colimit over $f Et$. □

Example 5.14. This is explicitly encoded in μ for the free category monad, expressed as a sum in \mathbb{N} . For the other recurring examples, it is straightforward to check that this is the case. For instance, the set of multi-edges in a composite tree $E\mu(t, f)$ is indeed the disjoint union of the multi-edges in the subtrees $Ef(x)$ for x ranging over multi-edges in Et .

5.3 Restriction of Cell Shapes

We will often be interested in restricting a familial monad on $\widehat{\mathcal{C}}$ to a monad on diagrams over a subcategory of \mathcal{C} . To this end, we fix a functor $u : \mathcal{C}' \rightarrow \mathcal{C}$, which will typically be the inclusion of a full subcategory, and write $u^* : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}'}$ for the corresponding restriction functor and $u_!, u_*$ for its left and right adjoints, respectively.

Definition 5.15. Given a familial endofunctor T on $\widehat{\mathcal{C}}$ represented by (S, E) , its restriction along u is the endofunctor on $\widehat{\mathcal{C}'}$ denoted u^*T and represented by (u^*S, u^*E) where u^*E is the composite functor

$$\int u^*S \rightarrow \int S \xrightarrow{E} \widehat{\mathcal{C}} \xrightarrow{u^*} \widehat{\mathcal{C}'},$$

where the leftmost functor is the pullback of u along the discrete fibration $\int S \rightarrow \mathcal{C}$.

In particular, for X in $\widehat{\mathcal{C}'}$ and c' in \mathcal{C}' ,

$$(u^*T)X_{c'} = \coprod_{t \in Su(c')} \text{Hom}_{\widehat{\mathcal{C}'}}(u^*Et, X).$$

It is straightforward to check that u^* forms a functor $\mathbf{Rep}_{\mathcal{C}} \rightarrow \mathbf{Rep}_{\mathcal{C}'}$.

Definition 5.16. A familial endofunctor T represented by (S, E) is *u-graded* if the natural transformation $u_!u^*E \rightarrow E$ is an isomorphism when restricted along $\int u^*S \rightarrow \int S$.

Lemma 5.17. *If X is in $\widehat{\mathcal{C}}$ and $u_!u^*X \rightarrow X$ is an isomorphism, then the functor $\int u^*X \rightarrow \int X$ is final (in the sense of [29, Definition 2.1.4]).*

Proof. Recall that $f u^* X$ is the pullback of $f X \rightarrow \mathcal{C}$ along u , and $f u_! u^* X$ is the final- (discrete fibration) factorization ([29, Proposition 2.1.5]) of the composite functor $f u^* X \rightarrow \mathcal{C}' \rightarrow \mathcal{C}$. As the latter is isomorphic to $f X$ by assumption, the functor $f u^* X \rightarrow f X$ must be final. \square

Proposition 5.18. $u^* : \mathbf{Rep}_{\mathcal{C}} \rightarrow \mathbf{Rep}_{\mathcal{C}'}$, restricted to u -graded monads is a monoidal functor when u is fully faithful.

Proof. Restriction preserves terminal presheaves, so $u^* S^0 = S^0$ in $\widehat{\mathcal{C}'}$. $u^* E$ sends $*_{c'} \in u^* S^0 c'$ to $u^* y(u(c')) \cong u^* u_! y(c')$, which is isomorphic to $y(c')$ via the unit of the adjunction precisely when u is fully faithful.

$$\begin{aligned} (u^* S)(u^* S')c' &= \coprod_{t \in Su(c')} \mathrm{Hom}_{\widehat{\mathcal{C}'}}(u^* Et, u^* S') \cong \coprod_{t \in Su(c')} \mathrm{Hom}_{\widehat{\mathcal{C}'}}(u_! u^* Et, S) \\ &\cong \coprod_{t \in Su(c')} \mathrm{Hom}_{\widehat{\mathcal{C}'}}(Et, S) = (u^* S S')c', \end{aligned}$$

the last isomorphism given by the u -gradedness condition. We then have

$$\begin{aligned} (u^* E)(u^* E')(t, f : u^* Et \rightarrow u^* S') &= \mathrm{colim}_{x \in f u^* Et} u^* E' f(x) \cong u^* \left(\mathrm{colim}_{x \in f u^* Et} E' f(x) \right) \\ &\rightarrow u^* \left(\mathrm{colim}_{x \in f Et} E' f(x) \right) \cong u^* (EE'(t, f)) = (u^* EE')(t, f). \end{aligned}$$

The above is therefore an isomorphism if the functor $f u^* Et \rightarrow f Et$ is final (as by definition then the induced map on colimits is an isomorphism), which follows from Lemma 5.17 (in fact, this functor is an isomorphism). Verifying the unitality and associativity equations is tedious but straightforward. \square

Note that the assumption that u is fully faithful is only used to show that u^* preserves units; without that assumption u^* is *nearly* colax-monoidal, but the map between units would be a non-cartesian maps between familial representations.

The following is an immediate consequence of Proposition 5.18

Corollary 5.19. $u^* : \mathbf{Rep}_{\mathcal{C}} \rightarrow \mathbf{Rep}_{\mathcal{C}'}$ lifts to a functor from u -graded familial monads on $\widehat{\mathcal{C}}$ to familial monads on $\widehat{\mathcal{C}'}$ when u is fully faithful.

Finally, we show that restriction along u induces a functor between algebras.

Proposition 5.20. For T a u -graded familial monad on $\widehat{\mathcal{C}}$ with u fully faithful, $u^* : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}'}$ lifts to a functor from T -algebras to u^*T -algebras.

Proof. Given X in $\widehat{\mathcal{C}}$, observe that

$$\begin{aligned} (u^*T)(u^*A)_{\mathcal{C}'} &= \coprod_{t \in \text{Su}(\mathcal{C}')} \text{Hom}_{\widehat{\mathcal{C}'}}(u^*Et, u^*A) \\ &\cong \coprod_{t \in \text{Su}(\mathcal{C}')} \text{Hom}_{\widehat{\mathcal{C}}}(u!u^*Et, A) \cong \coprod_{t \in \text{Su}(\mathcal{C}')} \text{Hom}_{\widehat{\mathcal{C}}}(Et, A) \cong u^*(TA)_{\mathcal{C}'}, \end{aligned}$$

so $(u^*T)(u^*A) \cong u^*(TA)$.

Now given a T -algebra $TA \rightarrow A$, applying u^* gives a map $(u^*T)(u^*A) \cong u^*(TA) \rightarrow u^*A$ in $\widehat{\mathcal{C}'}$, and it is straightforward to check that this map satisfies the properties of a u^*T -algebra. \square

Example 5.21. Consider a small category \mathcal{C} with a set $e = \{e_k\}$ of endpoint objects, with $u : \mathcal{C} \setminus e \rightarrow \mathcal{C}$ the canonical inclusion. For X in $\widehat{\mathcal{C}}$, we write $\partial_e X$ for u^*X .

A familial monad T on $\widehat{\mathcal{C}}$ is then u -graded precisely when it is e -graded (Definition 5.11). We write $\partial_e T$ for the familial monad u^*T on $\widehat{\mathcal{C}\setminus e}$, which has the same operations and arities as T excluding the e -operations. As T is e -graded, the arities Et for $t \in Sc$, $c \notin e$, do not include any e -cells, so applying ∂_e leaves them entirely unchanged.

Example 5.22. For the free category or free multicategory monads with e containing all endpoint objects and u as in Example 5.21, u^* produces the identity monad on sets, as $\mathcal{C}\setminus e$ is the one-object category and μ^*T has just one operation with the singleton as its arity. The functor u^* on algebras sends a small (multi)category to its set of objects.

Example 5.23. Let u be the inclusion $\mathbf{G}_m \rightarrow \mathbf{G}_n$ for $m \leq n$ sending i to i , s to s , t to t . The free n -category monad T on n -globular sets is u -graded as $u_!u^*X$ simply deletes the cells in X above dimension m , and operations outputting cells of at most dimension m have no such higher dimensional cells to begin with. The restricted monad u^*T on m -globular sets is the free m -category monad and the functor u^* on algebras sends an n -category to its underlying m -category, forgetting the higher dimensional cells.

Example 5.24. Consider $\mathbf{G}_1 \times \mathbf{G}_1$ and the free double category monad T on double graphs. For u as in Example 5.21 with e the endpoint object \square , T is u -graded and restricts to the monad on graphs with two types of edges that applies the free category monad to both types of edges, with algebras pairs of categories which have the same objects. T is also u -graded when $u : \mathbf{G}_1 \rightarrow \mathbf{G}_1 \times \mathbf{G}_1$ sends 0 to \bullet and 1 to \twoheadrightarrow (resp. $\circ \rightarrow$). In this case the restriction of T is the free category monad, and the

restriction functor on algebras sends a double category to its underlying category of horizontal (resp. vertical) arrows.

$u : \mathbf{G}_1 \rightarrow \mathbf{G}_1 \times \mathbf{G}_1$ could also send 0 to \rightrightarrows and 1 to \square . In this case u^* sends a double graph to the graph with vertices the horizontal arrows and edges the squares between them. $u_!$ sends a graph to the double graph with a horizontal edge for each vertex and a square for each edge, where no two horizontal edges or squares are horizontally adjacent. T is not u -graded, as applying $u_!u^*$ to an $n \times m$ grid of squares separates the grid into n disjoint $1 \times m$ grids. One interpretation for this failure is that u can “see” the horizontal composition of squares and model it as composition of edges in a graph, but vertical composition can’t be preserved in any way by this restriction as there is no way to check in a graph whether two vertices or edges are “vertically adjacent” in a sense compatible with u .

6 Lawvere Theories and Nerves

The construction of the nerve functor $\mathbf{Cat} \rightarrow \widehat{\Delta}$ can be replicated for any familial monad T , with the role of Δ replaced by a category Θ_T . When T is a finitary monad on sets (meaning all of its arities are finite), Θ_T is the opposite of the Lawvere Theory of T . We review Lawvere Theories and show how the category Θ_T generalizes key features of both Lawvere Theories and Δ .

6.1 Lawvere Theories

Definition 6.1. Let T be a finitary monad on \mathbf{Set} . The *Lawvere Theory of T* is the category L_T , where L_T is the full subcategory of T -algebras spanned by the free T -algebras $T\underline{n}$.

In L_T the object $T\underline{n}$ is a finite coproduct of $T\underline{1}$, as $T : \mathbf{Set} \rightarrow \mathbf{alg}(T)$ is a left adjoint and hence preserves coproducts.

Remark 6.2. Lawvere theories are often considered via the opposite of L_T in which every object is a finite product of $T\underline{1}$, but like in [25, Chapter II, Proposition III.2] we call L_T the theory.

Definition 6.3. ([25, First definition of Chapter III]) An *L_T -model* is a product preserving functor $L_T^{op} \rightarrow \mathbf{Set}$.

Example 6.4. Let T be the free monoid monad on \mathbf{Set} . For a fixed monoid M , there is a product preserving functor $M^- : L_T^{op} \rightarrow \mathbf{Set}$ sending $T\underline{n}$ to M^n . The functor preserves projections, diagonals, and products of morphisms, so it suffices to specify the image of the maps $T\underline{2} \rightarrow T\underline{1}$ and $T\underline{0} \rightarrow T\underline{1}$ given by $x_1 \mapsto x_1x_2$ and $x_1 \mapsto e$ respectively (recall that a map of monoids from $T\underline{1} \cong \mathbb{N}$, which is a map to $T\underline{1}$ in L_T^{op} , is determined by the image of the generator which we denote x_1). These are sent to the functions $M^2 \rightarrow M$ and $1 \rightarrow M$ that define the product and unit in the monoid M .

In a similar fashion, for any finitary monad T , a T -algebra A gives rise to a

L_T -model A^- sending $T\underline{n}$ to A^n , with the action on morphisms determined by the cartesian product structure and algebra structure maps in A .

Theorem 6.5. *Up to isomorphism, every L_T -model is of the form A^- for some T -algebra A .*

In other words, the category of L_T -models is equivalent to the category of T -algebras. This can also be phrased as a fully faithful embedding of the category of T -algebras into the presheaf category $\widehat{L_T}$, where the essential image of this embedding is given by those functors $L_T^{op} \rightarrow \mathbf{Set}$ which preserve products.

6.2 The Theory Category Θ_T and Nerves

We first recall the definition of the nerve of a category.

Example 6.6. Δ can be defined as the full subcategory of \mathbf{Cat} on the finite nonempty ordinal categories $[n]$ with $n + 1$ objects and n generating arrows for each natural number n . The nerve functor $\mathcal{N} : \mathbf{Cat} \rightarrow \widehat{\Delta}$ is then defined as

$$\mathcal{N}\mathcal{A}_n = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathcal{A}).$$

Notably, \mathcal{N} is fully faithful.

We now fix a familial monad T on $\widehat{\mathcal{C}}$ represented by (S, E) . The idea of Θ_T is to imitate L_T when T is a familial monad on any $\widehat{\mathcal{C}}$.

Definition 6.7. The *theory category* of T , denoted Θ_T , is the full subcategory of T -algebras spanned by the free algebras $T\mathit{Et}$ on each arity diagram. Likewise, Θ_0 (or $\Theta_{T,0}$ when T is not clear from context) is the full subcategory of $\widehat{\mathcal{C}}$ spanned by the arity diagrams Et , and the free algebra functor on $\widehat{\mathcal{C}}$ restricts to a functor $T_\Theta : \Theta_0 \rightarrow \Theta_T$.

The “theory” terminology is inspired by Lawvere theories and [5, Definition 1.5]. Θ_T could equivalently be defined as a full subcategory of the Kleisli category of T on the arity diagrams; using $\mathit{alg}(T)$ will be more convenient for defining the nerve functor, while the Kleisli category description is better suited for the constructions in Section 6.3.

Remark 6.8. We will treat all of the functors

$$\mathcal{C} \xrightarrow{\eta} \int S \xrightarrow{E} \Theta_0 \xrightarrow{T_\Theta} \Theta_T$$

as inclusions of subcategories, the latter two identity-on-objects. Hence in Θ_T we will write simply t for the free algebra $T\mathit{Et}$ ($t \in S\mathcal{C}$), and write c for $T\mathit{E}\eta(c)$ (which is isomorphic to $T\mathit{y}(c)$). To this end, any two operations in S are treated as distinct objects in Θ_0 and Θ_T even if their arities are isomorphic (though often it is more convenient to work with the skeleton, especially in the situations described in Section 6.4).

$\mathcal{C} \hookrightarrow \Theta_T$ will occasionally be called the *elementary subcategory*, and Θ_0 is called the *inert subcategory*. We show in Lemma 6.21 that T_Θ is in fact a faithful inclusion of Θ_0 into Θ_T .

Example 6.9. When T is the free category monad on graphs, Θ_T is the full subcategory of \mathbf{Cat} on the ordinal categories $[n]$ which are the free categories on the string graphs \xrightarrow{n} , but by our convention on objects $[0]$ is counted twice, for both $0 \in S_0$ and $0 \in S_1$. Θ_T is then the usual category Δ but with two uniquely-isomorphic copies of $[0]$. This makes no meaningful difference to the theory of nerves of categories, but becomes more significant when considering active subcategories in Section 6.3 and Section 6.4.

We are now ready to generalize the nerve functor to T -algebras.

Definition 6.10. The *nerve functor* for T -algebras is defined as

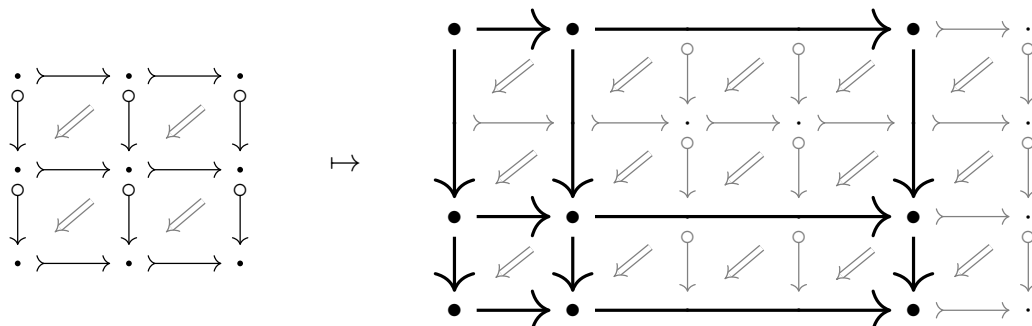
$$\mathcal{N}_T : \text{alg}(T) \rightarrow \widehat{\Theta}_T \quad \mathcal{N}_T(A)_t = \text{Hom}_{\text{alg}(T)}(TEt, A).$$

Example 6.11. For T the free n -category monad on n -globular sets, Θ_T is the full subcategory of free n -categories on the n -dimensional pasting diagrams such as in Example 1.12. This is precisely Joyal's category Θ_n ([23]), and the corresponding nerve functor from n -categories to $\widehat{\Theta}_n$ is the standard *cellular nerve* of n -categories (see [5, Theorem 1.12, Remark 1.13]). As with Δ , our definition of Θ_n includes $n-m+1$ different uniquely-isomorphic copies of each m -dimensional pasting diagram, $0 \leq m \leq n$, corresponding to the operations with that pasting diagram as arity outputting an ℓ -cell for $m \leq \ell \leq n$.

Similarly, the theory category for the free ω -category monad on globular sets is Joyal's category Θ , the sequential colimit of the categories Θ_n .

Example 6.12. For T the free double category monad on double graphs, Θ_T is the

category of free double categories on $n \times m$ grids of squares, whose morphisms are easily checked to correspond to linear inclusions of grids such as the one depicted below.



These morphisms are uniquely determined by the pair of morphisms in Δ describing the action on horizontal arrows and vertical arrows respectively. It is then straightforward to see that Θ_T is in fact $\Delta \times \Delta$ (this product even perfectly accounts for our convention on repeated objects). The corresponding nerve functor from double categories to bisimplicial sets is the standard one ([15, Definition 2.14]).

More generally, there is a free n -tuple category monad on diagrams over $\mathbf{G}_1^{\times n}$ whose arities are given by n -dimensional grids of n -cubes, with theory category $\Delta^{\times n}$ and nerve functor to n -simplicial sets.

Example 6.13. For T the free (symmetric/nonsymmetric) multicategory monad on multigraphs, Θ_T is the (ordinary/planar) *tree category* Ω of Moerdijk and Weiss ([30]), which is defined as the full subcategory of free (symmetric/nonsymmetric) multicategories on the arity trees. The nerve functor to the corresponding diagrams, called *dendroidal sets*, is precisely the dendroidal nerve.

Weber showed that just like the classical nerve of categories, \mathcal{N}_T is always fully

faithful and nerves are characterized among diagrams in $\widehat{\Theta}_T$ by certain limit conditions.

Definition 6.14. $X : \Theta_T^{op} \rightarrow \mathbf{Set}$ satisfies the *Segal condition* if for all t in S ,

$$X_t \cong \lim_{x:y(c) \rightarrow Et} X_c.$$

Theorem 6.15 ([35, 4.10]). \mathcal{N}_T is fully faithful, and a diagram in $\widehat{\Theta}_T$ is isomorphic to the nerve of a T -algebra precisely when it satisfies the Segal condition.

Theorem 6.15 can be seen as a generalization of the perspective of models for Lawvere theories, by rephrasing the Segal condition in terms of limit preservation.

Lemma 6.16. For any operation $t \in Sc$ and $f : Et \rightarrow S$, we have in Θ_T

$$\mu(t, f) \cong \text{colim}(f Et \xrightarrow{Jf} f S \xrightarrow{E} \Theta_0 \xrightarrow{T_\Theta} \Theta_T).$$

Proof. By assumption $E\mu(t, f) \cong \text{colim}(E \circ f f)$ in $\widehat{\mathcal{C}}$, and the left adjoint $T : \widehat{\mathcal{C}} \rightarrow \text{alg}(T)$ preserves colimits. As Θ_0 and Θ_T are full subcategories the same colimits are reflected in Θ_0 and Θ_T respectively and hence preserved by T_Θ . \square

By Lemma 6.16, the Segal condition is equivalent to the functor $X : \Theta_T^{op} \rightarrow \mathbf{Set}$ preserving the limits in Θ_T^{op} described in the lemma.

Definition 6.17. A Θ_T -model is a functor $X : \Theta_T^{op} \rightarrow \mathbf{Set}$ which satisfies the Segal condition, and a morphism of Θ_T -models is a natural transformation of functors.

The following is then an equivalent rephrasing of Theorem 6.15.

Corollary 6.18. *The category of T -algebras is equivalent via \mathcal{N}_T to the category of Θ_T -models.*

Example 6.19. When T is the free monoid monad with e its unique cell shape (the vertex), Θ_T is the category of free monoids on the sets \underline{n} , opposite the Lawvere theory for monoids. The nerve of a monoid M is the corresponding functor M^- sending $T\underline{n}$ to M^n , and the Segal condition is that $X : \Theta_T \rightarrow \mathbf{Set}$ sends \underline{n} to $X(1)^n$, equivalent to the product preservation condition for models.

Example 6.20. (See also [5, Remark 1.7] for related commentary.) In general, any familial monad on \mathbf{Set} with finite arities has Θ_T given by the opposite of the Lawvere theory, L_T . Θ_T -models are then the same as models for the Lawvere theory, and the Segal condition is simply product preservation from L_T^{op} to \mathbf{Set} . The fact that Lawvere theories apply more broadly than to just familial monads on \mathbf{Set} suggests that theories and nerves can be defined for a broader class of monads than just the familial ones, though this is not needed for our purposes (see [8]).

6.3 Active and Inert Subcategories

Θ_T carries a factorization system, where morphisms factor as *active maps* followed by *inert maps* (this is a reformulation of ideas primarily from [34], following the style of [16, 2.2,2.3]).

Inert maps are those of Θ_0 , which we now prove is a subcategory of Θ_T .

Lemma 6.21. *$T_\Theta : \Theta_0 \rightarrow \Theta_T$ is faithful.*

Proof. For any $g, h : Et \rightarrow Et'$ with $Tg = Th$, the naturality square below commutes for both g and h . As η is a natural monomorphism (either by cartesianness of T or the explicit form of η as the inclusion of a component of a coproduct), this means $g = h$.

$$\begin{array}{ccc} Et & \xrightarrow{\eta_{Et'}} & TEt \\ g \downarrow & & \downarrow Tg \\ Et' & \xrightarrow{\eta_{Et'}} & TEt' \end{array}$$

□

Example 6.22. In Δ , the inert morphisms are the linear maps $[m] \rightarrow [n]$ (which send adjacent objects in $[m]$ to adjacent objects in $[n]$), as these are the functors between ordinal categories that arise from the underlying maps between their generating graphs $\xrightarrow{m}, \xrightarrow{n}$.

Similarly, inert morphisms in Θ_n are those which are “linear in each dimension” (see for an alternative description of these inert maps). Inert morphisms in $\Delta \times \Delta$ are pairs of inert morphisms in Δ .

Example 6.23. In Ω (planar or otherwise), the inert morphisms are the “linear” inclusions of planar trees, meaning the morphisms which send single multi-arrows to single multi-arrows. Even in the case of symmetric multicategories, these do not include any maps of the corresponding free multicategories which don’t respect the planar order on the leaf objects, so the isomorphisms in Ω between trees with different orders on the leaves are not considered inert.

By the adjunction between $\widehat{\mathcal{C}}$ and $alg(T)$, a morphism $TEt \rightarrow TEt'$ in $alg(T)$ is

determined by a map $Et \rightarrow TEt'$ in $\widehat{\mathcal{C}}$. Each $x \in Et_{c'}$ is sent to an operation $t_x \in Sc'$ and a map $Et_x \rightarrow Et'$. When $t = \eta(c)$, $t_{\text{id}_c} = t'$, and $Et' \rightarrow Et'$ the identity, we write $a_t : c \rightarrow t$ for the corresponding morphism in Θ_T . These are the “cocomposition maps” which exhibit t -composition $A_t \rightarrow A_c$ in a Θ_T -model $A : \Theta_T^{\text{op}} \rightarrow \mathbf{Set}$. An active map $t \rightarrow t'$ is a colimit of cocomposition maps, exhibiting t' as a composite of operations with t .

Definition 6.24. An active map in Θ_T is of the form $a_f : t \rightarrow \mu(t, f)$ for some $t \in Sc$, $f : Et \rightarrow S$, and is given by

$$t \cong \operatorname{colim}_{x:y(c') \rightarrow Et} c' \xrightarrow{\operatorname{colim} a_{f(x)}} \operatorname{colim}_{x:y(c') \rightarrow Et} f(x) \cong \mu(t, f).$$

Example 6.25. The cocomposition maps in Δ are the endpoint-preserving maps $[1] \rightarrow [n]$, picking out the composite arrow of the n generating edges of the free category $[n]$. The arity colimits in Δ take ordinals $[n_1], \dots, [n_m]$ and adjoin them along their endpoints to get $[n_1 + \dots + n_m]$ as in [11, Section 2]. Hence on cocomposition maps, the arity colimits assemble m different cocomposition maps $[1] \rightarrow [n_i]$ into a single endpoint-preserving map $[m] \rightarrow [n_1 + \dots + n_i]$. In fact all endpoint-preserving maps arise in this way, so the active maps in Δ are precisely those that preserve endpoints.

Just as with inert maps, the active maps in Θ_n and $\Delta \times \Delta$ arise from those in Δ , the former being maps which “preserve endpoints” in every dimension and the latter being pairs of endpoint-preserving maps.

Example 6.26. In Ω , active maps are those which preserve both leaves and the root, but need not preserve the order on leaves (isomorphisms permuting the leaves

of a tree are active). As in Δ , these maps go from one tree with n leaves to another tree with n leaves whose multi-arrows can be composed or permuted into the domain tree.

Lemma 6.27. *For any c in \mathcal{C} , $a_{\eta(c)} = \text{id}_c$ in Θ_T , and for any $t \in Sc$ and $f : Et \rightarrow S$, $a_f \circ a_t = a_{\mu(t,f)} : c \rightarrow \mu(t, f)$.*

Proof. The first equation follows from the fact that a map $TEt \rightarrow TEt'$ in $\text{alg}(T)$ corresponds to the map $Et \rightarrow TEt'$ in $\widehat{\mathcal{C}}$ given by precomposing with $\eta_{Et} : Et \rightarrow TEt$. The second equation follows from the reverse correspondence: the map $Et \rightarrow TE\mu(t, f)$ corresponding to a_f sends x to the pair $(f(x), Ef(x) \rightarrow E\mu(t, f))$, and extends to a map $TEt \rightarrow TE\mu(t, f)$ sending $(t', g : Et' \rightarrow Et)$ in TEt to the pair

$$(\mu(t', fg), E\mu(t', fg)) \cong \text{colim}_{x' \in Et'} Efg(x') \rightarrow E\mu(t, f)$$

in $TE\mu(t, f)$. When t' is t and $g = \text{id}_{Et}$ (namely, the image of $(\eta(c), \text{id}_{y(c)})$ under $a_t : Ty(c) \rightarrow TEt$) a_f then sends the pair to $(\mu(t, f), \text{id}_{E\mu(t,f)})$, which shows the composite $Ty(c) \rightarrow TEt \rightarrow TE\mu(t, f)$ is given by $a_{\mu(t,f)}$ as desired. \square

Proposition 6.28. *Active maps form a subcategory Θ_a of Tht .*

Proof. Identities are active maps by choosing f to be $Et \rightarrow * \xrightarrow{\eta} S$, so a_f is a colimit of identities.

For composition, we start with active maps $t \rightarrow \mu(t, f) \rightarrow \mu(\mu(t, f), F)$ and use the associator to turn $F : E\mu(t, f) \rightarrow S$ into the conglomerate of morphisms

$F_x : Ef(x) \rightarrow E\mu(t, f) \rightarrow S$. The composite is then the morphism

$$t \rightarrow \mu(t, x \mapsto \mu(f(x), F_x)) = \mu(\mu(t, f), F),$$

as for each component of the colimit the composite of $a_{f(x)}$ with a_{F_x} is $a_{\mu(f(x), F_x)}$ by Lemma 6.27. \square

Note that this subcategory will have separate connected components for each c in \mathcal{C} , as all morphisms are between $t, \mu(t, f)$ which are both c -operations. We will sometimes write Θ_a^c for the connected component of c -operations, so Θ_a can be regarded as the coproduct of the categories Θ_a^c .

Example 6.29. In Δ , using our convention on the objects of Θ_T , the unique isomorphism between the two copies of $[0]$ is inert (as an isomorphism of the graph with only one vertex) but not active. Therefore the active subcategory of Δ as a theory category is the disjoint union of Θ_a^0 which is the terminal category consisting of $[0]$ and Θ_a^1 which is the subcategory of endpoint preserving maps in Δ . This subcategory of Δ is known to be isomorphic to Δ_+^{op} , the opposite of the augmented simplex category of all finite ordinals and monotone maps, where an endpoint preserving map in one direction corresponds to a monotone map in the opposite direction on the sets of generating edges (each edge in the codomain sent to the edge in the domain that “covers” it). (This can be deduced from, for instance, [23, 1.1].)

Example 6.30. The active subcategory of $\Delta \times \Delta$ as a theory category for the free double category monad consists of the terminal category Θ_a^\bullet , two copies of Δ_+^{op} as Θ_a^{\rightarrow} and Θ_a^{\leftarrow} , and $\Delta_+^{op} \times \Delta_+^{op}$ as Θ_a^\square .

We now construct the active-inert factorization system on Θ_T .

Theorem 6.31. *Each morphism in Θ_T factors uniquely as an active map followed by an inert map.*

In Δ (and mostly analogously in our other examples) this factors a map $g : [m] \rightarrow [n]$ as an endpoint preserving map, to the ordinal subcategory of $[n]$ from $g(0)$ to $g(m)$, followed by a linear map including this ordinal into $[n]$.

Proof. Consider $g : t \rightarrow t'$, and let $f_g : Et \rightarrow S$ be the composite $Et \xrightarrow{g} TEt' \xrightarrow{T!} T* \cong S$ in $\widehat{\mathcal{C}}$. $T! : TEt' \rightarrow T* \cong S$ sends a pair $(t'' \in Sc'', Et'' \rightarrow Et')$ to the underlying operation t'' in S , so f_g sends each $x \in Et_{c'}$ to the operation its image in TEt' is the output of. The data of g amounts to the operations $f_g(x)$ and compatible maps $Ef_g(x) \rightarrow Et'$.

g then factors as the active map $t \rightarrow \mu(t, f_g)$ followed by the inert map $g_0 : \mu(t, f_g) \rightarrow t'$ given by the map

$$E\mu(t, f_g) \cong \operatorname{colim}_{x:y(c') \rightarrow Et} Ef_g(x) \rightarrow Et'$$

induced by the maps $Ef_g(x) \rightarrow Et'$, as each $x \in Et_{c'}$ is sent by the active map to the pair

$$(f_g(x), Ef_g(x) \rightarrow E\mu(t, f_g))$$

in $TE\mu(t, f_g)_{c'}$, which is sent by g_0 to the pair

$$(f_g(x), Ef_g(x) \rightarrow Et')$$

in $TEt'_{c'}$, precisely the image of x under g . □

Example 6.32. When T is the free monoid monad Θ_T is the category of free monoids on the sets \underline{n} and Θ_a is Δ_+^{op} . The active-inert factorization on Θ_T factors the map $\langle x_1, \dots, x_m \rangle \rightarrow \langle y_1, \dots, y_n \rangle$ sending x_i to $y_{j_{i,1}} \cdots y_{j_{i,\ell_i}}$ into the composite

$$\langle x_1, \dots, x_m \rangle \rightarrow \langle z_{1,1}, \dots, z_{1,\ell_1}, \dots, z_{m,1}, \dots, z_{m,\ell_m} \rangle \rightarrow \langle y_1, \dots, y_n \rangle$$

where the first map sends x_i to $z_{i,1} \cdots z_{i,\ell_i}$ and the second map sends $z_{i,k}$ to $y_{j_{i,k}}$. The first map is active, with each generator in the codomain appearing in order exactly once in the images of the x_i , and the second map is inert, with each generator in the domain sent to a generator in the codomain. As such, the active maps from the free monoid on \underline{n} to the free monoid on \underline{m} correspond to ordered functions from \underline{m} to \underline{n} , which exhibits Θ_a as Δ_+^{op} .

6.4 Degenerated Unit Operations

In many standard examples of higher categories, each cell shape c has “degeneracy” or “unit” operations which output a c -cell for each lower-dimensional cell shape. These operations are typically those providing “identity cells” in a higher category, which arise from lower dimensional cells like the identity morphisms in a category. When a monad T has suitably many such unit operations, its active subcategory Θ_a can be regarded as a (more) connected subcategory of Θ_T than its description above as a disjoint union of the categories Θ_a^c .

Definition 6.33. A familial monad T on $\widehat{\mathcal{C}}$ has P -units if there is a poset P on the objects of \mathcal{C} and for each $c' \leq c$ in P an element $\eta_c(c') \in Sc$ and an isomorphism

$\eta_{c,c'} : E\eta_c(c') \cong y(c')$ such that

- $\eta_c(c) = \eta(c)$ and $E\eta_c(c) = E\eta(c) \rightarrow y(c)$ is given by the unit on T
- For $c'' \leq c' \leq c$ in P , $\mu(\eta_c(c'), \eta_{c'}(c'')) = \eta_c(c'')$ and the following diagram commutes:

$$\begin{array}{ccc}
 E\eta_c(c'') & \xrightarrow{\eta_{c,c''}} & y(c'') \\
 \cong \downarrow & & \uparrow \eta_{c',c''} \\
 \operatorname{colim}_{x:y(c'') \rightarrow E\eta_c(c') \cong y(c')} E(S_{\eta_{c,c'}x}\eta_{c'}(c'')) & \xrightarrow{\cong} & E\eta_{c'}(c'')
 \end{array}$$

Note that there are no conditions on how the operations $\eta_c(c')$ are acted on by \mathcal{C} in S , only that they are compatible with η, μ , though in practice they are often related in both ways. The notation here is meant to portray the operations $\eta_c(c')$ as an extension of the unit operations $\eta(c) \in Sc$ according to the poset P .

Example 6.34. For T the free category monad on graphs, let P be the poset $0 < 1$ on $\operatorname{Ob}(\mathbf{G}_1)$. In this simple case it suffices to specify $\eta_1(0) = 0 \in S1$ with $\eta_{1,0}$ the unique isomorphism $E\eta_1(0) \xrightarrow{0} \cong y(0)$.

In fact, whenever $c' \leq c$ any operation outputting c' has a corresponding operation outputting c with the same arity by composition with $\eta_c(c')$: for $t \in Sc'$ regarded as a map $E\eta_c(c') \cong y(c') \rightarrow S$, $\mu(\eta_c(c'), t) \in Sc$ has arity isomorphic (via $\eta_{c,c'}$ on the indexing category of the colimit) to Et . We will write $\eta_c(t) = \mu(\eta_c(c'), t)$ and $\eta_{c,t} : E\eta_c(t) \cong Et$.

Example 6.35. Let T be the free n -category monad on n -globular sets, and P the linear poset $0 < 1 < \dots < n$ on $\operatorname{Ob}(\mathbf{G}_n)$, (n can be ω here). Setting $\eta_m(\ell)$ for

$\ell \leq m$ to be the m -dimensional pasting diagram with a single ℓ -cell with the unique isomorphism $E\eta_m(\ell) \cong y(\ell)$, we get that for any ℓ -dimensional pasting diagram t , $\eta_m(t)$ is the corresponding m -dimensional pasting diagram with only cells up to dimension ℓ . $\eta_{m,t}$ is the unique isomorphism between the pasting diagrams Et and $E\eta_m(t)$.

Lemma 6.36. *The functions $\eta_c : Sc' \rightarrow Sc$ extend to fully faithful functors $\eta_c^\Theta : \Theta_a^{c'} \rightarrow \Theta_a^c$.*

Proof. η_c^Θ sends the morphism $(f : Et \rightarrow S) : t \rightarrow \mu(t, f)$ in $\Theta_a^{c'}$ to $f\eta_{c,t} : E\eta_c(t) \cong Et \rightarrow S$. This assignment is clearly fully faithful, but it remains to show it preserves codomains and is functorial:

- Preserving codomains means that $\mu(\eta_c(t), E\eta_c(t) \xrightarrow{\eta_{c,t}} Et \xrightarrow{f} S) = \eta_c(\mu(t, f))$, which follows from associativity of μ as $\eta_c(t) = \mu(\eta_c(c'), t)$
- Preserving identities is immediate as the identity on Et is given by $Et \rightarrow * \xrightarrow{\eta} S$
- Preserving composition means that for $f : Et \rightarrow S$ and $F : E\mu(t, f) \rightarrow S$,

$$E\eta_c(t) \xrightarrow{\eta_{c,t}} Et \xrightarrow{x \mapsto \mu(f(x), F_x)} S \quad \text{agrees with} \quad E\eta_c(t) \xrightarrow{x \mapsto \mu(f\eta_{c,t}(x), F'_x)} S.$$

This is clear from observation, noting that F'_x is the map

$$E(f\eta_{c,t}(x)) \rightarrow E\mu(\eta_c(t), f\eta_{c,t}) \cong E\mu(t, f) \xrightarrow{F} S.$$

□

Example 6.37. In the previous example these functors are the evident inclusion functors, but when T is the free double category monad they are (at least a little bit) less canonical. $\mathbf{G}_1 \times \mathbf{G}_1$ has the product poset structure induced by that on \mathbf{G}_1 , as shown below.

$$\begin{array}{ccc} \bullet & < & \blacktriangleright \\ \wedge & & \wedge \\ \circ\rightarrow & < & \square \end{array}$$

The corresponding inclusion functors

$$\eta_{\blacktriangleright}^{\Theta}, \eta_{\circ\rightarrow}^{\Theta} : \Theta_a^{\bullet} \cong * \rightarrow \Delta_+^{op} \cong \Theta_a^{\blacktriangleright/\circ\rightarrow}$$

are both the inclusion of the single-vertex operation in $S(\blacktriangleright)$ and $S(\circ\rightarrow)$, which corresponds to the empty set (of edges) in Δ_+^{op} . Similarly, the functors

$$\eta_{\square}^{\Theta} : \Theta_a^{\blacktriangleright/\circ\rightarrow} \cong \Delta_+^{op} \rightarrow \Delta_+^{op} \times \Delta_+^{op} \cong \Theta_a^{\square}$$

are given by inserting the empty set (of vertical or horizontal edges, respectively) into the second/first component of the product.

Lemma 6.38. *In Θ_T , the composite $\eta_c(t) \cong t \rightarrow \mu(t, f)$ has $\eta_c(t) \rightarrow \eta_c(\mu(t, f)) \cong \mu(t, f)$ as its active-inert factorization while $t \cong \eta_c(t) \rightarrow \mu(\eta_c(t), g)$ factors as $t \rightarrow \mu(t, g\eta_{c,t}^{-1}) \cong \eta_c(\mu(t, g\eta_{c,t}^{-1}))$.*

Proof. As discussed above, $\eta_c(\mu(t, f)) = \mu(\eta_c(t), f\eta_{c,t})$, and $\eta_c(t) \rightarrow \mu(\eta_c(t), f\eta_{c,t})$ is precisely the active map associated to this composite by the construction in Theorem 6.31. The maps which then induce the inert part are $Ef\eta_{c,t}(x) \rightarrow E\mu(t, f)$, which assemble precisely into $\eta_{c,\mu(t,f)}$. The second claim follows similarly. \square

Corollary 6.39. *The functors η_c^Θ commute over Θ_T up to natural isomorphism and vary functorially with respect to P .*

Proof. Functoriality of η_c^Θ over P follows from the facts that for $c'' \leq c' \leq c$ in P and $t \in Sc''$, $\eta_c(\eta_{c'}(t)) = \eta_c(t)$ (Definition 6.33 and associativity of μ) and $\eta_{c,\eta_{c'}(t)}\eta_{c',t} = \eta_{c,t}$ (consequence of Definition 6.33).

For $c' \leq c$ in P , the natural isomorphism in Θ_T has components given by $\eta_{c,t}$ for each $t \in Sc'$, which are natural over $\Theta_a^{c'}$ by Lemma 6.38. These are also functorial with respect to P by the same reasoning as above. \square

This system of inclusions $\Theta_a^c \rightarrow \Theta_a^{c'}$ for $c \leq c'$ in P allows Θ_a as a subcategory of Θ_T to be treated up to equivalence as either a single Θ_a^c or filtered colimit thereof, rather than a disjoint union of such categories.

Definition 6.40. For T with P -units, define $\overline{\Theta}_a$ as $\operatorname{colim}_{c \in P} \Theta_a^c$, the colimit of the functors η_c^Θ .

Proposition 6.41. *When T has P -units, $\overline{\Theta}_a$ is equivalent to the subcategory of Θ_T generated by Θ_a and the inert isomorphisms $\eta_{c,t} : E\eta_c(t) \cong Et$, along with their inverses.*

Note that by Lemma 6.38, this subcategory consists of precisely the maps with active-inert factorization

$$\eta_c(t) \rightarrow \eta_c(\mu(t, f)) \xrightarrow{\eta_{c,\mu(t,f)}} \mu(t, f) \quad \text{or} \quad t \rightarrow \mu(t, f) \xrightarrow{\eta_{c,\mu(t,f)}} \eta_c(\mu(t, f)),$$

since the composite

$$\eta_c(t) \xrightarrow{\eta_{c,t}} t \rightarrow \mu(t, f)$$

has the above active-inert factorization and the composite

$$t \xrightarrow{\eta_{c,t}^{-1}} \eta_c(t) \rightarrow \mu(\eta_c(t), g) = \eta_c(\mu(t, g\eta_{c,t}^{-1}))$$

factors as

$$t \rightarrow \mu(t, g\eta_{c,t}^{-1}) \xrightarrow{\eta_{c,\mu(t, g\eta_{c,t}^{-1})}} \eta_c(\mu(t, g\eta_{c,t}^{-1})).$$

Proof. For each $t \in Sc$, we will write $\bar{\eta}t$ for the corresponding object in $\bar{\Theta}_a$, and more generally denote the functor $\Theta_a^c \rightarrow \bar{\Theta}_a$ by $\bar{\eta}$. To extend $\bar{\eta}$ to a functor from the above subcategory to $\bar{\Theta}_a$, we define $\bar{\eta}$ on any isomorphism $\eta_{c,t} : \eta_c(t) \cong t$ in Θ_T as the identity on $\bar{\eta}(t) = \bar{\eta}(\eta_c(t))$ in $\bar{\Theta}_a$. This extended $\bar{\eta}$ is a functor by Corollary 6.39, fully faithful as each η_c^\ominus is, and essentially surjective as it is surjective on objects, so it is an equivalence from this subcategory of Θ_T to $\bar{\Theta}_a$. \square

When it does not cause confusion, we will sometimes write $\bar{\Theta}_a$ for this equivalent subcategory of Θ_T .

Example 6.42. When P has a maximal element, as is the case for \mathbf{G}_n and $\mathbf{G}_1 \times \mathbf{G}_1$, $\bar{\Theta}_a$ is precisely Θ_a^c for c the maximal cell shape. So for the free n -category monad $\bar{\Theta}_a$ is Θ_a^n and for the free double category monad $\bar{\Theta}_a$ is $\Theta_a^\square \cong \Delta_+^{op} \times \Delta_+^{op}$.

The main result of this section uses this equivalence to describe an alternative factorization system on Θ_T replacing Θ_a with $\bar{\Theta}_a$.

Theorem 6.43. *Every morphism in Θ_T factors as a morphism in $\overline{\Theta}_a$ followed by a morphism in Θ_0 , uniquely up to unique isomorphism.*

Proof. $g : t \rightarrow t'$ has a unique active-inert factorization

$$t \xrightarrow{f_g} \mu(t, f_g) \xrightarrow{g_0} t',$$

so it suffices to show that any two factorizations of an inert map g_0 through an isomorphism of the form $\eta_{c,t''}$ or $\eta_{c,t''}^{-1}$ are related by a unique isomorphism. But this is immediate, as if h_1, h_2 are two such isomorphisms and $g_0 = g_1 h_1 = g_2 h_2$ in Θ_0 , the two factorizations are related uniquely by $h_1 = (h_1 h_2^{-1}) h_2$ and $g_2 = g_1 (h_1 h_2^{-1})$. \square

This particular type of factorization system differs from those most common in the literature: Θ_0 and $\Theta_a/\overline{\Theta}_a$ do not generally contain all isomorphisms in Θ_T , so this is not an orthogonal factorization system. However, this type of factorization will nonetheless be suitable for our purposes, so we will from here onward use the otherwise ambiguous term *factorization system* as follows.

Definition 6.44. A *factorization system* on a category \mathcal{A} is a pair of wide subcategories $\mathcal{A}_1, \mathcal{A}_2$ such that $\mathcal{A}_1 \cap \mathcal{A}_2$ is a groupoid and every morphism in \mathcal{A} factors as a morphism in \mathcal{A}_1 followed by a morphism in \mathcal{A}_2 uniquely up to unique isomorphism in $\mathcal{A}_1 \cap \mathcal{A}_2$.

Remark 6.45. As the active and inert subcategories of Θ_T intersect only at the identity morphisms, for the factorization system of Theorem 6.31 to be orthogonal (as in, both categories contain all isomorphisms) both categories must contain no

non-identity isomorphisms. For Θ_0 this is the case precisely when the arity diagrams Et are rigid, and for Θ_a this happens when for all cell shapes c there are no operations in Sc with arity $y(c)$ other than $\eta(c)$.

For the factorization system in Theorem 6.43 to be orthogonal, there must still be no active isomorphisms, but as $\bar{\Theta}_a$ contains the maps $\eta_{c,t}$ and their inverses it suffices for these to be the only isomorphisms between the arity diagrams in Θ_0 .

Finally, we describe when a familial monad T has enough units for a collection of operations with arbitrary outputs to be considered by means of the functors η_c as all having the same output shape. Recall that for a cardinal κ , a poset has κ -small upper bounds if any subset of P of cardinality less than κ has an upper bound.

Definition 6.46. T has κ -enough units for a regular cardinal κ if there exists a poset P on $\text{Ob}(\mathcal{C})$ with κ -small upper-bounds such that T has P -units.

Example 6.47. When P has a maximal element T has κ -enough units for any κ . This includes the free n -category and free double category monads.

Example 6.48. When T is the free ω -category monad on globular sets and P is the linear order on $\text{Ob}(\mathbf{G})$, P has finite upper bounds but the set of all (ω -many) objects of \mathbf{G} has no upper bound, so T has only ω -enough units (or *finitely enough units*).

Example 6.49. The free multicategory monad T (symmetric or non-symmetric) does not have enough units of any size, as the only poset P on $\text{Ob}(\mathbf{M})$ for which T has P -units has $0 < (1, 1)$ and no other non-reflexive comparisons. $S(1, 1)$ has a single operation with arity isomorphic to $y(0)$, the height 0 tree with one leaf, but no

other pair of objects in \mathbf{M} have operations with isomorphic arities. This corresponds to how multicategories only have identity 1-to-1 arrows, not identity n -to-1 arrows nor canonical ways of constructing an n -to-1 arrow from a single m -to-1 arrow.

7 More Examples

Theorem 5.1 reduces the task of defining a familial monad to specifying its operations, their arities, how these relate to one another, the identity operations, and how operations compose. We demonstrate the convenience of this characterization with several less familiar examples exhibiting the three most common types of algebraic structure: units, symmetries, and compositions.

7.1 Adding Degeneracies

In most cases of interest, the category \mathcal{C} of cell shapes is made up only of morphisms from “lower dimensional” to “higher dimensional” cell shapes, resembling inclusions into each shape of its lower dimensional faces.

Definition 7.1. A category \mathcal{C} is *direct* if it admits an identity-reflecting functor to the linear order **Ord** of ordinals regarded as a category. Concretely, this amounts to a “degree” function from $\text{Ob}(\mathcal{C})$ to ordinals such that each morphism in \mathcal{C} strictly raises degree.

When \mathcal{C} is direct and all of the degrees are finite, the degree functor can always be taken to send each object c to the length n of the longest string of nontrivial composable morphisms $c_0 \rightarrow c_1 \rightarrow \cdots \rightarrow c_{n-1} \rightarrow c_n = c$ in \mathcal{C} .

Example 7.2. The *semicube category*, which we denote \square_{∂} , is the free monoidal category generated by a single object \square^1 and two maps $\partial_0, \partial_1: \square^0 \rightarrow \square^1$ where \square^0 is the monoidal unit. We further denote $\square^n := \square^1 \otimes \cdots \otimes \square^1$, and

$$\partial_{i,\varepsilon}^n := \text{id}_1 \otimes \cdots \otimes \text{id}_1 \otimes \partial_\varepsilon \otimes \text{id}_1 \otimes \cdots \otimes \text{id}_1: \square^n \rightarrow \square^{n+1}$$

for $i = 1, \dots, n$, $\varepsilon = 0, 1$. These maps are generators of \square_{∂} .

\square^n can be thought of as an n -dimensional cube with $\partial_{i,0}^n, \partial_{i,1}^n$ respectively its front and back $(n-1)$ -dimensional faces in the i th direction, where the edges of the cube are directed from front to back. We will also write \square^n for the semicubical set represented by \square^n .

\square_{∂} is direct, with degree functor sending \square^n to n and all of the morphisms inclusions of faces from a lower dimensional cube to a higher dimensional one.

Direct categories are most often discussed in the context of Reedy categories (see for instance [7, 2.2]).

Definition 7.3. A Reedy category is a category \mathcal{R} equipped with wide subcategories $\mathcal{R}_+, \mathcal{R}_-$ such that \mathcal{R}_+ is direct, the morphisms in \mathcal{R}_- are strictly degree-lowering, and every morphism factors uniquely as a morphism in \mathcal{R}_- followed by a morphism in \mathcal{R}_+ .

Example 7.4. The *cube category* $\square_{\partial\sigma}$ can be defined as the free monoidal category generated by objects \square^0, \square^1 (with \square^0 the monoidal unit), morphisms ∂_0, ∂_1 as above, and $\sigma: \square^1 \rightarrow \square^0$ with $\sigma\partial_0 = \sigma\partial_1 = \text{id}_{\square^0}$. $\square_{\partial\sigma}$ is generated by the maps $\partial_{i,\varepsilon}^n$ as above along with “degeneracy maps”

$$\sigma_i^n := \text{id}_1 \otimes \cdots \otimes \sigma \cdots \otimes \text{id}_1: \square^n \rightarrow \square^{n-1}$$

satisfying appropriate relations, which can be found in [2, Equation 2.1].

$\square_{\partial\sigma}$ is a Reedy category with \mathcal{R}_+ the subcategory \square_{∂} and \mathcal{R}_- the subcategory generated by the maps σ_i^n . As any map between \square^0 and/or \square^1 factors uniquely as a map in \mathcal{R}_- followed by a map in \mathcal{R}_+ (either potentially an identity), the same is true for monoidal products of these maps, which can be factored the same way in each component.

Our characterization of familial monads allows us to show the following by a simple construction, showing in particular that there is a monad on semicubical sets which freely adds in the degenerate cubes of a cubical set (presheaf on $\square_{\partial\sigma}$), whose algebras are cubical sets and whose theory category is $\square_{\partial\sigma}$.

Proposition 7.5. *For any Reedy category \mathcal{R} , $\widehat{\mathcal{R}}$ is equivalent to the category of algebras for a familial monad on $\widehat{\mathcal{R}}_+$, which has \mathcal{R} as its theory category.*

Proof. Let \mathcal{R} be a Reedy category. We define a familial representation (S, E) over \mathcal{R}_+ as follows:

- $Sc = \coprod_{b \in \text{Ob}(\mathcal{R})} \text{Hom}_{\mathcal{R}_-}(c, b)$ for $c \in \text{Ob}(\mathcal{R})$

- For $i: c' \rightarrow c$ in \mathcal{R}_+ and $t: c \rightarrow b$ in Sc , ti factors uniquely as $i't': c' \rightarrow b' \rightarrow b$ with i' in \mathcal{R}_+ and t' in \mathcal{R}_- . We then set $(Si)(t) = t'$
- For $t: c \rightarrow b$ in Sc , $Et = y(b)$
- For t, i, t', i' as above, Ei_t is given by $y(i'): y(b') \rightarrow y(b)$

The induced familial endofunctor on $\widehat{\mathcal{R}}_+$ sends X to TX where

$$TX_c = \coprod_{b \in \text{Ob}(\mathcal{R})} \text{Hom}_{\mathcal{R}_-}(c, b) \times X_b,$$

adding a “degenerate” c -cell for each b -cell and degree-lowering map $t: c \rightarrow b$. In the example above, this amounts to adding for each n -cube a degenerate m -cube for each projection from the m -cube to the n -cube.

As \mathcal{R}_+ has no nontrivial isomorphisms as a direct category, each presheaf $y(b)$ is rigid, so following Definition 5.2 we can specify a monoidal structure on (S, E) as follows:

- $e: S^0 \rightarrow S$ sends $*_c$ to $\text{id}: c \rightarrow c$ for each $c \in \text{Ob}(\mathcal{C})$
- $SSc \cong \coprod_{t: c \rightarrow b} Sb$, and $m: SS \rightarrow S$ sends $(t: c \rightarrow b, t': b \rightarrow a) \in SSc$ to $(t't: c \rightarrow a) \in Sc$
- Clearly $Ee(*_c) \cong y(c)$ and $Em(t: c \rightarrow b, t': b \rightarrow a) = y(a) \cong \text{colim}_{i: y(c') \rightarrow y(b)} E(Si(t'))$

An algebra of T is a presheaf A in $\widehat{\mathcal{R}}_+$ along with a map $TA \rightarrow A$, which amounts to functions $A_t: A_b \rightarrow A_c$ for each $t: c \rightarrow b$ in \mathcal{R}_- such that:

- $A_{\text{id}}: A_c \rightarrow A_c$ is the identity (unit law)
- $A_t \circ A_{t'} = A_{t't}$ (multiplication law)
- for $i: c' \rightarrow c$ in \mathcal{R}_+ $A_i A_t = A_{t'} A_{i'}$ for i', t' as described above (naturality of algebra structure map)

This is precisely the data of a presheaf over \mathcal{R} , and a map of algebras corresponds similarly to a morphism in $\widehat{\mathcal{R}}$.

The theory category Θ_T is the full subcategory of $\widehat{\mathcal{R}}$ consisting of the free algebras on the representable presheaves $y(d)$ of $\widehat{\mathcal{R}}_+$. $Ty(d)$ is simply the cells of $y(d)$ with a c -cell added for each pair of a non-identity arrow $c \rightarrow b$ in \mathcal{R}_- and an arrow $i: b \rightarrow d$ in \mathcal{R}_+ , which correspond to the morphisms $c \rightarrow d$ in \mathcal{R} that don't come from \mathcal{R}_+ . $Ty(d)$ is therefore the representable presheaf $y(d)$ in $\widehat{\mathcal{R}}$, so Θ_T agrees with \mathcal{R} as the full subcategory of representables in $\widehat{\mathcal{R}}$. \square

Example 7.6. Perhaps the most famous Reedy category is the simplex category Δ , whose direct subcategory contains only the face maps between simplices. Presheaves on this subcategory are called semisimplicial sets, and Proposition 7.5 shows that there is a familial monad on semisimplicial sets which adds in the degeneracies needed to form a simplicial set, with Δ as its theory.

Example 7.7. A much simpler example comes from the Reedy category $\mathbf{G}_{1,r}$, with $\mathbf{G}_1 = 0 \begin{matrix} \xrightarrow{s} \\ \rightrightarrows \\ \xleftarrow{t} \end{matrix} 1$ as its direct subcategory and a single map $\epsilon: 1 \rightarrow 0$ as \mathcal{R}_- , with $\epsilon \circ \sigma = \epsilon \circ \tau = \text{id}_0$. Here Proposition 7.5 produces the monad on graphs adding a new self-loop to every vertex, whose algebras are reflexive graphs.

Remark 7.8. While presheaves on direct categories commonly form the data of higher categorical structures, this construction in fact applies much more broadly. The degree raising and lowering properties of the factorization system in a Reedy category was never used in the construction, which therefore shows that for any category \mathcal{C} with wide subcategories $\mathcal{C}', \mathcal{C}''$ such that each morphism factors uniquely as a map in \mathcal{C}'' followed by a map in \mathcal{C}' , there is a familial monad on $\widehat{\mathcal{C}}$ whose algebras are presheaves over \mathcal{C} and with \mathcal{C} as its theory category.

In the reverse direction, starting with a familial monad T on $\widehat{\mathcal{C}}$ there is an “active-inert” factorization system on Θ_T such that the “inert” maps are precisely those arising from maps between the arity presheaves in $\widehat{\mathcal{C}}$. Writing Θ_0 for this subcategory, Θ_T is then also the theory of a familial monad on $\widehat{\Theta}_0$, though this was previously known (see [35, Lemma 4.5]).

7.2 Adding Symmetries

In types of higher categories with particularly symmetric cell shapes, cells are often equipped with reflected or permuted versions of themselves.

Example 7.9. The *cubical nerve* of a category \mathcal{C} is the cubical set with n -cubes the commutative n -dimensional cube diagrams in \mathcal{C} , faces given by restriction to the appropriate lower dimensional subcubes, and degeneracies given by inserting identities in the appropriate direction. For any square as below left in \mathcal{C} , there is

also a square as below right with the directions swapped.

$$\begin{array}{ccc}
 \cdot & \xrightarrow{f} & \cdot \\
 g \downarrow & & \downarrow h \\
 \cdot & \xrightarrow{k} & \cdot
 \end{array}
 \qquad
 \begin{array}{ccc}
 \cdot & \xrightarrow{g} & \cdot \\
 f \downarrow & & \downarrow k \\
 \cdot & \xrightarrow{h} & \cdot
 \end{array}$$

This type of operation in a cubical set is called a *symmetry*: a single cell is sent to another cell of the same shape with its faces permuted in some way. n -dimensional cubes in a cubical nerve admit symmetries for any permutation of the n directions. Symmetries can be added as automorphisms of the objects in any variety of cube category: the semicube category, the cube category, even the category \boxminus described below which incorporates composites of cubical cells.

Symmetries of various sorts are also found in other types of higher categories. For instance, in a symmetric multicategory any n -to-1 arrow is equipped with additional n -to-1 arrows for every permutation of the inputs. Much like the familial monads discussed above for adding degenerate cells, symmetries can be freely added to a presheaf on \mathcal{C} by a familial monad. The corresponding theory category is then constructed by adding new automorphisms to the objects of \mathcal{C} .

Definition 7.10 ([6, Definition 2.1, 2.3]). For \mathcal{C} a small category, a *crossed \mathcal{C} -group* is a functor $G: \mathcal{C}^{op} \rightarrow \mathbf{Set}$ equipped with a group structure on each set Gc denoted by (\cdot, e_c) and a left Gc -action on each set $\text{Hom}_{\mathcal{C}}(c', c)$ denoted by $(-)_*$, such that for each $g, h \in Gc$ and $i: c' \rightarrow c, i': c'' \rightarrow c'$ in \mathcal{C} :

a) $g_*(i \circ i') = g_*(i) \circ (Gi(g))_*(i')$

b) $g_*(\text{id}_c) = \text{id}_c$

c) $Gi(g \cdot h) = (Gh_*(i))(g) \cdot Gi(h)$

d) $Gi(e_c) = e_{c'}$

The *total category* \mathcal{CG} has the same objects as \mathcal{C} , with morphisms $c' \rightarrow c$ of the form (i, g) where $i : c' \rightarrow c$ in \mathcal{C} and $g \in Gc'$. Identities are of the form (id_c, e_c) and composition is given by $(i, g) \circ (i', h) = (i \circ g_*(i'), Gi'(g) \cdot h)$.

Crossed \mathcal{C} -groups are precisely the structure needed to describe a system of symmetries that can be freely added to a presheaf on \mathcal{C} by a familial monad, and the theory category for this monad is given by the total category.

Proposition 7.11. *For G a crossed \mathcal{C} -group, $\widehat{\mathcal{CG}}$ is equivalent to the category of algebras for a familial monad on $\widehat{\mathcal{C}}$, which has \mathcal{CG} as its theory category.*

Proof. \mathcal{CG} has a unique factorization system given by the subcategory of morphisms of the form $(i : c' \rightarrow c, e_{c'})$, which is isomorphic to \mathcal{C} , and the subcategory of morphisms of the form (id_c, g) . In particular, any morphism $(i : c' \rightarrow c, g)$ factors as $(i, e_{c'}) \circ (\text{id}_{c'}, g)$. The result then follows from Remark 7.8.

Concretely, the representation of the monad has $S = G$ and E sends $g \in Gc$ to $y(c)$ with its faces permuted by g_* . The unit and multiplication arise from the unit and multiplication in the groups Gc . □

Example 7.12. Symmetries in cubical sets are described by the crossed cubical group $G : \square \rightarrow \mathbf{Set}$ defined as follows:

- $G\Box^n = \Sigma_n$, the permutation group on n elements
- $G\partial_{i,\varepsilon}: \Sigma_n \rightarrow \Sigma_{n-1}$ sends a permutation γ on $\{1, \dots, n\}$ to the permutation on $\{1, \dots, n-1\}$ given by removing i from the domain of γ and reordering (e.g. $G\partial_{2,\varepsilon}(231) = (21)$ by removing the 2 and reordering)
- $G\sigma_i: \Sigma_n \rightarrow \Sigma_{n+1}$ sends γ to the permutation on $\{1, \dots, n+1\} \cong \{1, \dots, i, i', i+1, \dots, n\}$ treating i, i' as a single element (e.g. $G\sigma_2(321) = (322'1) = (4231)$ by moving 2 and 2' together and then relabeling)
- $\gamma_*(\partial_{i,\varepsilon}) = \partial_{\gamma(i),\varepsilon}$ and $\gamma_*(\sigma_i) = \sigma_{\gamma(i)}$. This is how each symmetry permutes the faces (and degeneracies) of a cube
- It is straightforward to check that these satisfy the axioms of a crossed cubical group

This monad takes a cubical set X and adds in a new n -cube for each permutation of the dimensions of each cube in X_n . An algebra A for this monad is equipped with a choice of these symmetries: for each $\gamma \in \Sigma_n$ and each n -cube $a \in A_n$, a choice of cube $\gamma(a) \in A_n$ whose faces and degeneracies are those of a permuted by γ . This is precisely a symmetric cubical set (in the language of [9], a presheaf on the symmetric cube category $\mathbb{C}_{(we, \cdot)}$). Symmetries can be similarly added to any other type of cubical sets that doesn't already have them.

There is also a crossed cubical group with $G\Box^n = \{1, \tau\}^n$, where each τ in the i th position reverses the source and target faces of the n -cube in the i th direction (as in, swaps $\partial_{i,0}, \partial_{i,1}$ and does the same for all maps that factor through them). Algebras

for the associated familial monad are cubical sets with reversals (in [9], presheaves on $\mathbb{C}_{(w, \cdot)}$).

Example 7.13. Symmetries are also often considered for permuting the sources of many-to-one arrows, such as in a multicategory. For the category \mathbf{M} of the vertex and n -to-1 arrows in Example 1.13, a crossed \mathbf{M} -group can be defined with $G0$ the trivial group, $G(n, 1) = \Sigma_n$, and each $\gamma \in \Sigma_n$ permuting the n different source maps $0 \rightarrow (n, 1)$. Algebras for this monad are symmetric multigraphs: multigraphs equipped with, for each n -to-1 edge and each permutation γ , a choice of n -to-1 edge with its sources permuted by γ (a restriction of the symmetric polygraphs discussed in [19, Section 2.4]).

Dendroidal sets [30] are presheaves over the tree category Ω , which is the theory category for the free symmetric multicategory monad on multigraphs [35, Example 4.19]. There is similarly a planar tree category Ω_{planar} which is the theory category for the free non-symmetric multicategory monad. The above crossed group structure for permutations of sources in an n -to-1 arrow extends to a crossed Ω_{planar} -group sending each tree of many-to-one arrows to its group of planar rearrangements, whose total category is equivalent to Ω [6, Example 2.8]. Hence dendroidal sets are algebras for a familial monad on planar dendroidal sets.

Example 7.14. Crossed \mathcal{C} -groups were originally defined for $\mathcal{C} = \Delta$ [14], and any crossed simplicial group provides a monad for adding symmetries to simplicial sets. For instance, the crossed simplicial group sending $[n]$ to $\mathbb{Z}/(n + 1)$ has as its total category Connes' cycle category Λ , so cyclic sets are algebras for a familial monad

on simplicial sets.

7.3 Cubical ω -Categories

The examples above all have representable arity diagrams, but we can also define familial monads on semicubical sets whose algebras have compositional structure similar to n -tuple categories.

Definition 7.15 ([2, Definition 2.1]). A *cubical ω -category* is a cubical set equipped with n composition operations for n -cubes in the n different directions satisfying unit (with respect to degeneracies), associativity, and interchange equations.

When restricted to cubes in dimensions up to n , cubical n -categories resemble n -tuple categories [15, Definition 2.1], but without the n distinct types of arrows and resulting $\binom{n}{m}$ distinct types of m -cubes.

Example 7.16. For a category \mathcal{C} , its cubical nerve is a cubical ω -category with composition of two compatible cubes given by composition in \mathcal{C} of the arrows in the i th direction:

$$f \mapsto \begin{array}{ccc} \cdot & \xlongequal{\quad} & \cdot \\ f \downarrow & & \downarrow f \\ \cdot & \xlongequal{\quad} & \cdot \end{array} \quad \begin{array}{ccccc} \cdot & \xrightarrow{f} & \cdot & \xrightarrow{g} & \cdot \\ h \downarrow & & i \downarrow & & \downarrow j \\ \cdot & \xrightarrow{k} & \cdot & \xrightarrow{\ell} & \cdot \end{array} \mapsto \begin{array}{ccc} \cdot & \xrightarrow{gf} & \cdot \\ h \downarrow & & \downarrow j \\ \cdot & \xrightarrow{\ell k} & \cdot \end{array}$$

Remark 7.17. Cubical ω -categories with connections, symmetries, etc. can be defined similarly using equations such as those for connections in [2, Equation 2.6]. The example above is in fact a cubical ω -category with both symmetries and connections.

[2] shows that the category of cubical ω -categories with connections is equivalent to the category of strict globular ω -categories.

Definition 7.18. The *Day convolution product* $\otimes: \widehat{\square}_{\partial} \times \widehat{\square}_{\partial} \rightarrow \widehat{\square}_{\partial}$ on semicubical sets is defined by left Kan extension of the functor

$$\square_{\partial} \times \square_{\partial} \xrightarrow{\otimes} \square_{\partial} \xrightarrow{y} \widehat{\square}_{\partial}$$

and has the 0-cube \square^0 as a unit.

Example 7.19. Consider the string \xrightarrow{k} of k composable 1-cubes (arrows) in $\widehat{\square}_{\partial}$. The Day product $\xrightarrow{k_1} \otimes \cdots \otimes \xrightarrow{k_n}$ is the standard $k_1 \times \cdots \times k_n$ grid of n -cubes, where each zero among the natural numbers k_i reduces the top dimension of the cubes in the grid by one. The inclusions s, t from \square^0 to \xrightarrow{k} sending the 0-cube to the source or target of the string of 1-cubes lets us define the source and target maps

$$s_i, t_i: \left(\xrightarrow{k_1} \otimes \cdots \otimes \xrightarrow{\widehat{k_i}} \otimes \cdots \otimes \xrightarrow{k_n} \right) \rightarrow \left(\xrightarrow{k_1} \otimes \cdots \otimes \xrightarrow{k_n} \right)$$

of an n -dimensional grid in each of the n directions.

Proposition 7.20. *Cubical ω -categories are algebras for a familial monad on semicubical sets.*

Cubical ω -categories are just as well algebras for a familial monad on cubical sets, but we prefer to use semicubical sets and treat degeneracies as algebraic structure rather than part of the underlying data. We show this in the simplest case of no connections or symmetries, though the same is true in those settings as well by a more complicated construction; a description of the monad for cubical ω -categories

with connections can be found in [24, Section 2.7]. Presenting this monad as familial provides an alternative proof of [24, Proposition 9], which shows that it is cartesian, and a familial representation of the free cubical ω -category with connections monad would similarly suffice to prove [24, Theorem 1].

Proof. The familial monad representation (S, E) is defined as follows:

- $S_n = \mathbb{N}^n$ and $S\partial_{i,c}^n$ is the projection map $\mathbb{N}^n \rightarrow \mathbb{N}^{n-1}$ omitting the i th component
- $E() = \square^0$, $E(k_1, \dots, k_n) = \left(\xrightarrow{k_1} \otimes \dots \otimes \xrightarrow{k_n} \right)$, and E sends the generating morphisms

$$\partial_{i,0}^n, \partial_{i,1}^n : (k_1, \dots, \hat{k}_i, \dots, k_n) \rightarrow (k_1, \dots, k_n) \text{ to } s_i, t_i \text{ respectively}$$

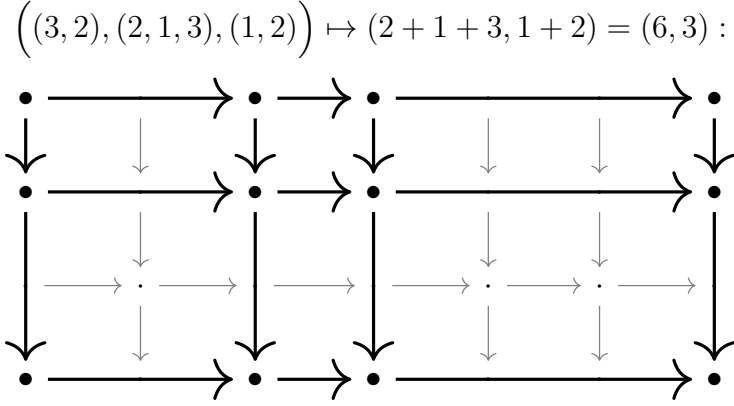
- The map $e : S^0 \rightarrow S$ sends $*_n$ to $(1, \dots, 1) \in S_n$, where $\square^n \cong E(1, \dots, 1)$ as $\xrightarrow{1}$ is precisely \square^1 and $\square^1 \otimes \dots \otimes \square^1 \cong \square^n$

- $SS_n \cong \coprod_{(k_1, \dots, k_n)} \text{Hom} \left(E(k_1, \dots, k_n), S \right)$, where as $S\partial_{i,0}^n = S\partial_{i,1}^n$ a map from $E(k_1, \dots, k_n)$ to S is determined by its values in $S1 = \mathbb{N}$ on the 1-cubes $(0, \dots, 0, j \rightarrow j+1, 0, \dots, 0)$ in the $k_1 \times \dots \times k_n$ grid. Therefore $SS_n \cong \coprod_{(k_1, \dots, k_n)} \mathbb{N}^{k_1} \times \dots \times \mathbb{N}^{k_n}$, and we define $m : SS \rightarrow S$ by

$$\left((k_1, \dots, k_n), (\ell_{1,1}, \dots, \ell_{1,k_1}), \dots, (\ell_{n,1}, \dots, \ell_{n,k_n}) \right) \mapsto \left(\sum_{i=1}^{k_1} \ell_{1,i}, \dots, \sum_{i=1}^{k_n} \ell_{n,i} \right).$$

For such (k, ℓ) , $\text{colim} \left(f E(k_1, \dots, k_n) \rightarrow f S \xrightarrow{E} \widehat{\square}_\partial \right)$ is the grid given by plugging an $\ell_{1,j_1} \times \dots \times \ell_{n,j_n}$ grid into the (j_1, \dots, j_n) th cube in the grid $E(k_1, \dots, k_n)$,

which is isomorphic to $E\left(\sum_{i=1}^{k_1} \ell_{1,i}, \dots, \sum_{i=1}^{k_n} \ell_{n,i}\right)$ as desired. Pictured below is the grid for the assignment



- The unitality and associativity equations follow from the analogous properties of multi-valued sums, and rigidity of the arity diagrams

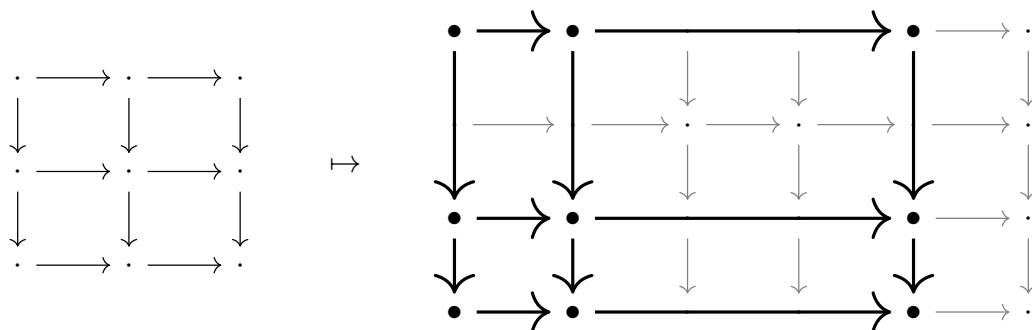
An algebra for T is a semicubical set A equipped with

- Degeneracy maps as in Example 7.4, where for each $1 \leq i \leq n$, $(1, \dots, 1, 0, 1, \dots, 1) \in S_n$ provides a map $s_i: A_{n-1} \rightarrow A_n$ satisfying the usual cubical identities
- n binary composition operations for n -cubes, where for each $1 \leq i \leq n$, $(1, \dots, 1, 2, 1, \dots, 1) \in S_n$ provides a map $\mu_i: A_n \times_{d_{i,1}} \times_{d_{i,0}} A_n \rightarrow A_n$
- As for each composable grid of cubes up to dimension n , there is a unique element of S_n sent to that grid by E , the multiplication law for the monad algebra A ensures that these compositions are unital (with respect to degeneracies), associative in each direction, and satisfy the interchange law between compositions in different directions

which makes the structure of A precisely that of a cubical ω -category. □

Uniqueness of each operation with respect to its arity makes T *shapely* in the sense of [19].

Remark 7.21. The theory associated to T has finite cubical grids as objects, with morphisms the homomorphisms of the free cubical ω -categories $TE(k_1, \dots, k_n)$ generated by those grids. Concretely, these are the maps between the vertices of the grids which send rows in each direction of the domain to rows in the codomain, such as the map $TE(2, 2) \rightarrow TE(3, 5)$ depicted below:



These grids are the cubical analogue of the pasting diagrams of Joyal's category Θ , the theory of the free strict ω -category monad on globular sets, and we call this category *cubical* Θ , written \boxplus . Concatenation of the lists (k_1, \dots, k_n) defines a tensor product \otimes on \boxplus , and all morphisms in \boxplus uniquely decompose under \otimes into morphisms between the 1-dimensional grids (k_1) . The full subcategory of 1-dimensional grids is isomorphic to Δ , so in this sense \boxplus is the free monoidal category generated by Δ with identity (0) (in \boxplus , (0) is isomorphic to $(\)$).

Remark 7.22. We can also consider the *n-truncated* semicube category, defined as the full subcategory of \square_∂ spanned by $\square^0, \dots, \square^n$, where \square_ω recovers \square_∂ . The construc-

tion above restricted to dimensions up to n gives a monad on n -truncated semicubical sets whose algebras are the analogous notion of cubical n -categories, with theory category \mathbb{E}_n of grids up to dimension n .

CHAPTER 3
SHAPE INDEPENDENT ENRICHMENT

8 Enrichment of Higher Categories

Here we define enrichment of T -algebras for a fixed familial monad T on $\widehat{\mathcal{C}}$ with a distinguished set of endpoint objects $e = \{e_k\}$.

8.1 (T, e) -Structured Categories

There are many formal descriptions of the relationship between categories and monoids. A category with a single object carries the same information as a monoid, its identity and composition providing the unit and multiplication for a monoid structure on its set of morphisms. Alternatively, the active subcategory Δ_a^1 of edge operations in Δ (as the theory for the free category monad on graphs) is isomorphic to Δ_+^{op} , the opposite of the augmented simplex category (Example 6.29). $(\Delta_a^1)^{op} \cong \Delta_+$ is the initial monoidal category with a monoid object, so a monoidal functor $M : (\Delta_a^1)^{op} \rightarrow \mathcal{A}$ determines a monoid in a monoidal category \mathcal{A} . A monoid in **Cat**, in the weak sense, is a monoidal category, which is precisely the structure a category is most easily enriched in.

The goal of this subsection is to generalize this picture from categories to algebras for any familial monad T on $\widehat{\mathcal{C}}$ with the role of Δ replaced by Θ_T and the role of

morphisms in a one-object category replaced by e -cells for $e = \{e_k\}$ a set of endpoint objects in \mathcal{C} . The algebraic structure on those e -cells can be described using the categories $\Theta_a^{e_k}$.

Definition 8.1. We will denote by Θ_a^e the full subcategory of Θ_a containing $\Theta_a^{e_k}$ for all k .

Functors $M : (\Theta_a^e)^{op} \rightarrow \mathcal{A}$ with monoidal-like properties will describe certain algebraic structures on the collection of objects $M(e_k)$, which when $\mathcal{A} = \mathbf{Cat}$ describe a kind of structured category in which T -algebras can be enriched.

Definition 8.2. Let T be a familial monad on $\widehat{\mathcal{C}}$ represented by (S, E, η, μ) . A *(weak) (T, e) -structured category* is a (pseudo-)functor $V : (\Theta_a^e)^{op} \rightarrow \mathbf{Cat}$ such that for each $t \in Se$,

$$V(t) \cong \prod_k V(e_k)^{Et_{e_k}},$$

and for each $(t \in Se, f : Et \rightarrow S)$, the functor $V(a_f) : V(\mu(t, f)) \rightarrow V(t)$ agrees with

$$\prod_k \prod_{x \in Et_{e_k}} V(a_{f(x)}) : V(\mu(t, f)) \cong \prod_k \prod_x V(f(x)) \rightarrow \prod_k \prod_x V(e_k) \cong V(t).$$

We write V_k for $V(e_k)$ and \otimes_t for $V(a_t) : V(t) \cong \prod_j V_j^{Et_{e_j}} \rightarrow V_k$, for each $t \in Se_k$.

Using this notation, the second condition reads as $V(a_f) = \prod_k \prod_{x \in Et_{e_k}} \otimes_{f(x)}$.

The first condition resembles the ‘‘Segal conditions’’ associated to a finite product sketch on Θ_a^e , where each $V(t)$ is the product of the diagram $Et_e \rightarrow (\Theta_a^e)^{op} \rightarrow \mathbf{Cat}$, from the discrete category on Et_e , sending each e_k -cell in Et to $V(e_k)$. The second condition extends this limit preservation to morphisms.

By the equations $a_{\eta(c)} = \text{id}_c$ and $a_f \circ a_t = a_{\mu(t,f)}$ in Θ_a^e (Lemma 6.27) and Corollary 5.13, the data of a (T, e) -structured category amounts to the categories V_k and functors $\otimes_t : \prod_j V_j^{Et_{e_j}} \rightarrow V_k$ for each e_k -operation, subject to the equations $\otimes_{\eta(e_k)} = \text{id}_{V_k}$ and for each $t \in Se$ and $f : Et \rightarrow S$,

$$\otimes_{\mu(t,f)} = \left(\otimes_t \circ \prod_{j,x \in Et_{e_j}} \otimes_{f(x)} \right) : \prod_i V_i^{E\mu(t,f)_{e_i}} \cong \prod_j \prod_x \prod_i V_i^{Ef(x)_{e_i}} \rightarrow \prod_j \prod_x V_j \rightarrow V_k.$$

When V is weak, these equations are replaced by natural isomorphisms subject to coherence equations similar to those for weak monoidal functors.

Remark 8.3. Weak (T, e) -structured categories could be equivalently defined as strict functors with the Segal isomorphisms relaxed to equivalences. This follows a pattern that can be found in for instance [27, Proposition 3.1] for weak monoidal categories, where strict functors $V : \Delta_+ \rightarrow \mathbf{Cat}$ which are monoidal only up to equivalence ($V(n+m) \simeq V(n) \times V(m)$) agree with pseudofunctors $V : \Delta_+ \rightarrow \mathbf{Cat}$ which are monoidal up to isomorphism ($V(n+m) \cong V(n) \times V(m)$). In short, the natural isomorphisms that form the equivalences on one side correspond to the natural isomorphisms that provide for pseudofunctoriality. While the definition with the weaker Segal condition is closer to how weak structures are often defined in the literature, we prefer pseudofunctoriality as it directly provides the isomorphisms used to define enrichment in Definition 8.16.

Example 8.4. When T is the free category monad, Θ_a^1 is Δ_+^{op} as discussed above, $e = \{1\}$, and (weak) (T, e) -structured categories are unbiased (weak) monoidal categories as in [28, Definition 3.3.8].

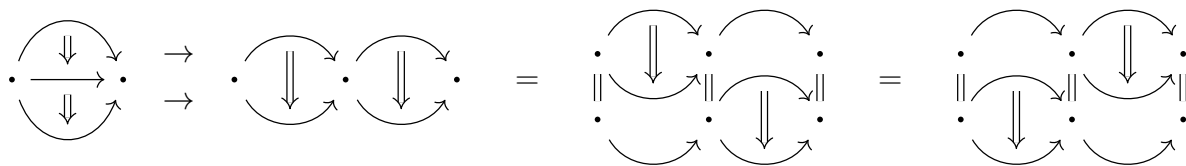
Example 8.5. When T is the free monoid monad on sets, Θ_a is also Δ_+^{op} , as the

active subcategory of the Lawvere theory L_T (see Example 6.32).

Example 8.6. When T is the free double category monad on double graphs with $e = \{\square = (1, 1)\}$ in $\mathbf{G}_1 \times \mathbf{G}_1$, $\Theta_a^e = \Theta_a^\square \cong \Delta_+^{op} \times \Delta_+^{op}$. A strict (T, e) -structured category consists of a category $V = V_{1,1}$, two strict monoidal structures on it, corresponding to the horizontal and vertical composition operations in Θ_a^e , which satisfy the interchange law. By the Eckmann-Hilton argument, this is precisely a strict symmetric monoidal category. By similar reasoning, a weak (T, e) -structured category is a braided weak monoidal category.

Example 8.7. When T is the free 2-category monad on 2-globular sets with $e = \{2\}$, Θ_a^e is the subcategory of active maps in Θ_2 and $e = \{2\}$ in \mathbf{G}_2 . Much like in the previous example, the operations in $S2$ for horizontal and vertical composition of 2-cells ensure that a strict/weak (T, e) -structured category has two monoidal structures satisfying interchange, so by Eckmann-Hilton corresponds to a symmetric/braided monoidal category.

The main difference between Θ_a^e for 2-categories and for double categories, aside from the former including pasting diagrams with uneven numbers of 2-cells in each column, is that for 2-categories Θ_a^e also includes active maps from vertical to horizontal composites of 2-cells.



Writing V for V_2 in a (T, e) -structured category, these two maps in Θ_a^e are sent to isomorphisms $\sigma, \tau : V^2 \rightarrow V^2$ which differ by a swap. Also in Θ_a^e , with t the

binary vertical composition operation and t' the binary horizontal composition, both of these maps composed with a_t give $a_{t'}$. A contravariant pseudofunctor from Θ_a^e to \mathbf{Cat} then includes a natural isomorphism

$$\otimes_t \circ \sigma \cong \otimes_{t'} \cong \otimes_t \circ \tau : V^2 \rightarrow V,$$

which provides a choice of braiding for the monoidal product \otimes_t (after precomposing with σ , this isomorphism relates \otimes_t and its composite with the swap $\tau \circ \sigma$). This explicit encoding of the swap and braiding isomorphisms makes this Θ_a^e more expressive in some sense, even though here it encodes the same structure as $\Delta_+^{op} \times \Delta_+^{op}$.

Example 8.8. When T is either the free n -category or free n -tuple category monad (on diagrams over \mathbf{G}_n and $\mathbf{G}_1^{\times n}$ respectively with the unique choice of endpoint object e), a (strict/weak) (T, e) -structured category amounts to a category V with n different (strict/weak) monoidal structures all satisfying interchange laws in every dimension. When $n > 2$, the higher dimensional Eckmann-Hilton argument enforces that all of the different monoidal structures agree (up to equality/isomorphism) and are (strict/weak) symmetric.

Remark 8.9. We could just as well define (T, e) -structured n -categories, where both the variety of m -categories and the functor from Θ_a^e can be weak in a variety of ways. When the m -categories are fully weak (such as bicategories or tricategories) and the functor $V : \Theta_a^e \rightarrow m - \mathbf{Cat}$ is fully weak (with the same Segal condition via isomorphisms), varying n recovers via Eckmann-Hilton the full range of symmetries in monoidal weak m -categories ([3, Table 21]).

More specifically, when T is the free n -category or n -tuple category monad a

fully weak (T, e) -structured m -category has the structure of a weak $(n + m)$ -category with just a single ℓ -cell for $\ell = 0, \dots, n - 1$. The n different monoidal operations come from composition in the $1, \dots, n$ -cell directions, and by higher Eckmann-Hilton this structure is precisely a (mere/braided/sylleptic/.../symmetric, according to n) monoidal m -category.

Example 8.10. When T is the free (nonsymmetric/symmetric) multicategory monad on multigraphs and $e = \{(0, 1), (1, 1), (2, 1), \dots\}$, Θ_a^e is the subcategory of (Ω / the planar variant) consisting of morphisms trees which preserve the root and leaves. The morphisms in Θ_a^e are generated under composition and arity colimits by the cocomposition maps $a_t : (n, 1) \rightarrow t$ for t a height-2 tree with n leaves, so the operations of a (T, e) -structured category in the non-symmetric case are generated by the functors

$$\otimes_t : V_{n,1} \times V_{k_1,1} \times \dots \times V_{k_n,1} \rightarrow V_{k_1+\dots+k_n,1}.$$

In the symmetric case there are also generating structure maps $\otimes_t : V_{n,1} \rightarrow V_{n,1}$ for each height-1 tree, corresponding to each permutation of the n vertices. Equations between these operations are the same as those for a (nonsymmetric/symmetric) operad; accordingly, a (weak) (T, e) -structured category is a (weak) (nonsymmetric/symmetric) operad object in **Cat**.

Remark 8.11. As the structure of a (T, e) -structured category is given by functors $\prod_j V_j^{Et_{e_k}} \rightarrow V_k$ satisfying certain equations (weakly or strictly), it is tempting to define the structure more simply as an algebra in **Cat** of a symmetric multicategory with object set e and multi-morphisms $((e_x)_{x \in Et_e}; e_k)$ for each $t \in Se_k$ and $e_x = e_j$ when $x \in Et_{e_j}$. However, composition in such a multicategory is not fully defined,

as not every combination of operations in Se can be plugged into the cells of Et_e to form a new operation in Se_k ; they must be compatible in the sense of arising from a map $Et \rightarrow S$. This structure would also neglect the μ -composition of operations with no e -cells, which is significant in the many examples where Θ_a^e contains multiple different operations with no e -cells.

Example 8.12. For M a weak symmetric monoidal category, there is a weak (T, e) -structured category with $V_k = M$ for all k , and each $\otimes_t : M^{Et_e} \rightarrow M$ given by the symmetric monoidal structure. All equations between the operations \otimes_t are guaranteed by the uniqueness-up-to-unique-isomorphism of all functors $M^n \rightarrow M$ derived from the symmetric monoidal structure. We will be particularly interested in the case when M is the cartesian monoidal category of T' -algebras.

8.2 Enrichment

We now define T -algebras enriched in a (T, e) -structured category. Just like a monoidal category has the structure needed to define composition maps between generalized “Hom objects” in a category, a (T, e) -structured category V has the structure needed to define compositions between V -objects describing the e_k -cells with fixed boundary in a T -algebra.

Recall from Definition 5.11 that a familial representation (S, E) is e -graded if Et_e is empty for all $t \in Sc$, c in $\mathcal{C} \setminus e$. This is the case for all common types of higher categories, where a diagram typically composes to a cell of the same dimension, or

yields a higher dimensional coherence cell rather than a lower dimensional cell. $\partial_e X$ denotes the restriction of X in $\widehat{\mathcal{C}}$ to $\widehat{\mathcal{C}} \setminus e$, and $\partial_e T$ denotes the familial monad on $\widehat{\mathcal{C}} \setminus e$ which forgets the e -cell operations (Corollary 5.19).

Example 8.13. For the free category, 2-category, and double category monads and their top-dimensional cells, this construction recovers the identity monad on sets, the free category monad on graphs, and the monad on graphs with two types of arrows whose algebras are pairs of categories with the same objects. For the free multicategory monad with all n -to-1 arrow shapes as endpoints, this also yields the identity monad on sets.

A T -algebra enriched in a (T, e) -structured category V is then a $\partial_e T$ -algebra with a V -object for each “ e -cell boundary” and composition maps in V satisfying appropriate axioms. To state this definition, we use the following notation for a $\partial_e T$ -algebra A :

- For $\beta : \partial_e Et \rightarrow A$ and $x \in Et_{e_k}$, α_x is the map $\partial_e y(e_k) \xrightarrow{x} \partial_e Et \xrightarrow{\beta} A$
- For $\beta : \partial_e Et \rightarrow A$ with $t \in Se_k$, $\alpha_\beta : \partial_e y(e_k) \rightarrow A$ sends the cell $i : c \rightarrow e_k$ in $\partial_e y(e_k)_c$ to the composite c -cell in A of $E(S_i(t)) \xrightarrow{Ei_t} Et \xrightarrow{\beta} A$ (A admits c -cell composites of such diagrams as a $\partial_e T$ -algebra, sufficiently functorial to make α_β natural)
- For $\gamma : \partial_e E\mu(t, f) \rightarrow A$ and $x \in Et_e$, β_x is the map $\partial_e Ef(x) \rightarrow \partial_e E\mu(t, f) \xrightarrow{\gamma} A$
- For $\gamma : \partial_e E\mu(t, f) \rightarrow A$, $\beta_\gamma : \partial_e Et \rightarrow A$ sends $x' \in \partial_e Et_e$ to the composite c -cell in A of $Ef(x') \rightarrow E\mu(t, f) \xrightarrow{\gamma} A$

Example 8.14. When T is the free category monad, e the arrow cell shape, and A a set (or rather, a $\partial_e T$ -algebra), a map $\alpha : \partial_e y(1) \rightarrow A$ is simply a pair of elements in A as $\partial_e y(1)$ is the set with two elements. $\partial_e \xrightarrow{n}$ is the set with $n + 1$ elements, and a map $\beta : (\partial_e \xrightarrow{n}) \rightarrow A$ amounts to elements a_0, \dots, a_n . For x the i th edge in \xrightarrow{n} , $\alpha_x : \partial y(1) \rightarrow A$ is the pair a_{i-1}, a_i , while α_β is the pair a_0, a_n .

Similarly for a map $\gamma : (\partial_e \xrightarrow{n_1 + \dots + n_m}) \rightarrow A$ which picks out elements $a_{i,j}$ for $1 \leq i \leq n$ and $0 \leq j \leq n_i$ with $a_{i,n_i} = a_{i+1,0}$, $\beta_x : (\partial_e \xrightarrow{n_i}) \rightarrow A$ picks out the elements $a_{i,0}, \dots, a_{i,n_i}$ and $\beta_\gamma : (\partial_e \xrightarrow{m}) \rightarrow A$ picks out $a_{1,0}, a_{2,0}, \dots, a_{m,0}, a_{m,n_i}$.

Example 8.15. This notation works similarly for the other examples. For instance let T be the free double category monad, t is a grid in $S\Box$, and $\beta : \partial_e Et \rightarrow A$ is a “functor” from the “1-skeleton” of the grid into a $\partial_e T$ -algebra (here a pair of categories with the same objects). The maps $\alpha_x : \partial_e y(\Box) \rightarrow A$ pick out each of the squares inside the grid β in A , and α_β is the square in \mathcal{A} given by composing the four strings of arrows on the boundary of the grid in \mathcal{A} .

Definition 8.16. Let T be an e -graded familial monad on $\widehat{\mathcal{C}}$ represented by (S, E) , and $V : (\Theta_a^e)^{op} \rightarrow \mathbf{Cat}$ a weak (T, e) -structured category. A V -enriched T -algebra consists of the following:

- A $\partial_e T$ -algebra A
- For each $\alpha : \partial_e y(e_k) \rightarrow A$ in $\widehat{\mathcal{C} \setminus e}$, an object $\text{Hom}(\alpha)$ of V_k
- For each $t \in Se_k$, $\beta : \partial_e Et \rightarrow A$ in $\widehat{\mathcal{C} \setminus e}$, a morphism in V_k

$$\text{comp}_\beta : \otimes_t (\text{Hom}(\alpha_x))_{x \in Et_e} \rightarrow \text{Hom}(\alpha_\beta)$$

- For $\alpha : \partial_e y(e) \rightarrow A$, $comp_\alpha : \otimes_{\eta(e)}(\text{Hom}(\alpha)) \rightarrow \text{Hom}(\alpha)$ agrees with the identitor isomorphism $\otimes_{\eta(e)}(\text{Hom}(\alpha)) \cong \text{Hom}(\alpha)$ of V
- For each $t \in Se$, $f : Et \rightarrow S$, $\gamma : \partial_e E\mu(t, f) \rightarrow A$, the following diagram commutes:

$$\begin{array}{ccc}
\otimes_t(\otimes_{f(x)}(\text{Hom}(\alpha_{x'}))_{x' \in Ef(x)_e})_{x \in Et_e} & \xrightarrow{\otimes_t(comp_{\beta_x})} & \otimes_t(\text{Hom}(\alpha_{\beta_x}))_{x \in Et_e} \\
\Downarrow & & \downarrow comp_{\beta_\gamma} \\
\otimes_{\mu(t, f)}(\text{Hom}(\alpha_{x'}))_{x' \in E\mu(t, f)_e} & \xrightarrow{comp_\gamma} & \text{Hom}(\alpha_\gamma)
\end{array}$$

Example 8.17. When T is the free category monad on graphs and V is a monoidal category, a V -enriched T -algebra is precisely a V -enriched category. For a set A and elements x, y, z in A , let $\beta_0 : (\partial_e \xrightarrow{0}) \rightarrow A$ pick out $a \in A$, and $\beta_2 : (\partial_e \xrightarrow{2}) \rightarrow A$ pick out a_0, a_1, a_2 . The maps $comp_{\beta_0}, comp_{\beta_2}$ in V then look like

$$comp_{\beta_0} : I \rightarrow \text{Hom}(a, a) \qquad comp_{\beta_2} : \text{Hom}(a_0, a_1) \otimes \text{Hom}(a_1, a_2) \rightarrow \text{Hom}(a_0, a_2),$$

just as in the classical definition of a V -enriched category. The equations in the definition above specialize to the same equations satisfied by a category enriched in V , but presented in an “unbiased” way: rather than only defining nullary and binary composition on the Hom objects, there are n -ary composition maps of the form

$$\text{Hom}(a_0, a_1) \times \cdots \times \text{Hom}(a_{n-1}, a_n) \rightarrow \text{Hom}(a_0, a_n).$$

The associativity and unit equations of a V -enriched category are all subsumed by the last equation in Definition 8.16, which shows that any way of associating n -ary composition using binary and nullary composites agrees with the given n -ary composite maps above up to the unitors and/or associators in V .

Example 8.18. When T is the free monoid monad, $\mathcal{C}\setminus e$ is the empty category and $\partial_e T$ is the identity monad on the one-object category $\widehat{\mathcal{C}\setminus e}$. In the above definition of a V -enriched monoid for a monoidal category V , A is the empty presheaf on the empty category and so is $\partial_e y(e)$, so there is a unique map $\alpha : \partial_e y(e) \rightarrow A$. A monoid enriched in V therefore contains a single object $\text{Hom}(\alpha)$. The maps $\beta : \partial_e Et \rightarrow A$ are also unique (being maps from the empty presheaf to itself), so for each $n \in S_0$ there is a map $\text{Hom}(\alpha)^{\otimes n} \rightarrow \text{Hom}(\alpha)$ in V . The equations in Definition 8.16 ensure that these maps endow $\text{Hom}(\alpha)$ with the structure of a monoid object in V , and any monoid object in V has this structure, so a V -enriched monoid is precisely a monoid object in V .

This has interesting philosophical implications for category theory, as it shows that a strict monoidal category is merely a **Cat**-enriched monoid, which broadens the scope of categorical structures which can be defined using enrichment. In Example 11.17, we use this definition to give a concise construction of the free strict monoidal category monad on graphs. Were we to consider weak enrichment in a (T, e) -structured bicategory (or even just 2-category), we would recover weak monoidal categories as monoids weakly enriched in **Cat**.

Example 8.19. A double category enriched in a braided monoidal category V consists of a pair of categories A_1, A_2 with the same objects, for each square α in A_1, A_2 an object $\text{Hom}(\alpha)$ in V , and composition maps $\otimes_{i,j} \alpha_{i,j} \rightarrow \alpha_\beta$ for β a grid in A_1, A_2 made up of squares indexed by i, j , satisfying the usual unit, associativity, and interchange laws.

In [1], Aguiar defines double categories enriched in a *duoidal* category V , which is essentially a category with a pair of monoidal structures which only satisfy *lax* interchange, as in for objects a_1, a_2, a_3, a_4 there is a morphism

$$(a_1 \otimes_1 a_2) \otimes_2 (a_3 \otimes_1 a_4) \rightarrow (a_1 \otimes_2 a_3) \otimes_1 (a_2 \otimes_2 a_4),$$

among other relaxations of the compatibility conditions conferred by a (T, e) -structured category when T is the free double category monad. In particular, when these lax interchange morphisms are invertible V is precisely a braided monoidal category by the usual Eckmann-Hilton argument. This notion of enrichment for double categories is therefore more general than ours, which suggests the question of whether or not this theory of enrichment can encompass it.

There are several potential extensions of this theory that could reproduce duoidal enrichment, though they are beyond the current scope of this work. The basic idea is that a duoidal category is a highly specialized relaxation of a functor $\Delta_+ \times \Delta_+ \rightarrow \mathbf{Cat}$, which results in a diagram in \mathbf{Cat} as below left, where $d_1 : [2] \rightarrow [1]$ and $s_0 : [0] \rightarrow [1]$ are the unique such maps in Δ_+ . Using the Segal condition this diagram is equivalent to that below right, where the duoidal category V denotes $V([1], [1])$, the two monoidal products \diamond, \star denote the maps $V(\text{id}, d_1), V(d_1, \text{id})$, and

the two monoidal units I, J denote the maps $V(\text{id}, s_0), V(s_0, \text{id})$.

$$\begin{array}{ccccc}
 V([2], [2]) & \xrightarrow{V(\text{id}, d_1)} & V([2], [1]) & \xleftarrow{V(\text{id}, s_0)} & V([2], [0]) \\
 \downarrow V(d_1, \text{id}) & \nearrow & \downarrow V(d_1, \text{id}) & \nwarrow & \parallel V(d_1, \text{id}) \\
 V([1], [2]) & \xrightarrow{V(\text{id}, d_1)} & V([1], [1]) & \xleftarrow{V(\text{id}, s_0)} & V([1], [0]) \\
 \uparrow V(s_0, \text{id}) & \searrow & \uparrow V(s_0, \text{id}) & \swarrow & \parallel V(s_0, \text{id}) \\
 V([0], [2]) & \xrightarrow{V(\text{id}, d_1)} & V([0], [1]) & \xleftarrow{V(\text{id}, s_0)} & V([0], [0])
 \end{array}
 \qquad
 \begin{array}{ccccc}
 V^4 & \xrightarrow{\diamond \times \diamond} & V^2 & \xleftarrow{I \times I} & * \\
 \downarrow * \times * & \nearrow & \downarrow * & \nwarrow & \parallel \\
 V^2 & \xrightarrow{\diamond} & V & \xleftarrow{I} & * \\
 \uparrow J \times J & \searrow & \uparrow J & \swarrow & \parallel \\
 * & \xrightarrow{\quad} & * & \xleftarrow{\quad} & *
 \end{array}$$

We do not currently have the tools to canonically extract the appropriate definition of duoidal category from merely the free double category monad, but it is possible that reconstructing this monad as an “internalization” of the free category monad with itself would provide such a toolset.

Example 8.20. When T is the free n -category monad and V is a symmetric monoidal category (or merely braided when $n = 2$), a V -enriched n -category consists of an $(n - 1)$ -category A along with an object $\text{Hom}(\alpha)$ of V for α any pair of parallel $(n - 1)$ -cells. There are then n -different units and binary composition operations for each of the usual n directions in which n -cells can be composed, satisfying unit, associativity, and interchange equations in all dimensions. It is straightforward to check that when V is the symmetric monoidal category of m -categories, a V -enriched n -category is precisely an $(n + m)$ -category. This is further discussed in Section 11, where we show that this fact generalizes to a wide variety of pairs of familial monads: enriching one type of higher category in the symmetric monoidal category of another type of higher category produces a new type of higher category.

Example 8.21. When V is symmetric monoidal, this definition of enriched multicategories agrees with existing notions in the literature, such as [13, Section 2]. It

also specializes the notion of enriched multicategory in [26]. A V -enriched multicategory consists of an object set A and for each tuple $(a_1, \dots, a_n; a)$ in A an object $\text{Hom}(a_1, \dots, a_n; a)$ of V equipped with composition maps for each tree in $S(n, 1)$ labeled with elements of A , satisfying equations in V corresponding to the usual equations for composition in a multicategory (and symmetries in the case of symmetric multicategories).

However, for T the free (nonsymmetric/symmetric) multicategory monad a (T, e) -structured category can be any (nonsymmetric/symmetric) operad object in \mathbf{Cat} (which is, incidentally, a 1-object \mathbf{Cat} -enriched multicategory in the previous sense), which significantly broadens the examples.

If each V_n is a set (aka discrete category), a V -enriched multicategory consists on a set of objects A and, for each tuple $(a_1, \dots, a_n; a)$, an n -ary operation $\text{Hom}(a_1, \dots, a_n; a) \in V_n$. The composition maps are all identities, which means that for any height-2 tree labeled with elements of A with leaves a_1, \dots, a_n and root a , the choices of operations in V for each multi-edge in the tree must compose in V to $\text{Hom}(a_1, \dots, a_n; a)$. For A to be symmetric, permutation of the domains in A must agree with the corresponding symmetries in V .

When A has a single object, this amounts to a choice of a single operation in V_n for all n such that these operations are closed under composition in V_n . This is precisely a morphism of operads from the terminal (nonsymmetric/symmetric) operad to V , which picks out a (ordinary/commutative) monoid structure on any V -algebra. When V is any operad in \mathbf{Cat} , a 1-object V -enriched multicategory amounts

to a lax-operadic functor from the terminal (nonsymmetric/symmetric) operad to V .

Remark 8.22. A (T, e) -structured n -category would allow for “weak” enrichment of T -algebras completely analogously to the notion of weak enrichment in a bicategory in [18, Above Definition 11].

We now define morphisms of enriched T -algebras primarily for the sake of Theorem 11.14, which shows that for V the symmetric monoidal category of T' -algebras, under quite general circumstances the category of V -enriched T -algebras is equivalent to the category of algebras for another familial monad.

Definition 8.23. A morphism of V -enriched T -algebras A, A' consists of a morphism $\phi_0 : A \rightarrow A'$ of the underlying $\partial_e T$ -algebras along with, for all $\alpha : \partial y(e_k) \rightarrow A$, a morphism $\phi_\alpha : \text{Hom}_A(\alpha) \rightarrow \text{Hom}_{A'}(\phi_0 \circ \alpha)$ in V such that the morphisms ϕ_α commute with the composition maps in A, A' .

9 Generalized Wreath Products

Classically, when the objects of a category \mathcal{A} are equipped with a finite number of “slots” and the morphisms “send” these slots of the domain to disjoint subsets of the slots of the codomain, we can form the *wreath product* category $\mathcal{A} \wr \mathcal{B}$ for any category \mathcal{B} , whose objects consist of an object in \mathcal{A} with each of its slots occupied by an object of \mathcal{B} , and morphisms those of \mathcal{A} equipped with a collection of maps between the adorning objects of \mathcal{B} .

Definition 9.1. The category Γ has objects the finite sets $\underline{n} := \{1, \dots, n\}$ ($n \geq 0$) and morphisms $\underline{n} \rightarrow \underline{m}$ functions $\underline{n} \rightarrow \mathcal{P}(\underline{m})$ such that the images of distinct elements of \underline{n} are disjoint subsets of \underline{m} . Identity morphisms send $i \in \underline{n}$ to $\{i\} \in \mathcal{P}(\underline{n})$ and the composite of $f : \underline{n} \rightarrow \underline{m}$ and $g : \underline{m} \rightarrow \underline{\ell}$ sends $i \in \underline{n}$ to $\coprod_{j \in f(i)} g(j)$.

Definition 9.2. Given a functor $\gamma : \mathcal{A} \rightarrow \Gamma$ and a category \mathcal{B} , the wreath product $\mathcal{A} \wr \mathcal{B}$ is the category with objects tuples

$$(a \in \text{Ob}(\mathcal{A}); (b_i \in \text{Ob}(\mathcal{B}))_{i \in \gamma(a)})$$

and morphisms tuples

$$(p : a \rightarrow a'; (p_{i,j} : b_i \rightarrow b'_j)_{i \in \gamma(a), j \in \gamma(p)(i)}).$$

Identities are given by the morphisms $(\text{id}_a; (\text{id}_{b_i}))$ and the composite of morphisms $(p; (p_{i,j}))$ and $(q; (q_{i',j'}))$ is given by $(qp; (q_{j,j'} p_{i,j})_{i,j'})$ where j is the unique element of $\gamma(p)(i)$ with $j' \in \gamma(q)(j)$ for all j' in $\gamma(qp)(i) = \coprod_{j \in \gamma(p)(i)} \gamma(q)(j)$.

This wreath product forms the technical basis for nearly every step in our construction of familial monads for enriched structures. Before using it to construct new cell shapes and operations on cell diagrams, we describe a generalization of the wreath product dropping the finiteness assumption from the sets in the definition of γ , dropping the disjointness condition from its morphisms, and replacing the subsets of \underline{m} with multisets. The result is a wreath product construction in which morphisms can consist of any arrangement of arrows between the sets of objects from \mathcal{B} .

9.1 The Category Span

The morphisms in Γ between sets \underline{n} and \underline{m} admit several equivalent descriptions:

- Functions $\underline{n} \rightarrow \mathcal{P}(\underline{m})$ landing in disjoint subsets
- Isomorphism classes of injective relations between \underline{n} and \underline{m} , which is to say, subsets $R \subseteq \underline{n} \times \underline{m}$ such that if $(i, j) \in R$ for $i \in \underline{n}$ and $j \in \underline{m}$, (i', j) is not in R for any $i' \neq i$
- Isomorphism classes of spans $\underline{n} \leftarrow R \hookrightarrow \underline{m}$ whose second leg is an injection.

To see that these are equivalent, an injective relation R uniquely determines both the function

$$\underline{n} \rightarrow \mathcal{P}(\underline{m}) \quad i \mapsto \{j \in \underline{m} \mid (i, j) \in R\}$$

and the span $\underline{n} \leftarrow R \hookrightarrow \underline{m}$, where the disjointness and injectivity conditions are equivalent, and composition in Γ described above agrees with the usual compositions of relations and spans.

Our desired generalization replaces the objects of Γ with arbitrary sets, and extends the morphisms in a manner expressible in terms of any of the three above descriptions of the morphisms in Γ , though we will use the description in terms of spans which is easy to work with and more present in the literature. For sets N and M , our morphisms from N to M will be, equivalently:

- Functions from N to the set of multisets consisting of (potentially many copies of) elements of M
- Isomorphism classes of “proof-relevant” relations between N and M , meaning (isomorphism classes of) functions $R \rightarrow N \times M$. Intuitively, this is like a relation except instead of elements $i \in N$ and $j \in M$ being merely related or not related, there is now a (potentially empty) set $R_{i,j}$ of “proofs” or “witnesses” to the relation of i and j , given by the preimage in R of the pair (i, j)
- Isomorphism classes of spans $N \leftarrow R \rightarrow M$, where we again write $R_{i,j}$ for the subset of R mapped both to $i \in N$ and $j \in M$.

We abuse notation by using R to denote both the set R as above and the entire span. An isomorphism between two spans R, R' is an isomorphism of sets $R \cong R'$ which commutes with the functions to N and M .

Definition 9.3. **Span** is the category whose objects are sets and whose morphisms are isomorphism classes of spans between them. Identity spans are given by pairs of identity functions, and composition is given by pullback, which is unital and associative on isomorphism classes of spans.

Note that **Span** is not even locally small, though it could be made so by restricting the cardinalities of the object sets and spans. However, we will primarily use only certain locally small subcategories of **Span**, through which we can factor the functors to **Span** we are interested in.

Example 9.4. An isomorphism class of spans R from N to M such that the sets $R_{i,j}$ have cardinality 1 or 0 is precisely a relation between N and M . In the second description above of the morphisms of **Span** as proof-relevant relations, these are the relations R in which there is only one “proof” of (i, j) in R , if any. However, this property is not preserved by composition in **Span**, as composing a relation from N to M which is not co-injective (each element in N relates to at most one element in M) with a relation from M to L which is not injective (each element in L relates to at most one element in M) often results in a relation from N to L where a pair of elements is related via more than one element in M , and these multiple witnesses are recorded by the composition in **Span**.

Example 9.5. While **Rel** is not a subcategory of **Span**, both Γ and **Set** are subcategories of **Rel** which are closed under composition in **Span**, as Γ consists only of injective relations (between finite sets) and **Set** consists only of co-injective relations (functions between sets are precisely the relations which are co-injective and co-surjective). In other words, Γ is the subcategory consisting of finite sets and spans of the form $N \leftarrow R \hookrightarrow M$, while **Set** is the subcategory consisting of spans of the form $N = N \rightarrow M$. **Set**^{op} is also a subcategory of **Span** consisting of the spans of the form $N \leftarrow M = M$.

Example 9.6. If we restrict to spans those whose sets $R_{i,j}$ are finite sets, their isomorphism classes are determined by the natural numbers $|R_{i,j}|$ and composition corresponds to matrix multiplication of these arrays of numbers. If we further restrict to spans such that R itself is finite (even if the source and target sets N, M are not), this subcategory is equivalent to the category $\mathbf{L}_{\text{comon}}$ of free abelian monoids and

monoid homomorphisms between them.

For any such span $N \leftarrow R \rightarrow M$, we get a homomorphism of free abelian monoids $\langle N \rangle \rightarrow \langle M \rangle$ defined on generators by

$$i \mapsto \prod_{j \in M} j^{|R_{i,j}|},$$

where the product is guaranteed to be finite by assumption. Conversely, given a monoid homomorphism $\langle N \rangle \rightarrow \langle M \rangle$ we can recover $|R_{i,j}|$ as the multiplicity of j in the image of the generator i , and composition and identities in the two categories agree.

Finally, recall (from, for instance, [31]) that **Span** extends to a double category with sets as objects, isomorphism classes of spans as horizontal morphisms, functions as vertical morphisms, and squares given by pairs of commuting squares as in the figure below.

$$\begin{array}{ccccc} X & \longleftarrow & Z & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longleftarrow & Z' & \longrightarrow & Y' \end{array}$$

This will allow us to consider vertical natural transformations between functors into **Span** whose components are functions rather than spans and whose naturality squares contain the additional data of the function $Z \rightarrow Z'$ above.

9.2 Wreath Products

A functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ can be interpreted as assigning to each object a of \mathcal{A} a set of available “slots” and to each morphism $p : a \rightarrow a'$ in \mathcal{A} a collection of arrows between those slots where $\gamma(p)_{i,j}$ describes the set of arrows from the i th slot in $\gamma(a)$ to the j th slot in $\gamma(a')$. When γ factors through Γ , for instance, each object has finitely many slots and each morphism is sent to a finite arrangement of arrows between the slots with the property that no slot of the codomain has more than one incoming arrow.

Given such a functor, the wreath product of \mathcal{A} and a category \mathcal{B} has as objects the tuple of an object a of \mathcal{A} with each of its slots filled with an object of \mathcal{B} , and a morphism between such tuples amounts to a morphism $p : a \rightarrow a'$ in \mathcal{A} and a morphism in \mathcal{B} for each of the corresponding arrows between the slots indexed by the span $\gamma(p)$. This recovers the classical wreath product when γ factors through Γ .

Definition 9.7. Given a functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ and a category \mathcal{B} , the wreath product $\mathcal{A} \wr_{\gamma} \mathcal{B}$ (or simply $\mathcal{A} \wr \mathcal{B}$) is the category with:

- Objects of the form $(a \in \text{Ob}(\mathcal{A}); (b_i \in \text{Ob}(\mathcal{B}))_{i \in \gamma(a)})$
- Morphisms of the form $(p : a \rightarrow a'; (p_{i,j,k} : b_i \rightarrow b'_j)_{i \in \gamma(a), j \in \gamma(a'), k \in \gamma(p)_{i,j}})$
- Identities of the form $(id_a; (id_{b_i})_{i \in \gamma(a)})$
- The composite of morphisms

$$(p; (p_{i,j,k})) : (a; (b_i)) \rightarrow (a'; (b'_j)) \quad \text{and} \quad (q; (q_{i',j',k'})) : (a'; (b'_{i'})) \rightarrow (a''; (b''_{j'}))$$

is given by

$$(qp; (q_{j,j',k'} \circ p_{i,j,k})_{i \in \gamma(a), j' \in \gamma(a''), (j,(k,k')) \in \gamma(qp)_{i,j'}}).$$

noting that

$$\gamma(qp)_{i,j'} \cong (\gamma(q)\gamma(p))_{i,j'} \cong \coprod_{j \in \gamma(a')} \gamma(q)_{j,j'} \times \gamma(p)_{i,j}.$$

Associativity and unit laws follow from the same for \mathcal{A} , \mathcal{B} , and **Span**. Intuitively, composition in $\mathcal{A}\mathcal{B}$ should be regarded as sending an arrangement of arrows between the slots of a and a' , and one between the slots of a' and a'' , to the arrangement consisting of all possible composites of those arrows.

Remark 9.8. This is why it is necessary to have γ land in **Span** rather than simply **Rel**: composing a one-to-many arrangement of arrows with a many-to-one arrangement of arrows can result in multiple different composite arrows between a fixed pair of slots, and **Span** keeps track of not just which pairs of slots have a composite arrow between them but also how many such composites should be expected. Classically, using Γ avoids this issue by allowing only one-to-many arrangements.

This generalization allows for the description of many new types of wreath products, though we also recall the most famous example of wreath products of the simplex category Δ .

Example 9.9. In the prototypical example of a classical wreath product, there is a functor $\Delta \rightarrow \Gamma \hookrightarrow \mathbf{Span}$ sending the n -simplex $[n]$ to the set of its n “spinal” edges of the form $\{i, i + 1\} \subseteq [n]$. A morphism f in Δ sends each spinal edge $\{i, i + 1\}$ to

the (possibly empty) set of spinal edges edges contained within $\{f(i), \dots, f(i + 1)\}$, a one-to-many relation on the edges of the domain and codomain simplices.

The wreath product $\Delta \wr \mathcal{B}$ then has as objects simplices with each spinal edge decorated with an object of \mathcal{B} , and as morphisms those of Δ equipped with a map $b_i \rightarrow b_j$ in \mathcal{B} for each j such that the i th spinal edge of the domain simplex covers the j th spinal edge of the codomain simplex.

Example 9.10. For a fixed set of objects $e = \{e_k\} \subseteq \text{Ob}(\mathcal{C})$, there is a functor $\widehat{\gamma}_e : \widehat{\mathcal{C}} \rightarrow \mathbf{Set} \hookrightarrow \mathbf{Span}$ sending a cell diagram X to its set of “ e -cells” $\sqcup_k X_{e_k}$. The wreath product $\widehat{\mathcal{C}} \wr \mathcal{B}$ has as objects cell diagrams over \mathcal{C} with each e -cell equipped with an object of \mathcal{B} , and as morphisms those of $\widehat{\mathcal{C}}$ equipped with a morphism in \mathcal{B} from the decorating object of each e -cell to the decorating object of its image. All of the wreath products in the next section on cell shapes and cell diagrams are of this form, though more complicated functors to \mathbf{Span} arise when taking wreath products of familial theories, such as the examples above and below.

Example 9.11. The discrete category \mathbb{N} has a functor to \mathbf{Span} sending n to the n -element set. For a category \mathcal{A} , $\mathbb{N} \wr \mathcal{A}$ has objects of the form $(n, (a_1, \dots, a_n))$ and morphisms of the form $(\text{id}_n, (f_1 : a_1 \rightarrow a'_1, \dots, f_n : a_n \rightarrow a'_n))$, making it precisely the free strict monoidal category on \mathcal{A} .

In a similar fashion, replacing \mathbb{N} with $\text{iso}(\mathbf{FinSet})$ with the functor $\text{iso}(\mathbf{FinSet}) \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$ produces the free weakly symmetric strict monoidal category on \mathcal{A} , replacing \mathbb{N} with the disjoint union over $n \in \mathbb{N}$ of the contractible groupoid on nullary/binary or unbiased parenthesizations of n letters (projecting

to \mathbb{N} for its functor to **Span**) produces the free biased or unbiased weak monoidal category on \mathcal{A} , and replacing \mathbb{N} with the disjoint union over $n \in \mathbb{N}$ of the braid group on n letters (as a 1-object groupoid, with the functor to **Span** coming from the projection to $iso(\mathbf{FinSet})$) gives the free braided monoidal category on \mathcal{A} .

These are all in fact classical wreath products as $iso(\mathbf{FinSet}) \rightarrow \mathbf{Span}$ factors through Γ , but in the following examples we use the generalized wreath product to produce more complicated free constructions on a category \mathcal{A} .

Example 9.12. For the functor $\mathbf{FinSet} \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$, $\mathbf{FinSet} \wr \mathcal{D}$ has objects given by finite sets of objects in \mathcal{D} and morphisms $(d_1, \dots, d_n) \rightarrow (d'_1, \dots, d'_m)$ given by a function $f : \underline{n} \rightarrow \underline{m}$ and morphisms $d_i \rightarrow d'_{f(i)}$. This is easily checked to be the free finite coproduct completion of \mathcal{D} , as

$$\mathrm{Hom}((d_1, \dots, d_n), (d'_1, \dots, d'_m)) \cong \prod_{i \in \underline{n}} \mathrm{Hom}(d_i, (d'_1, \dots, d'_m)).$$

Similarly, $\mathbf{Set} \wr \mathcal{D}$ is the free coproduct completion of \mathcal{D} . By completely dual reasoning, $\mathbf{FinSet}^{op} \wr \mathcal{D}$ and $\mathbf{Set}^{op} \wr \mathcal{D}$, defined using the subcategory inclusion of \mathbf{Set}^{op} into **Span**, give the free finite product completion and free product completion of \mathcal{D} , respectively. This is also a consequence of the observation that $(\mathcal{A} \wr \mathcal{B})^{op} \cong \mathcal{A}^{op} \wr \mathcal{B}^{op}$.

Example 9.13. Consider (the opposite of) the Lawvere theory for monoids, namely the category \mathbf{L}_{mon} of finitely generated free monoids. There is a functor $\mathbf{L}_{mon} \rightarrow \mathbf{Span}$ factoring through the subcategory \mathbf{L}_{cmon} (Example 9.6), given by abelianization. The abelianization of the free monoid on a finite set A is the free abelian monoid on A , and the abelianization of a monoid homomorphism sending each generator i

to some finite word on the elements j of B forgets the order of the generators in each word and sees only the multiplicity of each j .

The wreath product $\mathbf{L}_{mon} \wr \mathcal{D}$ then has as objects finitely generated free monoids with each generator i decorated by an object d_i of \mathcal{D} , and as morphisms monoid homomorphisms f equipped with $f_{i,j}$ -many morphisms $d_i \rightarrow d'_j$ in \mathcal{D} , where $f_{i,j}$ is the multiplicity of j in $f(i)$. The objects of $\mathbf{L}_{mon} \wr \mathcal{D}$ amount to finite lists of objects in \mathcal{D} , and so $\text{Ob}(\mathbf{L}_{mon} \wr \mathcal{D})$ is the free monoid on $\text{Ob}(\mathcal{D})$. From this perspective, the morphisms $(d_1, \dots, d_n) \rightarrow (d'_1, \dots, d'_m)$ consist of for each i from 1 to n a finite list of morphisms

$$(f_{i,1} : d_i \rightarrow d'_{j_1}, \dots, f_{i,k_i} : d_i \rightarrow d'_{j_{k_i}}).$$

This is certainly a monoidal category, though the large number of morphisms makes it much larger than the free monoidal category on \mathcal{A} . We plan to further investigate monoidal categories of this sort in future work, as they appear to encode a vast amount of additional structure.

Example 9.14. Similar to the above, $\mathbf{L}_{cmon} \wr \mathcal{D}$ has objects given by the free monoid on $\text{Ob}(\mathcal{D})$ and morphisms $(d_1, \dots, d_n) \rightarrow (d'_1, \dots, d'_m)$ given by, for each i from 1 to n and j from 1 to m , maps

$$(f_{i,j,1}, \dots, f_{i,j,k_{i,j}} : d_i \rightarrow d'_j)$$

in \mathcal{D} for some natural number $k_{i,j}$. Compared to the morphisms in the previous example, these do not keep track of the order of the codomains but only the number of maps from each d_i to each d'_j .

This is also a monoidal category with a lot of additional structure. By the inclusions of both **FinSet** and **FinSet**^{op} into the category of finite spans \mathbf{L}_{cmon} , the objects (d_1, d_2) can be checked to be both the product and coproduct of (d_1) and (d_2) . There is a unique morphism from any object to or from the empty list $()$. For any pair of morphisms $f, g : d \rightarrow d'$, there is a morphism $(f, g) : (d) \rightarrow (d')$ in $\mathbf{L}_{\text{cmon}} \wr \mathcal{D}$ as well as a canonical morphism $() : (d) \rightarrow (d')$. From this, it is straightforward to check that $\mathbf{L}_{\text{cmon}} \wr \mathcal{D}$ is the free finite biproduct completion of \mathcal{D} .

We now proceed to describe how wreath products interact with various constructs of category theory, including colimits of categories, monoidal structures, factorization systems, and colimits in a category.

9.3 Functoriality and Wreath Products of Colimits

We now describe how $\mathcal{A} \wr_{\gamma} \mathcal{B}$ is functorial with respect to $\mathcal{A}, \gamma, \mathcal{B}$, and prove that this functor preserves coproducts in the \mathcal{A}, γ position and filtered colimits in the \mathcal{B} position.

Definition 9.15. Let $\mathbf{CAT} // \mathbf{Span}$ denote¹ the category with objects of the form $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ for a category \mathcal{A} and morphisms $(\mathcal{A}_1, \gamma_1) \rightarrow (\mathcal{A}_2, \gamma_2)$ given by functors $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ which are “vertically colax over **Span**” in the sense of the diagram below left. In other words, for $f : a \rightarrow a'$ in \mathcal{A}_1 , there is a square in the

¹This is a slight abuse of the notation for lax overcategories in Section 2 as the natural transformations are double categorical rather than those in **CAT**.

double category **Span** as below right and composition of morphisms in \mathcal{A}_1 goes to horizontal composition of such squares.

$$\begin{array}{ccc}
\mathcal{A}_1 & \xrightarrow{F} & \mathcal{A}_2 \\
\searrow \gamma_1 & \longleftarrow & \swarrow \gamma_2 \\
& \mathbf{Span} &
\end{array}
\qquad
\begin{array}{ccccc}
\gamma_2(Fa) & \longleftarrow & \gamma_2(Ff) & \longrightarrow & \gamma_2(Fa') \\
\phi_a \downarrow & & \downarrow \phi_f & & \downarrow \phi_{a'} \\
\gamma_1(a) & \longleftarrow & \gamma_1(f) & \longrightarrow & \gamma_1(a')
\end{array}$$

Proposition 9.16. *The wreath product \wr forms a functor $\mathbf{CAT} // \mathbf{Span} \times \mathbf{CAT} \rightarrow \mathbf{CAT}$.*

Proof. Given a functor $G : \mathcal{B} \rightarrow \mathcal{B}'$, for fixed $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ we define the functor $\mathcal{A} \wr_\gamma \mathcal{B} \rightarrow \mathcal{A} \wr_\gamma \mathcal{B}'$ sending $(a, (b_i))$ to $(a, (Gb_i))$ and similarly applying G to morphisms (functoriality follows from functoriality of G).

For $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ vertically lax over **Span** and fixed category \mathcal{B} , we define the functor $\mathcal{A}_1 \wr_{\gamma_1} \mathcal{B} \rightarrow \mathcal{A}_2 \wr_{\gamma_2} \mathcal{B}$ sending $(a, (b_i))$ to $(Fa, (b_{\phi_a(i')})_{i' \in \gamma_2(Fa)})$ and $(f : a \rightarrow a', (f_k : b_i \rightarrow b'_j)_{k \in \gamma_1(f)_{i,j}})$ to

$$(Ff : Fa \rightarrow Fa', (f_{\phi_f(k')} : b_{\phi_a(i')} \rightarrow b'_{\phi_{a'}(j')})_{k' \in \gamma_2(f)_{\phi_a(i'), \phi_{a'}(j')}}).$$

These two types of functors are easily checked to commute for a pair of such F and G . □

Remark 9.17. Indeed, like the classical wreath product (see for instance [10, 4.1]) this functor lifts to one of the form

$$\mathbf{CAT} // \mathbf{Span} \times \mathbf{CAT} // \mathbf{Span} \rightarrow \mathbf{CAT} // \mathbf{Span}$$

which forms a monoidal product.

Proposition 9.18. *For a category \mathcal{B} , the functor $- \wr_{\mathcal{B}} : \mathbf{CAT} // \mathbf{Span} \rightarrow \mathbf{CAT}$ preserves coproducts.*

Proof. This is immediate by definition of the wreath product, as for $\gamma_1 : \mathcal{A}_1 \rightarrow \mathbf{Span}$ and $\gamma_2 : \mathcal{A}_2 \rightarrow \mathbf{Span}$ the category $(\mathcal{A}_1 \amalg \mathcal{A}_2) \wr_{\gamma_1, \gamma_2} \mathcal{B}$ contains isomorphic copies of $\mathcal{A}_1 \wr_{\gamma_1} \mathcal{B}$ and $\mathcal{A}_2 \wr_{\gamma_2} \mathcal{B}$ and no morphisms between them. \square

Proposition 9.19. *For $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ which factors through the subcategory of κ -small sets and κ -small spans for a regular cardinal κ (see Example 9.6 for the case of $\kappa = \omega$), the functor $\mathcal{A} \wr_{\gamma} - : \mathbf{CAT} \rightarrow \mathbf{CAT}$ preserves κ -filtered colimits.*

Proof. For a κ -filtered diagram $\mathcal{B} : \mathcal{I} \rightarrow \mathbf{CAT}$, an object in $\operatorname{colim}_{I \in \mathcal{I}} (\mathcal{A} \wr_{\gamma} \mathcal{B}_I)$ is an equivalence class of tuples $(a, (b_i)_{i \in \gamma(a)})$, where all of the objects b_i live in the same \mathcal{B}_I for some I in \mathcal{I} , while an object in $\mathcal{A} \wr_{\gamma} (\operatorname{colim}_{I \in \mathcal{I}} \mathcal{B}_I)$ is an equivalence class of tuples $(a, (b_i)_{i \in \gamma(a)})$ where each b_i has a representative in some category \mathcal{B}_I in the diagram, though these choices of I need not be the same. However, as \mathcal{I} is filtered this κ -small set of objects I has a cocone I' in \mathcal{I} and this object in $\mathcal{A} \wr_{\gamma} (\operatorname{colim}_{I \in \mathcal{I}} \mathcal{B}_I)$ has a representative in which all of the objects b_i belong to $\mathcal{B}_{I'}$. This, along with a completely analogous argument for morphisms, shows that these two categories are equivalent. \square

9.4 Wreath Products of Monoidal Categories

When \mathcal{A} and \mathcal{B} are monoidal categories, to put a monoidal structure on $\mathcal{A} \wr_{\gamma} \mathcal{B}$ requires additional structure involving γ . This requires a choice of monoidal structure on **Span**, which we take to be the cartesian product to accommodate the example in Section 11.1 but could just as well be given by disjoint union. It would suffice to ask that γ be a monoidal functor, but again for the sake of Section 11.1 we describe a weaker condition on γ which is also sufficient.

Definition 9.20. For a monoidal category \mathcal{A} , a functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ is *colax-monoidal in the double categorical sense* if the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is vertically colax over **Span** in the sense of the following diagram:

$$\begin{array}{ccc}
 \mathcal{A} \times \mathcal{A} & \xrightarrow{\quad \otimes \quad} & \mathcal{A} \\
 \searrow \gamma \times \gamma & \leftarrow \leftarrow \leftarrow & \swarrow \gamma \\
 & \mathbf{Span} &
 \end{array}$$

In other words, there is a vertical natural transformation $\phi_{a,a'} : \gamma(a \otimes a') \rightarrow \gamma(a) \times \gamma(a')$ in the double category **Span** satisfying unit and associativity equations to ensure that these compose to unique maps $\gamma(a_1 \otimes \cdots \otimes a_n) \rightarrow \gamma(a_1) \times \cdots \times \gamma(a_n)$ for all $n \geq 0$. Concretely, the naturality amounts to, for $f : a_1 \rightarrow a_2$ and $f' : a'_1 \rightarrow a'_2$ in \mathcal{A} , a commuting diagram as below

$$\begin{array}{ccccc}
 \gamma(a_1 \otimes a'_1) & \longleftarrow & \gamma(f \otimes f') & \longrightarrow & \gamma(a_2 \otimes a'_2) \\
 \phi_{a_1, a'_1} \downarrow & & \downarrow \phi_{f, f'} & & \downarrow \phi_{a_2, a'_2} \\
 \gamma(a_1) \times \gamma(a'_1) & \longleftarrow & \gamma(f) \times \gamma(f') & \longrightarrow & \gamma(a_2) \times \gamma(a'_2)
 \end{array}$$

such that ϕ sends composite morphisms in $\mathcal{A} \times \mathcal{A}$ to horizontal composition of squares in the double category **Span**.

The definition of a colax-monoidal functor would typically also include a vertical morphism $\gamma(I) \rightarrow *$, but this is redundant for the cartesian monoidal structure on **Span**. In the case when γ factors through **Set**, the action of ϕ on morphisms f is determined by the $\phi_{a,a'}$

Proposition 9.21. *Given monoidal categories \mathcal{A}, \mathcal{B} and a functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$ which is colax-monoidal in the double categorical sense, $\mathcal{A} \lambda_\gamma \mathcal{B}$ has a monoidal structure with identity $(I, (I)_{i \in \gamma(I)})$, product on objects given by*

$$(a, (b_i)_{i \in \gamma(a)}) \otimes (a', (b'_{i'})_{i' \in \gamma(a')}) = (a \otimes a', (b_i \otimes b'_{i'})_{k \in \gamma(a \otimes a'), \phi_{a,a'}(k) = (i, i')}),$$

and on morphisms given by

$$\begin{aligned} (f : a_1 \rightarrow a_2, (f_k : b_{1,i} \rightarrow b_{2,j})_{k \in \gamma(f)}) \otimes (f' : a'_1 \rightarrow a'_2, (f'_{k'} : b'_{1,i'} \rightarrow b'_{2,j'})_{k' \in \gamma(f')}) \\ = (f \otimes f', (f_k \otimes f'_{k'})_{\ell \in \gamma(f \otimes f'), \phi_{f,f'}(\ell) = (k, k')}). \end{aligned}$$

Proof. This is tedious but straightforward to check, where functoriality of \otimes follows from naturality of ϕ . Unitality and associativity (weak or strict corresponding to those of \mathcal{A}, \mathcal{B}) follow from the same on \mathcal{A}, \mathcal{B} and the analogous properties of ϕ . \square

9.5 Factorization Systems on Wreath Products

Similarly, factorization systems (in the sense of Definition 6.44) on \mathcal{A} and \mathcal{B} , preserved by γ , extend to a unique factorization system on $\mathcal{A} \lambda_\gamma \mathcal{B}$. We fix a factorization system on **Span**, chosen to accommodate Section 11.5, in which any isomorphism

class of spans $X \leftarrow Z \rightarrow Y$ factors uniquely up to isomorphism as $X \leftarrow Z = Z$ followed by $Z = Z \rightarrow Y$. For factorization in \mathcal{A} of a morphism $f : A \rightarrow A'$ we write $A \xrightarrow{f^1} A_f \xrightarrow{f^2} A'$.

Proposition 9.22. *Given factorization systems on categories \mathcal{A} and \mathcal{B} , and a factorization-preserving functor $\gamma : \mathcal{A} \rightarrow \mathbf{Span}$, $\mathcal{A} \wr_\gamma \mathcal{B}$ has a factorization system in which any morphism $(f : A \rightarrow A', (f_k : B_i \rightarrow B'_j)_{k \in \gamma(f)_{i,j}})$ factors as*

$$(A, (B_i)_{i \in \gamma(A)}) \xrightarrow{(f^1, (f_k^1 : B_i \rightarrow B_{f_k})_{k \in \gamma(f)_i})} (A_f, (B_{f_k})_{k \in \gamma(A_f)}) \xrightarrow{(f^2, (f_k^2 : B_{f_k} \rightarrow B'_j)_{k \in \gamma(f)_j})} (A', (B'_j)_{j \in \gamma(B)}),$$

noting that $\gamma(A_f) \cong \gamma(f)$ as sets.

Proof. All that remains to show is uniqueness of the factorizations, which follows from the same for $\mathcal{A}, \mathcal{B}, \mathbf{Span}$. □

Remark 9.23. It is worth noting that a similar result can be proven for classical wreath products, where the same factorization system restricted to $\Gamma \subseteq \mathbf{Span}$ replaces the class of forward-pointing functions with forward-pointing injections. For wreath products with $\mathcal{A} = \Delta$ for instance, the active-inert factorization system of Theorem 6.31 (or in other words, endpoint preserving maps followed by linear inclusions of finite nonempty ordinals) is preserved by the standard functor into Γ .

9.6 Diagram Decomposition and Colimits in a Wreath Product

Functors of the form $\mathcal{C} \rightarrow \mathcal{A} \wr_{\gamma} \mathcal{B}$ can be decomposed into functors landing in \mathcal{A} and \mathcal{B} respectively. In order to describe this decomposition, we first define an analogue of the Grothendieck construction for functors $\mathcal{A} \rightarrow \mathbf{Span}$.

Definition 9.24. Given a functor $\gamma : \mathcal{C} \rightarrow \mathbf{Span}$, let $\mathcal{F}\gamma$ be the category with objects pairs $(c \in \text{Ob}(\mathcal{C}), i \in \gamma(c))$ and morphisms $(c, i) \rightarrow (c', j)$ given by pairs $(f : c \rightarrow c', k \in \gamma(f)_{i,j})$. Identities and composites are determined by those of $\mathcal{C}, \mathbf{Span}$ respectively.

There is a canonical forgetful functor $\mathcal{F}\gamma \rightarrow \mathcal{C}$. While this Grothendieck construction can be extended to an equivalence between a certain category of functors $\mathcal{C} \rightarrow \mathbf{Span}$ (using lax vertical transformations rather than the usual natural transformations) and a category of certain functors into \mathcal{C} with discrete fibers, we will instead be interested in natural transformations between functors $\mathcal{C} \rightarrow \mathbf{Span}$ and categories constructed from them.

Definition 9.25. Given a natural transformation $\pi : \gamma \rightarrow \gamma' : \mathcal{C} \rightarrow \mathbf{Span}$, let $\mathcal{F}\pi$ denote the category over $\mathcal{C} \times 2$ given by applying \mathcal{F} to $\pi : \mathcal{C} \times 2 \rightarrow \mathbf{Span}$.

This category has objects the disjoint union of $\mathcal{F}\gamma$ and $\mathcal{F}\gamma'$ and morphisms generated by those of $\mathcal{F}\gamma, \mathcal{F}\gamma'$ along with maps $(c, i \in \gamma(c)) \rightarrow (c, j \in \gamma'(c))$ for every

$k \in \gamma(\pi_c)_{i,j}$. Hence there are disjoint inclusions of both $\mathcal{F}\gamma$ and $\mathcal{F}\gamma'$ into $\mathcal{F}\pi$. We call the morphisms between those maps *transition morphisms*.

Given a functor $F : \mathcal{C} \rightarrow \mathcal{A} \lambda_\gamma \mathcal{B}$, we will write F_1 for the composite functor $\mathcal{C} \rightarrow \mathcal{A} \lambda_\gamma \mathcal{B} \rightarrow \mathcal{A}$, where the second functor is the canonical forgetful functor $\lambda : \mathcal{A} \lambda_\gamma \mathcal{B} \rightarrow \mathcal{A}$. Furthermore we write $\gamma_1 : \mathcal{C} \rightarrow \mathbf{Span}$ for the composite $\gamma \circ F_1$.

Lemma 9.26. *A functor $F : \mathcal{C} \rightarrow \mathcal{A} \lambda_\gamma \mathcal{B}$ determines a functor $F_2 : \mathcal{F}\gamma_1 \rightarrow \mathcal{B}$.*

Proof. F_2 sends $(c, i \in \gamma(F(c)))$ to $F(c)_i$ where $F(c) = (F_1(c), (F(c)_i)_{i \in \gamma(F_1(c))})$, and sends

$$(f : c \rightarrow c', k \in \gamma(F(f))_{i,j})$$

to $F(f)_k$, where

$$F(f) = (F_1(f) : F_1(c) \rightarrow F_1(c'), (F(f)_k : F(c)_i \rightarrow F(c')_j)_{k \in \gamma(F_1(f))_{i,j}}).$$

Functoriality of this assignment follows from that of F . □

We are now ready to state the main result of this section characterizing functors $\mathcal{C} \rightarrow \mathcal{A} \lambda_\gamma \mathcal{B}$.

Proposition 9.27. *A pair of functors $F_1 : \mathcal{C} \rightarrow \mathcal{A}$ and $F_2 : \mathcal{F}\gamma_1 \rightarrow \mathcal{B}$ determine a functor $F : \mathcal{C} \rightarrow \mathcal{A} \lambda_\gamma \mathcal{B}$, forming an equivalence of categories between $\mathbf{Fun}(\mathcal{C}, \mathcal{A} \lambda_\gamma \mathcal{B})$ and the category with objects pairs (F_1, F_2) and morphisms pairs $(\pi_1 : F_1 \rightarrow F'_1, \pi_2 : \mathcal{F}\pi_1 \rightarrow \mathcal{B})$ with π_2 restricting to F_2, F'_2 on $\mathcal{F}\gamma_1, \mathcal{F}\gamma'_1$ respectively.*

In this latter category, identities are defined by $\text{id} : F_1 \rightarrow F_1$ and the map $\mathcal{f} \text{id} \cong \mathcal{f} \gamma_1 \times 2 \rightarrow \mathcal{B}$ sending all of the transition morphisms to identities. Composition is given by observing that for transformations $\pi_1 : F_1 \rightarrow F'_1$ and $\pi'_1 : F'_1 \rightarrow F''_1$, $\mathcal{f}(\pi'_1 \circ \pi_1)$ is isomorphic to the pushout of the inclusions of $\mathcal{f} \gamma'_1$ into $\mathcal{f} \pi_1$ and $\mathcal{f} \pi'_1$ respectively, with the objects from $\mathcal{f} \gamma'_1$ removed. So functors $\pi_2 : \mathcal{f} \pi_1 \rightarrow \mathcal{B}$ and $\pi'_2 : \mathcal{f} \pi'_1 \rightarrow \mathcal{B}$ assemble into a functor $\mathcal{f}(\pi'_1 \circ \pi_1) \rightarrow \mathcal{B}$ in a unital and associative manner with transition morphisms sent to the compositions of those from π_2, π'_2 .

Proof. Given such a pair F_1, F_2 , for c in \mathcal{C} and $f : c \rightarrow c'$, define

$$F(c) = (F_1(c), (F_2(c, i))_{i \in \gamma(F_1(c))}) \quad F(f) = (F_1(f), (F_2(f, k))_{k \in \gamma(F_1(f))}).$$

To see that this correspondence between F and (F_1, F_2) extends to an equivalence of categories, we observe that a natural transformation $\pi : F \Rightarrow F' : \mathcal{C} \rightarrow \mathcal{A} \wr_{\gamma} \mathcal{B}$ consists of morphisms

$$\pi_{1,c} : F_1(c) \rightarrow F'_1(c), \quad \pi_{c,k} : F_2(c, i) \rightarrow F'_2(c, j)$$

for c in \mathcal{C} and $k \in \gamma(\pi_1)_{i,j}$ satisfying a naturality condition for morphisms in \mathcal{C} . This amounts to precisely the data of a natural transformation $\pi_1 : F_1 \rightarrow F'_1$ and a functor $\pi_2 : \mathcal{f} \pi \rightarrow \mathcal{B}$, and composition can be easily checked to agree with that described above in the category of pairs (F_1, F_2) . \square

We now turn to how colimits in \mathcal{A}, \mathcal{B} interact with colimits in $\mathcal{A} \wr_{\gamma} \mathcal{B}$.

Definition 9.28. Given a functor $\gamma_1 : \mathcal{C} \rightarrow \mathbf{Span}$ equipped with a cocone (X, ι_c) , let $(\mathcal{f} \gamma_1)_j$ for $j \in X$ denote the category with objects of the form (c, i, k) for (c, i) in

$\mathcal{F}\gamma_1$ and for the span $\iota_c : \gamma_1(c) \rightarrow X$, $k \in \gamma(\iota_c)_{i,j}$. Morphisms $(c, i, k) \rightarrow (c', i', k')$ are given by those in $\mathcal{F}\gamma$ that commute with k, k' in $\mathcal{F}\gamma$.

Note that for all such j there is a forgetful functor $(\mathcal{F}\gamma_1)_j \rightarrow \mathcal{F}\gamma_1$.

Proposition 9.29. *If $F_1 : \mathcal{C} \rightarrow \mathcal{A}$ has a colimit in \mathcal{A} and $F_2 : \mathcal{F}\gamma_1 \rightarrow \mathcal{B}$ is such that $F_{2,j} : (\mathcal{F}\gamma_1)_j \rightarrow \mathcal{F}\gamma_1 \rightarrow \mathcal{B}$ has a colimit for all $j \in \gamma(\text{colim } F_1)$, then the corresponding $F : \mathcal{C} \rightarrow \mathcal{A} \wr_{\gamma} \mathcal{B}$ has a colimit given by*

$$(\text{colim } F_1, (\text{colim } F_{2,j})_{j \in \gamma(\text{colim } F_1)}).$$

Proof. We first define the cocone maps in $\mathcal{A} \wr_{\gamma} \mathcal{B}$

$$(F_1(c), (F_2(c, i))_{i \in \gamma_1(c)}) \rightarrow (\text{colim } F_1, (\text{colim } F_{2,j})_{j \in \gamma(\text{colim } F_1)})$$

as

$$(\iota_c, (\iota_{F_{2,j}(c,i,k)} : F_{2,j}(c, i, k) \rightarrow \text{colim } F_{2,j})_{k \in \gamma(\iota_c)_{i,j}}),$$

which commute with the maps in \mathcal{C} as ι_c and $\iota_{F_{2,j}(c,i,j)}$ commute with the maps in \mathcal{C} and $(\mathcal{F}\gamma_1)_j$ respectively.

Next, observe that a morphism

$$f : (\text{colim } F_1, (\text{colim } F_{2,j})_{j \in \gamma(\text{colim } F_1)}) \rightarrow (a, (b_{i'})_{i' \in \gamma(a)})$$

amounts to maps

$$f_0 : \text{colim } F_1 \rightarrow a \qquad f_{k'} : \text{colim } F_{2,j} \rightarrow b_{i'}$$

for all $j \in \gamma(\operatorname{colim} F_1)$, $i' \in \gamma(a)$, and $k' \in \gamma(f_0)_{j,i'}$, which in turn amount to maps

$$f_c : F_1(c) \rightarrow a, \quad f_{j,k,k'} : F_{2,j}(c, i, k) \rightarrow b_{i'}$$

for all c in \mathcal{C} , $j \in \gamma(\operatorname{colim} F_1)$, $i' \in \gamma(a)$, (c, i, k) in $(\oint \gamma_1)_j$, $k' \in \gamma(f_0)_{j,i'}$ commuting with the maps in \mathcal{C} and each $(\oint \gamma_1)_j$. These assemble into, for each c in \mathcal{C} , morphisms

$$(f_c, (f_{j,k,k'} : F_{2,j}(c, i, k) \rightarrow b_{i'})_{j \in \gamma(\operatorname{colim} F_1), k \in \gamma(\iota_c)_{i,j}, k' \in \gamma(f_0)_{j,i'}}) : (F_1(c), (F_2(c, i))_{i \in \gamma_1(c)}) \rightarrow (a, (b_{i'})),$$

which is well defined as $f_c = f_0 \circ \iota_c$ and so

$$\gamma(f_c)_{i,i'} = \coprod_{j \in \gamma(\operatorname{colim} F_1)} \gamma(f_0)_{j,i'} \times \gamma(\iota_c)_{i,j}.$$

Therefore, as the corresponding commutativity conditions between these morphisms agree, cocones under F correspond to morphisms out of $(\operatorname{colim} F_1, (\operatorname{colim} F_{2,j})_{j \in \gamma(\operatorname{colim} F_1)})$ in $\mathcal{A} \wr_{\gamma} \mathcal{B}$, exhibiting this as the colimit of F . \square

10 Wreath Products of Cell Shapes and Diagrams

We now describe how to use the generalized wreath product to describe operations on cell shapes and cell diagrams. The motivating idea is to generalize the process by which categories enriched in categories, combining objects built primarily from arrows, form 2-categories which are built primarily from a new higher dimensional cell shape, globular 2-cells.

$$\bullet \longrightarrow \bullet \quad \wr \quad \begin{array}{c} \cdot \\ \downarrow \\ \cdot \end{array} = \begin{array}{c} \bullet \\ \circlearrowleft \downarrow \circlearrowright \\ \bullet \end{array}$$

Globular 2-cells, or more precisely the category \mathbf{G}_2 describing their shape, can be built out of two arrows (which is to say, copies of \mathbf{G}_1) through a process we call *stuffing*. Intuitively, a globular 2-cell is obtained by “stuffing” one arrow into another via a construction given in terms of wreath products, and this construction will allow us to model the cell shapes which make up much more general enriched structures. The key idea, evident from the picture above, is that while the body of the arrow is replaced by an inner arrow of arrows, the source and target vertices remain fixed. In this sense the inner arrow is “stuffed” into the outer one.

We then extend this construction to cell diagrams, define an *external wreath product functor* from the wreath product of two cell diagram categories to the category of cell diagrams over the corresponding stuffed cell shapes, and prove several convenient properties of this functor. This will facilitate in the following section a construction of familial monads on these stuffed cell diagrams out of familial monads on the inner and outer cell diagrams.

10.1 Stuffed Cell Shapes

In the picture above, stuffing the inner arrow into the outer one only affected the top dimensional cell shape, the arrow, leaving unaffected the lower dimensional cells in its boundary. We model this procedure using a wreath product, where the top dimensional cell shapes in \mathcal{C} are treated as having a single slot to be filled by a cell shape from \mathcal{D} .

Definition 10.1. For e an endpoint of \mathcal{C} , we write γ_e for the functor $\mathcal{C} \rightarrow \mathbf{Span}$ sending e to the singleton set, all other objects to the emptyset, and each morphism into e to the unique span from \emptyset to $\{*\}$.

γ_e can be interpreted as assigning a single slot to the object e and no slots to any other objects.

Definition 10.2. For \mathcal{C} a category with endpoint e and \mathcal{D} any category, the *stuffing of \mathcal{D} into \mathcal{C}* is the wreath product category $\mathcal{C} \wr_{\gamma_e} \mathcal{D}$, which we will write as simply $\mathcal{C} \wr_e \mathcal{D}$.

We can unwind this definition to give an explicit description of $\mathcal{C} \wr_e \mathcal{D}$. Its objects are either of the form $(c; ())$ for $c \neq e$ in \mathcal{C} , which we abbreviate as simply c , or $(e; (d))$ for d in \mathcal{D} which we abbreviate as simply d . The morphisms are as follows, for $c, c' \in \text{Ob}(\mathcal{C}) \setminus \{e\}$ and $d, d' \in \text{Ob}(\mathcal{D})$:

- $\text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(c, c') = \text{Hom}_{\mathcal{C}}(c, c')$
- $\text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(d, d') = \text{Hom}_{\mathcal{D}}(d, d')$
- $\text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(d, c) = \emptyset$
- $\text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(c, d) = \text{Hom}_{\mathcal{C}}(c, e)$, and for $p : c \rightarrow e$ in \mathcal{C} we write p_d for the corresponding morphism $c \rightarrow d$ in $\mathcal{C} \wr_e \mathcal{D}$ (unless d is clear from context)

Composition is defined as in \mathcal{C}, \mathcal{D} everywhere appropriate, and for $p : c \rightarrow e$ in \mathcal{C} and $q : d \rightarrow d'$ in \mathcal{D} we have $q \circ p_d = p_{d'}$. In more geometric terms, the faces from \mathcal{C}

of the cell shape $(e; (d))$ are the same for all d , unaffected by composition with the morphisms from \mathcal{D} . This is the precise sense in which the shapes of \mathcal{D} are “stuffed” inside \mathcal{C} , without affecting the boundary of the cell shape e , and is reminiscent of the relations in the categories \mathbf{G}_n .

Example 10.3. The example motivating this definition is when $\mathcal{C} = \mathbf{G}_n$ and $\mathcal{D} = \mathbf{G}_m$, with e the unique endpoint object n of \mathbf{G}_n . $\mathbf{G}_n \wr_n \mathbf{G}_m$ is precisely the category \mathbf{G}_{n+m} , encoding that an n -cell stuffed with an m -cell is an $(n+m)$ -cell. The equations $s \circ s = t \circ s$ and $s \circ t = t \circ t$ for all s and t morphisms in \mathbf{G}_n and \mathbf{G}_m follow from the description of composition above. These equations express that the composition of an “outer” morphism in the wreath product followed by an “inner” morphism simply ignores the content of the inner morphism. Geometrically, it encodes that the outer faces of a cell shape are unaffected by the inner faces, just as the source and target vertices of a 2-cell are also the source and target vertices of both its 1-cell faces.

Example 10.4. $\mathbf{G}_1 \wr (\mathbf{G}_1 \times \mathbf{G}_1)$ describes a square stuffed inside an arrow, which resembles a more square-ish version of a lemon with 4 arrows, 4 globular faces connecting them in a square formation with fixed source and target vertices, and a 3-dimensional cell filling in those faces. In other words, the unreduced suspension of a square.

Example 10.5. $(\mathbf{G}_1 \times \mathbf{G}_1) \wr \mathbf{G}_1$ describes an arrow stuffed inside a square, as depicted below.



Such a cell consists of two squares with shared boundary (as in the pinched boundary above) and a 3-dimensional “arrow” stuffed between them, which can be interpreted as various sorts of fillings.

In some cases of interest (see Example 10.7 below) the category \mathcal{C} has multiple endpoint objects (e_1, e_2, \dots) . After stuffing \mathcal{C} with \mathcal{D} at e_1 , the objects e_2, e_3, \dots remain endpoints and so $\mathcal{C} \wr_{e_1} \mathcal{D}$ can be stuffed again with \mathcal{D} or any other category at e_2 , and so on and so forth. However for convenience, we extend our notation to allow for stuffing an entire set of endpoint objects with the same category \mathcal{D} all at once.

Definition 10.6. Let $e = \{e_k\}$ be a set of endpoint objects in a category \mathcal{C} . We define the functor $\gamma_e : \mathcal{C} \rightarrow \mathbf{Span}$ as sending each e_k to the singleton set and all other objects to the empty set (the action on morphisms is then uniquely determined). In this case $\mathcal{C} \wr_e \mathcal{D}$ denotes the wreath product of \mathcal{C} and \mathcal{D} with respect to this γ_e .

In this case, we again write c for the object $(c; ())$ of $\mathcal{C} \wr_e \mathcal{D}$ when $c \in \text{Ob}(\mathcal{C})$ is not in e , and we write d^k for the objects $(e_k; (d))$ when $d \in \text{Ob}(\mathcal{D})$.

Example 10.7. For the category \mathbf{M} with endpoint objects $\{(n, 1)\}$ for all n , the wreath product $\mathbf{M} \wr_e \mathbf{G}_1$ has objects $0, 0^{(n,1)}, 1^{(n,1)}$ for all n . The cell shape 0 is as before just a point, $0^{(n,1)}$ is a point stuffed into an n -to-1 arrow, which is just an n -to-1 arrow, and $1^{(n,1)}$ is an arrow stuffed into an n -to-1 arrow, which resembles a globular 2-cell with n source vertices rather than 1.

10.2 Diagrams of Stuffed Cells

We now turn to studying cell diagrams over the stuffed cell shapes $\mathcal{C} \wr_e \mathcal{D}$, for e a set of endpoints. $\mathcal{C} \wr_e \mathcal{D}$ is related to \mathcal{C} and \mathcal{D} via the functors

$$\mathcal{D} \xrightarrow{\rho_k} \mathcal{C} \wr_e \mathcal{D} \xrightarrow{\lambda} \mathcal{C},$$

where ρ_k is fully faithful sending d to d^k , and λ sends c to c and d^k to e_k .

Given a diagram X over $\mathcal{C} \wr_e \mathcal{D}$, we can extract from it diagrams over \mathcal{C} and \mathcal{D} . First, each $\rho_k : \mathcal{D} \rightarrow \mathcal{C} \wr_e \mathcal{D}$ gives rise to the restriction functor $\rho_k^* : \widehat{\mathcal{C} \wr_e \mathcal{D}} \rightarrow \widehat{\mathcal{D}}$, which sends a diagram X to the cell diagram over \mathcal{D} with d -cells given by the d^k -cells of X (and the corresponding structure maps). On the other hand, $\lambda : \mathcal{C} \wr_e \mathcal{D} \rightarrow \mathcal{C}$ left Kan extends to $\lambda_! : \widehat{\mathcal{C} \wr_e \mathcal{D}} \rightarrow \widehat{\mathcal{C}}$.

Example 10.8. For $\mathbf{M} \wr_e \mathbf{G}_1$ described above, $\lambda : \mathbf{M} \wr_e \mathbf{G}_1 \rightarrow \mathbf{M}$ sends 0 to 0 and both $0^{(n,1)}, 1^{(n,1)}$ to $(n, 1)$, while $\rho_{n,1} : \mathbf{G}_1 \rightarrow \mathbf{M} \wr_e \mathbf{G}_1$ sends 0 and 1 to $0^{(n,1)}, 1^{(n,1)}$ respectively. Given a cell diagram X over $\mathbf{M} \wr_e \mathbf{G}_1$, $\rho_{n,1}X$ is the graph whose vertices are the n -to-1 arrows of X and whose edges are the stuffed arrows between them, while $\lambda_!X$ is the diagram over \mathbf{M} with the same vertices and whose n -to-1 arrows are the connected components of n -to-1 arrows in X with respect to the stuffed n -to-1 arrows between them. When X only has 1-to-1 arrows it resembles a 2-globular set, and $\lambda_!X$ is the truncation of X to the graph whose set of edges is that of X quotiented out by the 2-cells connecting them.

Recall that a diagram in $\widehat{\mathcal{D}}$ is connected if any pair of cells is related by a zigzag of structure maps.

Lemma 10.9. *For X a diagram over $\mathcal{C} \wr_e \mathcal{D}$, $\lambda_!(X)$ has c -cells given by those of X and e_k -cells given by the set of connected components of $\rho_k(X)$.*

Proof. For each c in $\mathcal{C} \setminus e$, as λ^* , $\lambda_!$ both preserve colimits which are computed componentwise, we have

$$X_c \cong \operatorname{colim}_{y(a) \rightarrow X} y(a)_c \rightarrow \operatorname{colim}_{y(a) \rightarrow X} \lambda^* \lambda_!(y(a))_c \cong \lambda^* \lambda_!(X)_c = \lambda_!(X)_c$$

where the map

$$y(a)_c \rightarrow \lambda^* \lambda_!(y(a))_c = \lambda^*(y(\lambda(a)))_c = y(\lambda(a))_c$$

is given by $\operatorname{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(c, a) \rightarrow \operatorname{Hom}_{\mathcal{C}}(c, \lambda(a))$. But this map is always an isomorphism, as $\operatorname{Hom}(c, a)$ is defined to agree with $\operatorname{Hom}(c, \lambda(a))$ for all a , so $X_c \rightarrow \lambda_!(X)_c$ is also an isomorphism.

The e_k -cells of $\lambda_!(X)$ are given by

$$\lambda_!(X)_{e_k} \cong \operatorname{colim}_{y(a) \rightarrow X} y(\lambda(a))_{e_k}.$$

As e_k is an endpoint object, $y(\lambda(a))_{e_k}$ is empty if $a \neq d^k$ and singleton if $a = d^k$ for some d in \mathcal{D} . This colimit is therefore given by the set of connected components in the fiber over e_k of $f X \rightarrow \mathcal{C} \wr_e \mathcal{D} \rightarrow \mathcal{C}$, which are precisely the connected components of $\rho_k(X)$. □

We now proceed to show how a diagram X over $\mathcal{C} \wr_e \mathcal{D}$ is determined by $\lambda_!(X)$ and each $\rho_k(X)$.

10.3 External Wreath Product

The definition of $\mathcal{C} \wr_e \mathcal{D}$ describes the cell shapes in \mathcal{D} isolated inside the boundary of the cell shape e_k in \mathcal{C} . This isolation allows diagrams over $\mathcal{C} \wr_e \mathcal{D}$ to be characterized by the outer diagram over \mathcal{C} with separate diagrams over \mathcal{D} inserted into each e -cell (that is, each e_k -cell for all k). This characterization is mediated by an *external wreath product* functor

$$\wr : \widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C} \wr_e \mathcal{D}},$$

where $\widehat{\gamma}_e : \widehat{\mathcal{C}} \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$ sends X to $\sqcup_k X_{e_k}$, which we will sometimes abbreviate as X_e . An object of $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}$ then consists of a diagram X in $\widehat{\mathcal{C}}$ and for each $x \in X_e$ a diagram Y_x in $\widehat{\mathcal{D}}$.

Definition 10.10. Define $X \wr (Y_x)$ in $\widehat{\mathcal{C} \wr_e \mathcal{D}}$ as the diagram Z with $Z_c = X_c$ for c in $\mathcal{C} \setminus e$, and $Z_{d^k} = \coprod_{x \in X_{e_k}} (Y_x)_d$. On morphisms, Z restricts to X over $\mathcal{C} \setminus e$, and restricts to $\coprod_{x \in X_{e_k}} Y_x$ on the objects d^k . For $p : c \rightarrow e_k$, write $p_d : c \rightarrow d^k$ for the corresponding morphism to d^k in $\mathcal{C} \wr_e \mathcal{D}$. Then Z_{p_d} sends $(x, y) \in Z_{d^k}$ to $X_p(x)$. Functoriality of Z follows from functoriality of X and each Y_x , using the definition of $\mathcal{C} \wr_e \mathcal{D}$.

On maps, a morphism $(X, (Y_x)) \rightarrow (X', (Y'_{x'}))$ in $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}$ amounts to the tuple of $\phi_0 : X \rightarrow X'$ with $\phi_x : Y_x \rightarrow Y'_{\phi_0(x)}$ for all e_k and x in X_{e_k} , which suffices to define the map $\phi_0 \wr (\phi_x) : X \wr (Y_x) \rightarrow X' \wr (Y'_{x'})$.

Example 10.11. When \mathcal{C} and \mathcal{D} are both \mathbf{G}_1 , $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}$ is the category whose objects are graphs X with each edge x labelled by another graph Y_x . $X \wr (Y_x)$ is then the

2-globular set with the same vertices as X and for each edge x in X , a Y_x -shaped graph of edges and 2-cells between the source and target vertices of x .

Lemma 10.12. *When X in $\widehat{\mathcal{C}}$ has no e -cells, $X \sqsupset (\) \cong \lambda^*(X)$ naturally in X .*

Proof. Both have, by definition, c -cells X_c and no d^k -cells, with the same structure maps as X . □

Proposition 10.13. *The action of \sqsupset on morphisms*

$$\sqsupset : \text{Hom}_{\widehat{\mathcal{C}}\widehat{\lambda}_{\widehat{\rho}_e}\widehat{\mathcal{D}}}((X, (Y_x)), (X', Y'_{x'})) \rightarrow \text{Hom}_{\widehat{\mathcal{C}}\widehat{\lambda}_e\widehat{\mathcal{D}}}(X \sqsupset (Y_x), X' \sqsupset (Y'_{x'}))$$

is always injective, and is surjective if either Y_x is connected for all $x \in X_{e_k}$, or no distinct $x'_0, x'_1 \in X'_{e_k}$ share a boundary.

Proof. Injectivity follows immediately, as if parallel maps ϕ_0, ϕ'_0 disagree away from X_{e_k} then $\phi_0 \sqsupset (\phi_x), \phi'_0 \sqsupset (\phi'_x)$ disagree on the same component, if ϕ_0, ϕ'_0 disagree on X_{e_k} then $\phi_0 \sqsupset (\phi_x), \phi'_0 \sqsupset (\phi'_x)$ disagree on some Y_x , and if any ϕ_x, ϕ'_x disagree, then $\phi_0 \sqsupset (\phi_x), \phi'_0 \sqsupset (\phi'_x)$ disagree on Y_x in $X \sqsupset (Y_x)$.

For surjectivity, we consider a map $\psi : X \sqsupset (Y_x) \rightarrow X' \sqsupset (Y'_{x'})$ in $\widehat{\mathcal{C}}\widehat{\lambda}_e\widehat{\mathcal{D}}$ and construct $(\phi_0, (\phi_x))$ in $\widehat{\mathcal{C}}\widehat{\lambda}_{\rho_e}\widehat{\mathcal{D}}$ such that $\psi = \phi_0 \sqsupset (\phi_x)$. The maps $\psi_{d^k} : \coprod_{x \in X_{e_k}} (Y_x)_d \rightarrow \coprod_{x' \in X'_{e_k}} (Y'_{x'})_d$ assemble into a map $\psi_k : \coprod_{x \in X_{e_k}} Y_x \rightarrow \coprod_{x' \in X'_{e_k}} Y'_{x'}$.

By naturality ϕ_k sends any two cells in Y_x related by a structure map to the same $Y'_{x'}$, so if Y_x is connected, ψ_k restricts to a map $\phi_x : Y_x \rightarrow Y'_{x'}$ for some x' . Similarly any two cells in Y_x share a boundary in X and must be sent to cells in $X' \sqsupset (Y'_{x'})$ that

share a boundary in X' , so if any two such $x' \in X'_{e_k}$ agree, then again ψ_k restricts to a map $\phi_x : Y_x \rightarrow Y'_{x'}$ for some x' . In either case then, we can define $(\phi_0)_{e_k}(x)$ as this particular x' . We then set $(\phi_0)_c = \psi_c$ for all other c in \mathcal{C} , which is natural in all maps $c \rightarrow e_k$ in \mathcal{C} by naturality of ψ , so the tuple $(\phi_0, (\phi_x))$ forms a map in $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}$ with $\phi_0 \sqcap (\phi_x) = \psi$. \square

Definition 10.14. Write $\widehat{\mathcal{D}}_{con}$ for the full subcategory of connected diagrams in $\widehat{\mathcal{D}}$.

Corollary 10.15. *The restriction of \sqcap to $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con} \rightarrow \widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$ is an equivalence.*

Proof. By the above proposition, this restriction of \sqcap is fully faithful. For essential surjectivity, consider a diagram Z in $\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$. Let $X = \lambda_!(Z)$ in $\widehat{\mathcal{C}}$ and for each $x \in X_e$, let Y_x in $\widehat{\mathcal{D}}_{con}$ be the corresponding connected component of $\rho_k^*(Z)$, as described in Lemma 10.9. Then by definition of \sqcap , $X \sqcap (Y_x)$ is isomorphic to Z . \square

Example 10.16. When \mathcal{C} and \mathcal{D} are both \mathbf{G}_1 , this equivalence shows that 2-globular sets X are uniquely determined by their 1-truncated graph $\lambda_!X$, with the same vertices and edges given by the connected components of edges between each fixed pair of vertices, along with the connected graphs Y_x which form those connected components, whose vertices and edges are edges and 2-cells in X (these are the connected components of ρ^*X).

Similarly \sqcap restricts to an equivalence on the full subcategory of $\widehat{\mathcal{C}}$ on diagrams whose e_k -cells are determined by their boundaries. However, for an alternative characterization of $\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$ that does not require any restrictions on the diagrams, we

can use a different wreath product where diagrams over \mathcal{D} are assigned to “ e -cell positions” in a diagram over $\mathcal{C}\setminus e$ rather than e -cells in a diagram over \mathcal{C} .

Let $\widehat{\gamma}_e^\partial : \widehat{\mathcal{C}\setminus e} \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$ send X to

$$\coprod_k \mathrm{Hom}_{\widehat{\mathcal{C}\setminus e}}(\partial y(e_k), X).$$

Each element of $\widehat{\gamma}_e^\partial(X)$ is a potential boundary of an e_k -cell in a diagram over \mathcal{C} restricting to X , or of a d^k -cell in a diagram over $\mathcal{C} \wr \mathcal{D}$ restricting to X . We can similarly define an external wreath product

$$\square : \widehat{\mathcal{C}\setminus e} \wr_{\widehat{\gamma}_e^\partial} \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C} \wr \mathcal{D}}$$

sending $(X, (Y_\alpha)_{\alpha: \partial y(e_k) \rightarrow X})$ to the diagram over $\mathcal{C} \wr \mathcal{D}$ restricting to X on $\mathcal{C}\setminus e$ and

$$\coprod_{\alpha: \partial y(e_k) \rightarrow X} Y_\alpha$$

on \mathcal{D} along ρ_k with the $\mathcal{C}\setminus e$ -boundary of all Y_α given by α .

Proposition 10.17. $\square : \widehat{\mathcal{C}\setminus e} \wr_{\widehat{\gamma}_e^\partial} \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C} \wr \mathcal{D}}$ is an equivalence of categories.

Proof. This external wreath product factors as

$$\widehat{\mathcal{C}\setminus e} \wr_{\widehat{\gamma}_e^\partial} \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{\mathrm{con}} \xrightarrow{\square} \widehat{\mathcal{C} \wr \mathcal{D}},$$

where the first functor sends $(X, (Y_\alpha))$ to $(X', (Y_{\alpha,x}))$, with X' restricting to X on $\mathcal{C}\setminus e$ and having for each $\alpha : \partial y(e_k) \rightarrow X$ an e_k -cell x with boundary α for each connected component $Y_{\alpha,x}$ of Y_α . The second functor is an equivalence by Corollary 10.15, and the first is an equivalence with inverse functor sending $(X', (Y_x))$ to

$$(\partial_e X', (\coprod_{x \in X_{e_k}, \partial x = \alpha} Y_x)_{\alpha: \partial y(e_k) \rightarrow X'}),$$

hence the composite is an equivalence. \square

Example 10.18. This equivalent characterization of 2-globular sets shows that a 2-globular set is determined by its set of vertices (which is in $\widehat{\mathcal{C}\setminus e}$) and for each ordered pair of vertices α , a graph Y_α of the edges between those two vertices and the 2-cells between them.

10.4 Properties of the External Wreath Product

The following lemmas show that \boxtimes preserves terminal objects, representables, and colimits in the appropriate senses.

Lemma 10.19. $* \boxtimes (*)_k \cong *$, where each terminal object $*$ is in the appropriate diagram category and $(*)_k$ indicates the constant tuple of the terminal diagram in $\widehat{\mathcal{D}}$ for each chosen endpoint e_k in \mathcal{C} .

Proof. By definition, $* \boxtimes (*)_k \cong *$ has a single a -cell for each a in $\mathcal{C} \wr_e \mathcal{D}$. \square

Lemma 10.20. $y(c) \boxtimes () \cong y(c)$ in $\widehat{\mathcal{C}\setminus e} \wr_e \mathcal{D}$, naturally with respect to c in $\mathcal{C}\setminus e$.

Proof. First note that $\widehat{\gamma}_e$ sends $y(c)$ to the empty set, so $y(c) \boxtimes ()$ is well defined. By definition the two naturally agree on c' -cells for all c' in $\mathcal{C}\setminus e$, and as each e_k is an endpoint neither diagram has any d^k -cells. \square

Lemma 10.21. $y(e_k) \boxtimes (y(d)) \cong y(d^k)$ in $\widehat{\mathcal{C}\setminus e} \wr_e \mathcal{D}$, naturally in d .

Proof. As e_k is an endpoint object, $y(e_k)$ has only a single e_i -cell, so $\widehat{\gamma}_e$ sends $y(e_k)$ to a singleton set and $y(e_k) \sqcup (y(d))$ is well defined. For c in $\mathcal{C} \setminus e$,

$$(y(e_k) \sqcup (y(d)))_c = y(e_k)_c = \text{Hom}_{\mathcal{C}}(c, e_k) = \text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}(c, d^k) = y(d^k)_c$$

and for d' in \mathcal{D} ,

$$(y(e_k) \sqcup (y(d)))_{(d')^k} = y(d)_{d'} = \text{Hom}_{\mathcal{D}}(d', d) = \text{Hom}_{\mathcal{C} \wr_e \mathcal{D}}((d')^k, d^k) = y(d^k)_{(d')^k}$$

Naturality in d follows from the same naturality of y and the definition of composition in $\mathcal{C} \wr_e \mathcal{D}$. \square

We now describe how \sqcup interacts with certain colimits using Proposition 9.27 and Proposition 9.29. While \sqcup is an equivalence on $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con}$ and therefore preserves colimits, it will be useful to be able to decompose functors into $\widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con}$ using functors into $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}_{con}$ respectively in a colimit-preserving way.

Lemma 10.22. *Consider diagrams X in $\widehat{\mathcal{C}}$ and Y_x in $\widehat{\mathcal{D}}$ for each $x \in X_e$. Given functors $F_0 : \int X \rightarrow \widehat{\mathcal{C}}$ and $F_{x,x'} : \int Y_x \rightarrow \widehat{\mathcal{D}}_{con}$ for all $x \in X_e$, $x' \in F_0(x)_e$ such that $F_0(z)_e$ is empty for all $z \in X_c$, c in $\mathcal{C} \setminus e$, there is a functor $F_0 \sqcup (F_{x,x'}) : \int (X \sqcup (Y_x)) \rightarrow \widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$, which we write as simply F , where for $z \in X_c$, $F(z) = F_0(z) \sqcup ()$ and for $y \in (Y_x)_d$,*

$$F(x, y) = F_0(x) \sqcup (F_{x,x'}(y)).$$

Proof. Let $F_1 : \int (X \sqcup (Y_x)) \rightarrow \widehat{\mathcal{C}}$ send $z \in X_c$ to $F_0(z)$ and $(x, y) \in (X \sqcup (Y_x))_{d^k}$ to $F_0(x)$, with all morphisms sent according to F_0 or identities on $F_0(x)$ as appropriate.

$\gamma_1 : f(X \sqcup (Y_x)) \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$ sends z to \emptyset and (x, y) to $\widehat{\gamma}_e(F_0(x)) = F_0(x)_e$, from which it can be deduced that $\oint \gamma_1$ is the category

$$\coprod_{x \in X_e, x' \in F_0(x)_e} f Y_x.$$

Therefore a functor $F_2 : \oint \gamma_1 \rightarrow \widehat{\mathcal{D}}$ is determined by the hypothesized functors $F_{x,x'}$, and by *Proposition 9.27* this data determines a functor $f(X \sqcup (Y_x)) \rightarrow \widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con}$. Composing this functor with $\sqcup : \widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con} \rightarrow \widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$ completes the construction of the functor F , and it is easily checked that the resulting functor agrees with the definition of F on objects given above. \square

Proposition 10.23. *Given diagrams X in $\widehat{\mathcal{C}}$ and Y_x in $\widehat{\mathcal{D}}$ for each $x \in X_e$ and functors $F_0 : f X \rightarrow \widehat{\mathcal{C}}$, $F_{x,x'} : f Y_x \rightarrow \widehat{\mathcal{D}}_{con}$ as above,*

$$colim(F_0 \sqcup (F_{x,x'})) \cong colim(F_0) \sqcup (colim(F_{x,x'})).$$

The right hand side of this isomorphism is well defined as, under the assumption that $F_0(z)_e$ is empty, $\widehat{\gamma}_e(colim(F_0)) = \coprod_{x \in X_e} F_0(x)_e$.

Proof. With $\gamma_1 : f(X \sqcup (Y_x)) \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$ sending z to \emptyset and (x, y) to $\widehat{\gamma}_e(F_0(x)) = F_0(x)_e$, observe that for each $(x, x') \in colim(F_0)_e \cong \widehat{\gamma}_e(colim(F_0))$, $(\oint \gamma_1)_{x,x'} \cong f Y_x$, so by *Proposition 9.29* the colimit of the functor $f(X \sqcup (Y_x)) \rightarrow \widehat{\mathcal{C}} \wr_{\widehat{\gamma}_e} \widehat{\mathcal{D}}_{con}$ defined above from $F_0, (F_{x,x'})$ is precisely $(colim(F_0), (colim(F_{x,x'})))$. This completes the proof, as \sqcup is an equivalence and therefore preserves colimits. \square

11 Familial Monads for Enriched Algebras

We now define an external wreath product functor on categories of familial representations. We then show that a slight restriction of this functor is lax monoidal, which implies that suitable familial monads T on $\widehat{\mathcal{C}}$ and T' on $\widehat{\mathcal{D}}$ induce a familial monad on $\widehat{\mathcal{C} \wr_e \mathcal{D}}$. We then show that algebras for this monad agree with T -algebras enriched in the symmetric monoidal category of T' -algebras, and under quite general conditions its theory category is the wreath product of Θ_T and $\Theta_{T'}$.

11.1 Monoids in $\mathbf{Rep}_{\mathcal{C}} \wr \mathbf{Rep}_{\mathcal{D}}$

For e a collection of endpoints in \mathcal{C} , we have a functor²

$$\sigma_e : \mathbf{Rep}_{\mathcal{C}} \rightarrow \mathbf{Set} \rightarrow \mathbf{Span}$$

$$(S, E) \mapsto \coprod_{t \in Se} Et_e.$$

This lets us define the wreath product category $\mathbf{Rep}_{\mathcal{C}} \wr_{\sigma_e} \mathbf{Rep}_{\mathcal{D}}$. This category carries a monoidal structure induced by the monoidal structures on $\mathbf{Rep}_{\mathcal{C}}$ and $\mathbf{Rep}_{\mathcal{D}}$ using Proposition 9.21 as follows. $\sigma_e : \mathbf{Rep}_{\mathcal{C}} \rightarrow \mathbf{Span}$ is colax-monoidal in the double categorical sense via the functions

$$\sigma_e(SS', EE') = \{t \in Se, f : Et \rightarrow S', x \in Et_e, x' \in E'f(x)_e\} \rightarrow \sigma(S, E) \times \sigma(S', E')$$

$$(t, f, x, x') \mapsto ((t, x), (f(x), x'))$$

²Note that if the morphisms of representations were not cartesian, this functor would not factor through \mathbf{Set} .

which determine the colax structure on morphisms as σ_e factors through **Set**.

The unit is $((S^0, E^0), ((S^0, E^0))_k)$ made up of the units of **Rep \mathcal{C}** and **Rep \mathcal{D}** . $S^0 e_k$ and $E^0(*_{e_k})_e$ are all singleton, so σ_e sends (S^0, E^0) to the set e and thus $((S^0, E^0), ((S^0, E^0))_k)$ is well defined in **Rep \mathcal{C}** λ_{σ_e} **Rep \mathcal{D}** .

The product is given by

$$\begin{aligned} & ((S, E), ((S_x, E_x))_{t \in S_e, x \in Et_e}) ((S', E'), ((S'_{x'}, E'_{x'})_{t' \in S'_e, x' \in E't'_e}) := \\ & ((SS', EE'), ((S_x S'_{x'}, E_x E'_{x'})_{t \in S_e, f: Et \rightarrow S', x \in Et_e, x' \in E'f(x)_e})) \end{aligned}$$

noting that

$$EE'(t, f)_e = (\text{colim}_{x \in Et} E'f(x))_e \cong \coprod_{x \in Et_e} E'f(x)_e$$

by Corollary 5.13.

A monoid in this monoidal category amounts to a monoid structure (η, μ) on the representation (S, E) over \mathcal{C} , unit maps $(S^0, E^0) \rightarrow (S_x, E_x)$ on the representations over \mathcal{D} , and suitably compatible structure maps of the form

$$(S_x S_{x'}, E_x E_{x'}) \rightarrow (S_{\mu^E(t, f)(x, x')}, E_{\mu^E(t, f)(x, x')}).$$

Such a system of structured familial representations is generally complicated to specify, but we will only consider the case in which all of the representations (S_x, E_x) over \mathcal{D} are the same, and this representation carries a monoid structure in **Rep \mathcal{D}** which provides for all of the necessary structure maps. In other words, given familial monads represented by (S, E) over \mathcal{C} and (S', E') over \mathcal{D} , $((S, E), ((S', E'))_x)$ is a monoid in **Rep \mathcal{C}** λ_{σ_e} **Rep \mathcal{D}** .

Example 11.1. Let $\mathcal{C} = \mathbf{G}_1 \times \mathbf{G}_1$ and $\mathcal{D} = \mathbf{G}_1$. Letting (S, E) represent the free double category monad and $(S_x, E_x) = (S', E')$ represent the free category monad, $((S, E), ((S', E'))_x)$ is a monoid in $\mathbf{Rep}_{\mathcal{C}} \wr_{\sigma_e} \mathbf{Rep}_{\mathcal{D}}$. The monoidal structure maps described above arise as follows. Let:

- $t \in S\Box$ be a grid
- $f : Et \rightarrow S$ describe a compatible choice of grids $f(x)$ to plug into each square x in Et
- $t_x \in S_x 1$ be a string of arrows for each x
- $f_{x,x'} : Et_x \rightarrow S_{x'}$ describe a string of arrows $f_{x,x'}(x'')$ for each arrow x'' in Et_x

$\mu(t, f)$ is the grid with $Ef(x)$ plugged into each square x of Et , and $\mu^E(t, f)(x, x')$ is the square in $E\mu(t, f)$ corresponding to the square x' in the subgrid $f(x)$. The idea is that when the grid t has each square x stuffed with the string of arrows t_x , the squares x have new grids $f(x)$ plugged in, the string Et_x must have strings $f_{x,x'}(x'')$ plugged in separately at each square x' in $Ef(x)$ in order to imitate plugging operations over $(\mathbf{G}_1 \times \mathbf{G}_1) \wr \mathbf{G}_1$ into the stuffed square cells of $Et \boxtimes (E_x t_x)$. The morphism of representations from $(S_x S_{x'}, E_x E_{x'})$ to $(S_{\mu^E(t,f)(x,x')}, E_{\mu^E(t,f)(x,x')})$ then describes how to compose these strings of arrows (here by the usual concatenation) to get the appropriate string to plug into the square $\mu^E(t, f)(x, x')$ of $E\mu(t, f)$ when describing the composite of this total arrangement.

This compatibility of the structure maps described here with composition of operations over $(\mathbf{G}_1 \times \mathbf{G}_1) \wr \mathbf{G}_1$ is made precise in Proposition 11.3.

11.2 External Wreath Product of Familial Representations

We can now extend the external wreath product \boxtimes to familial representations:

$$\boxtimes : \mathbf{Rep}_{\mathcal{C}} \wr_{\sigma_e} \mathbf{Rep}_{\mathcal{D}} \rightarrow \mathbf{Rep}_{\mathcal{C} \wr_{\gamma} \mathcal{D}}$$

$$(S, E) \boxtimes ((S_x, E_x))_{t \in S_e, x \in Et_e} = (S \wr (S_x), E \wr (E_x)), \text{ where}$$

$$S \wr (S_x) := S \boxtimes \left(\prod_{x \in Et_e} S_x \right)_{t \in S_e} \text{ and}$$

$$E \wr (E_x) : (t, (t_x)_{x \in Et_e}) \mapsto Et \boxtimes (E_x t_x)_{x \in Et_e}$$

We will assume throughout that all representations on \mathcal{C} are e -graded (Definition 5.11) and all representations on \mathcal{D} have connected arities.

Example 11.2. In the example of (S, E) the free double category monad and S_x the free category monad for all x , $S \wr (S_x)$ has the same operations as S for the \cdot , \rightsquigarrow , and $\circ \rightarrow$ cell shapes. As $S_x 0$ has only the unit operation on vertices, $(S \wr (S_x)) 0$ has operations given by grids (as in, the same as $S \square$). The operations in $(S \wr (S_x)) 1$ are given by grids t as in $S \square$ with a string of arrows from $S_x 1$ for each square x in the grid Et . The arities of these are given by $Et \boxtimes (E_x t_x)_x$ which stuffs the string $E_x t_x$ into the square x of Et .



We now proceed to exhibit \square as a lax monoidal functor, which shows that when applied to a monad representation it yields a monad representation. \square can therefore construct familial monads denoted $T \wr_e T'$ on $\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$ from familial monads T on $\widehat{\mathcal{C}}$ and T' on $\widehat{\mathcal{D}}$, and we show in Section 11.4 that algebras for this monad agree with T -algebras enriched in T' -algebras.

Proposition 11.3. $\square : \mathbf{Rep}_{\mathcal{C} \wr_e \sigma_e} \mathbf{Rep}_{\mathcal{D}} \rightarrow \mathbf{Rep}_{\mathcal{C} \wr_e \mathcal{D}}$ is a lax monoidal functor when restricted to e -graded representations on \mathcal{C} (Definition 5.11) and representations on \mathcal{D} with connected arities.

\square therefore preserves monoids when the \mathcal{C} -part is e -graded and the \mathcal{D} -parts have connected arities.

Proof. We first show that \square preserves units, as in $(S^0, E^0) \square ((S^0, E^0)) \cong (S^0, E^0)$, each identity representation considered over the appropriate category. By Lemma 10.19, Lemma 10.20, and Lemma 10.21, $S^0 \square (S^0) \cong S^0$ as terminal diagrams, $E^0(*_{e_k}) \square (E^0(*_d)) \cong E^0(*_{d^k})$ as representables, $E^0(*_c) \square () \cong E^0(*_c)$ for c in $\mathcal{C} \setminus e$, and these isomorphisms are appropriately natural.

Next we construct the lax structure maps

$$\begin{aligned} & ((S, E) \square ((S_x, Et_x))_{t \in S_e, x \in Et_e}) ((S', E') \square ((S'_{x'}, E't'_{x'}))_{t' \in S'_e, x' \in E't'_e}) \\ & \rightarrow ((SS', EE') \square ((S_x S'_{x'}, E_x E'_{x'}))_{t \in S_e, f: Et \rightarrow S', x \in Et_e, x' \in E'f(x)_e}). \end{aligned}$$

On the c -operations, c in $\mathcal{C} \setminus e$, this map is the identity, as the c -operations on both sides are those of $SS'c$, and the isomorphism on arities for $(t, f) \in SS'c$ are

given by

$$EE'(t, f) \sqcup () = \left(\operatorname{colim}_{x:y(c') \rightarrow Et} E'f(x) \right) \sqcup () \cong \operatorname{colim}_{x:y(c') \rightarrow Et} (E'f(x) \sqcup ()) \cong \operatorname{colim}_{x:y(a) \rightarrow Et} (E'f(x) \sqcup ()),$$

where by Lemma 10.12 $(-) \sqcup () \cong \lambda^* : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}$ and λ^* preserves colimits and all maps $y(a) \rightarrow Et \sqcup ()$ have $a = c'$ for c' in $\mathcal{C} \setminus e$ as (S, E) is e -graded.

The map $(S \wr (S_x))(S' \wr (S'_{x'})) \rightarrow (SS' \wr (S_x S'_{x'}))$ is defined as follows, with the first isomorphism given by Corollary 10.15 and the second by the definition of morphisms in a wreath product:

$$\begin{aligned} & (S \wr (S_x))(S' \wr (S'_{x'}))d^k = \\ & \prod_{t \in Se_k, (t_x \in S_x d)_{x \in Et_e}} \operatorname{Hom}_{\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}} (Et \sqcup (E_x t_x), S' \sqcup (\prod_{x' \in E' t'_e} S'_{x'})) \\ & \cong \prod_{t \in Se_k, (t_x \in S_x d)_{x \in Et_e}} \operatorname{Hom}_{\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}}} ((Et, (E_x t_x)), (S', (\prod_{x' \in E' t'_e} S'_{x'}))) \\ & \cong \prod_{t \in Se_k, f: Et \rightarrow S'} \prod_{(t_x \in S_x d)_{x \in Et_e}} \prod_{x \in Et_e} \operatorname{Hom}_{\widehat{\mathcal{D}}} (E_x t_x, \prod_{x' \in E' f(x)_e} S'_{x'}) \\ & \cong \prod_{t \in Se_k, f: Et \rightarrow S'} \prod_{(t_x \in S_x d)_{x \in Et_e}} \prod_{x \in Et_e} \prod_{x' \in E' f(x)_e} \operatorname{Hom}_{\widehat{\mathcal{D}}} (E_x t_x, S'_{x'}) \\ & \cong \prod_{t \in Se_k, f: Et \rightarrow S'} \prod_{x \in Et_e} \prod_{t_x \in S_x d} \prod_{x' \in E' f(x)_e} \operatorname{Hom}_{\widehat{\mathcal{D}}} (E_x t_x, S'_{x'}) \\ & \rightarrow \prod_{t \in Se_k, f: Et \rightarrow S'} \prod_{x \in Et_e} \prod_{x' \in E' f(x)_e} \prod_{t_x \in S_x d} \operatorname{Hom}_{\widehat{\mathcal{D}}} (E_x t_x, S'_{x'}) \\ & \cong \prod_{t \in Se_k, f: Et \rightarrow S'} \prod_{x \in Et_e} \prod_{x' \in E' f(x)_e} S_x S'_{x'} d \\ & \cong (SS' \sqcup (\prod_{x, x'} S_x S'_{x'}))_{t, f} d^k \end{aligned}$$

The essence of this map is contained in the single non-invertible component above. Given an operation $t \in Se_k$, a map $f : Et \rightarrow S'$, and a cell $x \in Et_e$, the domain

requires a choice of operation $t_x \in S_x d$ and for each $x' \in E' f(x)_e$, a map $E_x t_x \rightarrow S'_{x'}$ picking out which operations from $S'_{x'}$ will be stuffed into the cells of $E_x t_x$ before being plugged into the cell (x, x') of

$$\left(\operatorname{colim}_{x:y(c) \rightarrow Et} E' f(x) \right)_e \cong \coprod_{x \in Et_e} E' f(x)_e.$$

The codomain allows each $x' \in E' f(x)_e$ to be equipped with a distinct choice of operation $t_x \in S_x d$ rather than each such operation applying over all $x' \in E' f(x)_e$ as in the domain. This is a quirk of the interaction between \square and composition of representations in $\mathbf{Rep}_{\mathcal{C}} \wr_{\sigma_e} \mathbf{Rep}_{\mathcal{D}}$ and not representative of how operations compose in wreath product representations, but this does not cause a problem as the lax structure map can simply pick out the elements in the codomain for which all $x' \in E' f(x)_e$ are equipped with the same operation $t_x \in S_x d$. The fact that this satisfies the conditions of a lax structure map follows from the abstract properties of this map described in Lemma 11.4 below.

Finally, we need for each tuple

$$(t \in Se_k, (t_x \in S_x)_{x \in Et_e}, f : Et \rightarrow S', (f_{x,x'} : E_x t_x \rightarrow S'_{x'})_{x \in Et_e, x' \in E' f(x)_e})$$

an isomorphism

$$\begin{aligned} & \left(\operatorname{colim}_{x:y(c') \rightarrow Et} E' f(x) \right) \square \left(\operatorname{colim}_{y:y(d') \rightarrow E_x t_x} E'_{x'} f_{x,x'}(y) \right)_{x \in Et_e, x' \in E' f(x)_e} \\ & \cong \operatorname{colim}_{z:y(a) \rightarrow Et \sqcup (E_x t_x)} (E' f(x) \square (E'_{x'} f_{x,x'}(y))_{x' \in E' f(x)_e}), \end{aligned}$$

where when $a = d^k z$ is taken to be the pair $(x \in Et_e, y \in (E_x t_x)_d)$. This isomorphism is precisely that of Proposition 10.23, with

$$X = Et, \quad Y_x = E_x t_x, \quad F_0 : \int Et \xrightarrow{f} \int S' \xrightarrow{E'} \widehat{\mathcal{C}}, \quad F_{x,x'} : E_x t_x \xrightarrow{f_{x,x'}} S'_{x'} \xrightarrow{E'_{x'}} \widehat{\mathcal{D}},$$

and it is tedious but straightforward to check that these isomorphisms on arities satisfy the appropriate unit and associativity conditions by the universal property of colimits. \square

Lemma 11.4. *Given sets A, B and for each $a \in A, b \in B$ a set $X_{a,b}$, there is an inclusion*

$$\coprod_{a \in A} \prod_{b \in B} X_{a,b} \rightarrow \prod_{b \in B} \coprod_{a \in A} X_{a,b}$$

sending $(a, (x_{a,b})_{b \in B})$ to $((a, x_{a,b}))_{b \in B}$. Furthermore, for sets

$$A, B, C, D, (X_{a,b,c,d})_{a \in A, b \in B, c \in C, d \in D}$$

the following diagram of the functions above commutes:

$$\begin{array}{ccc} \coprod_{a \in A} \prod_{b \in B} \prod_{c \in C} \prod_{d \in D} X_{a,b,c,d} & \longrightarrow & \prod_{b \in B} \coprod_{a \in A} \prod_{c \in C} \prod_{d \in D} X_{a,b,c,d} \\ \downarrow & & \downarrow \\ \coprod_{a \in A} \prod_{b \in B} \prod_{d \in D} \prod_{c \in C} X_{a,b,c,d} & \longrightarrow & \prod_{b \in B} \coprod_{a \in A} \prod_{d \in D} \prod_{c \in C} X_{a,b,c,d} \end{array}$$

Proof. On both sides of the diagram, an element $(a, ((c_b, (x_{a,b,c,d})_{d \in D}))_{b \in B})$ is sent by the composite to $((a, (c_b, x_{a,b,c,d})_{d \in D}))_{b \in B}$. \square

We can finally state the main result of this section, which follows immediately from Proposition 11.3.

Theorem 11.5. *If T is an e -graded familial monad on $\widehat{\mathcal{C}}$ represented by $(S, E, \eta^{\mathcal{C}}, \mu^{\mathcal{C}})$ and T' is a familial monad on $\widehat{\mathcal{D}}$ with connected arities represented by $(S', E', \eta^{\mathcal{D}}, \mu^{\mathcal{D}})$, then $T \wr_e T'$ inherits the structure of a familial monad on $\widehat{\mathcal{C} \wr_e \mathcal{D}}$ with structure maps η, μ where η, μ agree with $\eta^{\mathcal{C}}$ on c -operations for c in $\mathcal{C} \setminus e$,*

$$\eta(d^k) = (\eta^{\mathcal{C}}(e_k), (\eta^{\mathcal{D}}(d))) \in (S \wr S')d^k,$$

and for $(t \in Se_k, (t_x \in S'd)_{x \in Et_e}) \in (S \wr S')d^k$ and $(f : Et \rightarrow S, (f_{x,x'} : E't_x \rightarrow S')_{x \in Et_e, x' \in Ef(x_e)})$,

$$\mu((t, (t_x)), (f, (f_{x,x'}))) = (\mu^C(t, f), (\mu^D(t_x, f_{x,x'})_{(x,x') \in E\mu(t,f)_e})) \in (S \wr S')d^k.$$

Example 11.6. From the previous example when T is the free double category monad and T' is the free category monad, we get a monad on $(\mathbf{G}_1 \times \widehat{\mathbf{G}_1}) \wr \mathbf{G}_1$ which takes a double-graph-with-fillings and adds in all composites of grids of stuffed squares into a new wide one, as well as all composites of stacked stuffed squares into an extra-stuffed one.



These composites satisfy the *interchange law* that the grid-composite of a grid of stacked-composites of equal-height stacks agrees with the corresponding stack-composite of a stack of grid-composites. This interchange property is discussed further below, after which we show that an algebra for this monad is precisely a double category enriched in **Cat**.

Example 11.7. When T is the free n -category monad and T' is the free m -category monad, $T \wr_e T'$ is easily checked to be the free $(n + m)$ -category monad. Indeed, an n -dimensional pasting diagram with an m -dimensional pasting diagram stuffed into each n -cell is almost precisely the definition of an $(n + m)$ -dimensional pasting diagram, so these two monads have the same operations and arities.

11.3 Algebras and Interchange

As the most notable example of a higher category theory defined via enrichment, the theory of 2-categories offers several convenient properties that generalize to any enriched structures. For instance, in a 2-category the composite of any free globular pasting diagram can be decomposed into a horizontal composition of vertical composites.

For this section we fix an e -graded familial monad T on $\widehat{\mathcal{C}}$ represented by (S, E) and a familial monad T' on $\widehat{\mathcal{D}}$ with connected arities represented by (S', E') .

Lemma 11.8. *The following equation holds in $(S \wr_e S', E \wr_e E')$ for $t \in Se_k$ and $t_x \in S'd$ for all $x \in Et_e$:*

$$(t, (t')_{x \in Et_e}) = \mu((t, (\eta(d))_{x \in Et_e}), \eta \sqcap (t_x))$$

Proof. This follows immediately from Theorem 11.5 and unitality of η, μ . □

This equation warrants some explanation: on the right hand side, the outer operation looks like t with the unit operation on d -cells plugged into each e -cell of Et , and the composite applies this to the unit operation on each e -cell x of Et with t_x plugged into the single e -cell. Hence the map $Et \sqcap y(d) \rightarrow S \wr S'$ denoted $\eta \sqcap t'$ is given by $Et \rightarrow * \xrightarrow{\eta} S$ and repeated maps $y(d) \rightarrow S'$ picking out t_x .

When T, T' are both the free category monad, this composite corresponds to a horizontal composition of vertical compositions in a 2-category. Furthermore, when

each of the vertical composites are of the same number of 2-cells, the *interchange* property holds, which can be generalized as follows.

Lemma 11.9. *The following equation holds in $(S \wr_e S', E \wr_e E')$ for $t \in Se_k$ and $t' \in S'd$:*

$$(t, (t')_{x \in Et_e}) = \mu((t, (\eta(d))_{x \in Et_e}), \eta \boxtimes t') = \mu((\eta(e_k), (t')), t \boxtimes \eta)$$

Proof. This is a straightforward consequence of Lemma 11.8 and Theorem 11.5, using unitality of η and μ . □

The first composite in this equation is the special case of the previous lemma with $t_x = t'$ for all x . For the second composite, the outer operation is the unit on e_k -cells with t' plugged into the single e_k -cell of its arity $y(e_k)$, and the composite applies this to t and the unit operation on each cell from \mathcal{D} . Hence the map $y(e_k) \boxtimes E't' \rightarrow S \wr S'$ denoted $t \boxtimes \eta$ is given by $y(e_k) \rightarrow S$ picking out t and $E't' \rightarrow * \xrightarrow{\eta} S'$.

When both familial monads are the free category monad, the first composite describes a horizontal composition of vertical compositions in a 2-category, while the second describes a vertical composition of horizontal composites. The interchange law for 2-categories, and now $(T \wr_e T')$ -algebras, states that these two must agree. The leftmost operation in the statement of the proposition, $(t, (t'))$, describes the unbiased version of this operation, which does not need to specify whether it is obtained as a horizontal composition of vertical composites or vice versa.

One of the ways in which the interchange law is useful in the theory of 2-categories

is that in order to show that a choice of compositions for pasting diagrams of globular 2-cells assemble into a 2-category, it suffices to check that these compositions satisfy the unitality and associativity equations for both vertical and horizontal composition, along with the interchange equation. This will prove useful for comparing $(T \wr_e T')$ -algebras with T -algebras enriched in T' -algebras.

Proposition 11.10. *Consider a presheaf A in $\widehat{\mathcal{C} \wr_e \mathcal{D}}$ equipped with composition maps*

- $\text{Hom}(Et \sqcup (), A) \rightarrow A_c$ for $t \in Sc$, c in $\mathcal{C} \setminus e$
- $\text{Hom}(Et \sqcup (y(d)), A) \rightarrow A_{d^k}$ for $t \in Se_k$
- $\text{Hom}(y(e_k) \sqcup (t'), A) \rightarrow A_{d^k}$ for $t' \in S'd$

such that the following conditions hold:

- the first set of composition maps endow λ^*A with the structure of a $\partial_e T$ -algebra
- the second set of composition maps satisfy the unit and multiplication equations corresponding to e -operations of T
- the third set of composition maps satisfy the unit and multiplication equations corresponding to the operations of T'
- these compositions satisfy the interchange law, namely that for any map $Et \sqcup (E't') \rightarrow A$, applying the composition maps of the second and third type above in either order, both of which are possible by Lemma 11.9, yield the same result

Then this data suffices to define a $(T \wr_e T')$ -algebra structure on A .

Note that the “unit and multiplication” equations here in the statement of the theorem refer to the unit and multiplication maps of the monad, where the multiplication encodes both the unitality and associativity properties of classical structures such as those in 2-categories mentioned above.

Proof. It suffices to define composition maps $\text{Hom}(Et \sqcup (E't_x)_{x \in Et_e}, A) \rightarrow A_{d^k}$ for all operations $(t, (t_x))$ in $(S \wr S')d^k$ which satisfy the unit and multiplication equations. By Lemma 11.9,

$$Et \sqcup (E't_x)_{x \in Et_e} \cong \text{colim}_{x': y(a) \rightarrow Et \sqcup (y(d))} y(a) \sqcup (E't_{x'}),$$

where by an abuse of notation $(E't_{x'})$ denotes the singleton list $(E'S'i(t_x))$ when $x' : y(d'_k) \rightarrow Et \sqcup (y(d))$ corresponds to $x \in Et_{e_k}$ and $i : d' \rightarrow d$ in $y(d)_{d'}$, and denotes the empty list of presheaves over \mathcal{D} when $x' : y(c) \rightarrow Et \sqcup (y(d))$ for c in $\mathcal{C} \setminus e$.

Therefore, between this isomorphism and the hypothesized composition maps, we have

$$\begin{aligned} \text{Hom}(Et \sqcup (E't_x)_{x \in Et_e}, A) &\cong \lim_{x': y(a) \rightarrow Et \sqcup (y(d))} \text{Hom}(y(a) \sqcup (E't_{x'}), A) \\ &\rightarrow \lim_{x': y(a) \rightarrow Et \sqcup (y(d))} A_a \cong \text{Hom}(Et \sqcup (y(d)), A) \rightarrow A_{d^k}. \end{aligned}$$

Intuitively, these general composition maps first compose the $E't_x$ -diagrams into d -cells within each e -cell of the Et -diagram, then compose the Et -diagram of those d -cells.

As the unit equation for these compositions follows from the hypotheses (both of the maps making up the compositions above being identities up to Yoneda isomorphisms), it remains only to show that they satisfy the multiplication equation. For this we make use of the μ -factorization of operations from Lemma 11.8: any operation $(t, (t_x))$ factors uniquely as $(t, (\eta(d)))$ applied to the operations $(\eta(e_{k'}), (t_x))$ for each $x \in Et_{e_{k'}}$. For this reason, to check that the composition maps constructed above respect composition of operations in $T \wr_e T'$ via μ , it suffices to check this in the cases of $\mu(\bar{t}, \bar{f})$, where $\bar{t} \in (S \wr S')d$ and $\bar{f} : (E \wr E')\bar{t} \rightarrow S \wr S'$ are each concentrated in either \mathcal{C} or \mathcal{D} .

But when both are concentrated in \mathcal{C} , or both in \mathcal{D} , this is covered by the first three conditions in the proposition, and when \bar{t} is concentrated in \mathcal{C} and \bar{f} concentrated in \mathcal{D} this is implicit in the definition of the general composition operations. Hence it remains only to check that these compositions respect μ in the case when \bar{t} is concentrated in \mathcal{D} and \bar{f} is concentrated in \mathcal{C} .

Consider $\bar{t} = (\eta(e_k), (t')) \in (S \wr S')d^k$ and $\bar{f} : y(e_k) \boxtimes (E't') \rightarrow S \wr S'$ made up of $y(e_k) \rightarrow S$ picking out $t \in Se_k$ and $E't' \rightarrow * \xrightarrow{\eta} S'$. Then $\mu(\bar{t}, \bar{f}) = (t, (t'))$ and the composition maps respecting μ in this case is precisely the interchange law in the final condition of the proposition. \square

11.4 Wreath Product Monads Produce Enriched Algebras

We can now show that for familial monads T e -graded over $\widehat{\mathcal{C}}$ and T' with connected arities, the wreath product $T \wr_e T'$ has algebras which agree with T -algebras enriched in T' -algebras. T' -algebras form a cartesian (hence symmetric) monoidal category, and so give rise to a (T, e) -structured category. Throughout this section, we fix such monads T, T' represented by (S, E) and (S', E') respectively.

Lemma 11.11. *The restriction functor $\widehat{\mathcal{C}} \wr_e \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}} \setminus e$ extends to a functor from $(T \wr_e T')$ -algebras to $\partial_e T$ -algebras.*

We will abuse notation somewhat and denote this functor by ∂ .

Proof. This follows immediately from Proposition 5.20 and the observation that $(T \wr_e T')$ restricts to $\partial_e T$ on $\widehat{\mathcal{C}} \setminus e$ by the definition of its operations and arities. \square

Lemma 11.12. *Given a $(T \wr_e T')$ -algebra A and a map $\alpha : \partial y(e_k) \rightarrow \partial A$, the diagram A_α in $\widehat{\mathcal{D}}$ corresponding to α via Proposition 10.17 has the structure of a T' -algebra.*

Proof. By Corollary 5.19, $T \wr_e T'$ restricts along ρ_k to a familial monad T_k on $\widehat{\mathcal{D}}$ represented by (S_k, E_k) with

$$S_k = \coprod_{t \in S e_k} (S')^{E t_e} \quad E(t, (t_x)) = \coprod_{x \in E t_e} E' t_x,$$

and by Proposition 5.20

$$\rho_k^* A = \coprod_{\alpha : \partial y(e_k) \rightarrow \partial A} A_\alpha$$

is an algebra for this monad (it is a coproduct only of presheaves, not algebras; A_α is not generally a T_k -algebra).

In $\mathbf{Rep}_{\mathcal{D}}$, there is a morphism of monoids $(S', E') \rightarrow (S_k, E_k)$ sending $t' \in S'd$ to $(\eta(e_k), (t')) \in S_k d$, where as $E\eta(e_k)_e$ is a singleton the corresponding arities are isomorphic. This induces a functor from T_k -algebras to T' -algebras which is the identity on the underlying objects of $\widehat{\mathcal{D}}$.

For $t' \in S'd$, a map $(E \wr (E'))(\eta(e_k)(t')) = y(e_k) \sqcap (E't') \rightarrow A$ amounts to (by Proposition 10.17) a pair of maps $\alpha : \partial y(e_k) \rightarrow \partial A$ in $\widehat{\mathcal{C}} \setminus e$ and $E't' \rightarrow A_\alpha$ in $\widehat{\mathcal{D}}$, and its composite d^k -cell in A restricts to a d -cell in A_α . This is because the operation $(\eta(e_k), (t'))$ restricts to all unit operations on the cells in $\mathcal{C} \setminus e$, so its composition in A preserves the boundary $\partial_e y(e_k)$ -diagram.

Therefore, $\rho_k^* A$ regarded as a T' -algebra is a coproduct of T' -algebras A_α . \square

Lemma 11.13. *For A as above, each operation $t \in Se_k$ and $\beta : \partial Et \rightarrow \partial A$ induces a T' -algebra morphism*

$$\text{comp}_\beta : \prod_{x \in Et_e} A_{\alpha_x} \rightarrow A_{\alpha_\beta}.$$

Recall the maps $\alpha_x : \partial y(e_{k'}) \rightarrow \partial Et \rightarrow \partial A$ for $x \in Et_{e_{k'}}$ and $\alpha_\beta : \partial y(e_k) \rightarrow \partial A$ from Definition 8.16, where α_β is given by taking the composites in ∂A of the boundary diagrams of ∂Et .

Proof. For $t_x \in S'd$ for all $x \in Et_e$, a map $(E \wr (E'))(t, (t_x)) \cong Et \sqcap (E't_x) \rightarrow A$ amounts to, by Proposition 10.17, a map $\partial Et \rightarrow \partial A$ in $\widehat{\mathcal{C}} \setminus e$ and maps $E't_x \rightarrow A_{\alpha_x}$ in

$\widehat{\mathcal{D}}$. The composite in A is a d^k -cell, which restricts to α_β in $\widehat{\mathcal{C}} \setminus e$ and a d -cell in A_{α_β} .

The desired T' -algebra morphism is defined by setting $t_x = \eta(d)$ for all x : the maps $E'\eta(d) \rightarrow A_{\alpha_x}$ are merely elements of $(A_{\alpha_x})_d$, which are sent to an element of $(A_{\alpha_\beta})_d$. It is a morphism of presheaves over \mathcal{D} by naturality of η , and commuted with the T' -algebra structure by Lemma 11.9: indeed, for $t' \in S'd$ the two sides of the interchange equation describe respectively $comp_\beta$ applied to t' -composites in each A_{α_x} , and the t' -composite in A_{α_β} of $comp_\beta$ applied cell-wise to maps $E't' \rightarrow A_{\alpha_x}$ for all $x \in Et_e$. Both are, by the interchange law, equal to the composite cell described above in the case when $t_x = t'$ for all x . \square

We are finally ready to prove the main theorem.

Theorem 11.14. *$(T \wr_e T')$ -algebras are equivalent to T -algebras enriched in the cartesian (T, e) -structured category of T' -algebras.*

We describe the correspondence at the level of objects, constructing a T -algebra enriched in T' -algebras from a $(T \wr_e T')$ -algebra and vice versa in a mutually inverse manner, and leave the tedious but straightforward proof that they extend to equivalences of categories to the reader. All of the constructions making up this correspondence have a corresponding functoriality statement that we have omitted for the sake of space and clarity, which assemble into the desired equivalence in an unsurprising manner.

Proof. Let A be a $(T \wr_e T')$ -algebra. We construct an $alg(T')$ -enriched T -algebra as

follows:

- ∂A is a $\partial_e T$ -algebra by Lemma 11.11
- For each $\alpha : \partial y(e_k) \rightarrow \partial A$, A_α is a T' -algebra by Lemma 11.12
- For each $\beta : \partial Et \rightarrow \partial A$, $comp_\beta : \prod_{x \in Et_e} A_{\alpha_x} \rightarrow A_{\alpha_\beta}$ is a morphism of T' -algebras by Lemma 11.13
- The equations of an enriched T -algebra follow immediately from the definition of $comp_\beta$ and the fact that the algebra structure on A , from which these maps derive, respects unit and composite operations from T in the appropriate sense

Conversely, consider a T -algebra enriched in $alg(T')$ given by $(\bar{A}, \text{Hom}(\alpha), comp_\beta)$.

We construct a $(T \wr_e T')$ -algebra as follows:

- $A = \bar{A} \boxtimes (\text{Hom}(\alpha))$, the presheaf in $\widehat{\mathcal{C} \wr_e \mathcal{D}}$ given by plugging the presheaf $\text{Hom}(\alpha)$ over \mathcal{D} into the position α in \bar{A} for each $\alpha : \partial y(e_k) \rightarrow \bar{A}$
- Given $t \in (S \wr S')c = Sc$, $c \notin e$ in \mathcal{C} , t -composition in A is given by that in \bar{A}
- Given $(t, (t_x)_{x \in Et_e}) \in (S \wr S')d^k$ and a map $Et \boxtimes (Et_x) \rightarrow A$ made up of maps $\beta : \partial Et \rightarrow \bar{A}$ and $\beta_x : Et_x \rightarrow \text{Hom}(\alpha_x)$ for each $x \in Et_e$, define its composite in A to be $comp_\beta$ applied to the t_x -composites of β_x in $\text{Hom}(\alpha_x)$
- This choice of compositions endows A with the structure of a $(T \wr_e T')$ -algebra by Proposition 11.10, as the c -cell compositions form a $\partial_e T$ -algebra \bar{A} , the equations for enriched T -algebras ensure that the e_k -operations satisfy the algebra equations from T , the T' -algebra structure on $\text{Hom}(\alpha)$ ensures that the

d -operations satisfy the algebra equations from T' , and the fact that the maps $comp_\beta$ are T' -algebra morphisms ensures that the interchange equation holds. Composition defined above then agrees with that in Proposition 11.10.

□

Example 11.15. When T is the free category monad and T' is the free double category monad, algebras for the monad $T \lambda_e T'$ on “double-graph-enriched graphs” ($\mathbf{G}_1 \wr (\widehat{\mathbf{G}_1} \times \mathbf{G}_1)$) are precisely categories enriched in the symmetric monoidal category of double categories. The same is true when T' is replaced with any other familial monad with connected arities.

Example 11.16. When T is the free multicategory monad and T' is the free category monad, algebras for the monad $T \lambda_e T'$ on $\widehat{\mathbf{M}} \lambda_e \mathbf{G}_1$ have multicategorical compositions, both of trees and stuffed trees, as well as stacking-composition of stuffed n -to-1 trees. By the results above, these two types of composition satisfy an interchange law and these algebras are in fact the same as **Cat**-enriched multicategories.

Example 11.17. When T is the free monoid monad and T' is any familial monad with connected arities, algebras for the familial monad $T \lambda_e T'$ are then monoids enriched in $alg(T')$, which by Example 8.18 are precisely the monoids in the monoidal category $alg(T')$. This includes the free strict monoidal category monad on graphs, the free strict monoidal double category monad on double graphs, and so on for any T' with connected arities.

In particular, the construction of $T \lambda_e T'$ shows that the free strict monoidal category monad on graphs has a single operation outputting a vertex for each finite

set of vertices as the arity, and an operation outputting an edge for each finite set of disjoint string graphs as the arity. This agrees with the structure of a monoidal category, where any finite list of objects has a product, each string of arrows has a composite, and each finite list of arrows has a product which respects arrow composition. Functoriality of the tensor product follows from the fact that two disjoint strings of two arrows have a unique composite which by shapeliness of this monad agrees with both the product of the two arrow composites and the composite of the two arrow products.

Example 11.18. As an example of why T' should have connected arities, consider when both T and T' are the free monoid monad on sets. If $T \wr_e T$ were a familial monad on sets, its algebras would be monoids in the monoidal category of monoids. But these are by the Eckmann-Hilton argument precisely the commutative monoids, which are not (as a category) algebras for a familial monad.

11.5 Wreath Product of Theories

In this section we describe the theory category of a wreath product familial monad, generalizing the classical result that the theory category Θ_{n+1} for $(n+1)$ -categories, which arise as categories enriched in n -categories, is the wreath product of $\Theta_1 \cong \Delta$ and Θ_n .

First, we define the indexing functor for Θ_T , where T is an e -graded familial monad on $\widehat{\mathcal{C}}$. Recall that morphisms $g : t \rightarrow t'$ in Θ_T have an active-

inert factorization $t \rightarrow \mu(t, f_g) \rightarrow t'$ (Theorem 6.31) where $f_g : Et \rightarrow S$, and $E\mu(t, f_g)_e \cong \coprod_{x \in Et_e} Ef_g(x)_e$ (Corollary 5.13).

Definition 11.19. $\tau_e : \Theta_T \rightarrow \mathbf{Span}$ sends t to the set Et_e , and $g : t \rightarrow t'$ to the span

$$Et_e \leftarrow E\mu(t, f_g)_e \rightarrow Et'_e,$$

where the first map sends $(x \in Et_e, x' \in Ef(x)_e)$ to x and the second map is simply the action of the inert part of g on e -cells.

Intuitively, τ_e labels each arity with its number of e -cells, where an active map sends each e -cell to the set of e -cells in the operation it covers and an inert map sends each e -cell to its image under the map in $\widehat{\mathcal{C}}$.

Example 11.20. When T is the free category monad, τ_e recovers the classical functor $\Delta \rightarrow \Gamma \rightarrow \mathbf{Span}$. τ_e sends $[n]$ to the n -element set \underline{n} of its spinal edges between the vertices $i - 1$ and i for $1 \leq i \leq n$, and given a map $[n] \rightarrow [m]$ in Δ with active-inert factorization $[n] \xrightarrow{f} [\ell] \xrightarrow{g} [m]$, the span $\underline{n} \leftarrow \underline{\ell} \rightarrow \underline{m}$ sends the spinal edge between vertices $j - 1, j$ in $[\ell]$ to the unique edge from $i - 1$ to i in $[n]$ with $f(i - 1) \leq j - 1 < j \leq f(i)$, and to the spinal edge from $g(j - 1)$ to $g(j)$ in $[m]$.

As the forward map in the span is injective, this span is in the subcategory Γ and can equivalently be regarded as sending the edge from $i - 1$ to i in $[n]$ to the set of spinal edges from $gf(i - 1)$ to $gf(i)$ in $[m]$, as in the classical functor $\Delta \rightarrow \Gamma$ (see [5, Section 3]).

We now fix familial monads T on $\widehat{\mathcal{C}}$ which is e -graded and T' on $\widehat{\mathcal{D}}$ with connected arities.

Proposition 11.21. *For such T, T' , $\Theta_{T \wr T', 0} \cong \Theta_{T, 0} \wr \Theta_{T', 0}$.*

Proof. This follows from Corollary 10.15, as τ_e agrees with $\widehat{\gamma}_e$ when restricted to $\Theta_{T, 0}$ and the arities of $T \wr T'$ are defined as external wreath products of the arities of T, T' . \square

Example 11.22. When T, T' are both the free category monad on graphs and $\Theta_{T, 0}, \Theta_{T', 0}$ are both the full subcategory of graphs on the string graphs \xrightarrow{n} , this shows that the $\Theta_{T \wr T', 0}$, the full subcategory of 2-globular sets on the 2-dimensional pasting diagrams, agrees with $\Theta_{T, 0} \wr \Theta_{T', 0}$. In particular, this says that a 2-dimensional pasting diagram is precisely a string of arrows \xrightarrow{n} with each edge stuffed with a string $\xrightarrow{m_i}$, resulting in m_i vertically stacked 2-cells in the i th horizontal position in the pasting diagram.

Moreover, the inert subcategory of Θ_n is the n -fold wreath product of $\Theta_{T, 0}$ for T the free category monad.

Lemma 11.23. *For T, T' as above, $\Theta_{T \wr T', a}^{d^k} \cong \Theta_{T, a}^{e_k} \wr \Theta_{T', a}^d$.*

Proof. The objects are exactly the same, both written as $(t \in Se_k, (t_x \in S'd)_{x \in Et_e})$, so it suffices to check that the morphisms agree. Morphisms in $\Theta_{T \wr T', a}^{d^k}$ are given by maps

$$Et \boxtimes (E't_x)_{x \in Et_e} \rightarrow S \boxtimes (S'Et'_e)_{t' \in Se_e},$$

which by Proposition 10.13 (as the arities $E't_x$ are connected) correspond bijectively with pairs

$$f : Et \rightarrow S \quad (E't_x \rightarrow S')_{x \in Et_e, x' \in Ef(x)_e}.$$

These pairs, however, are precisely the data of morphisms in $\Theta_{T,a}^{e_k} \wr \Theta_{T',a}^d$, as τ_e sends f in $\Theta_{T,a}^{e_k}$ to the span

$$Et_e \leftarrow \coprod_{x \in Et_e} Ef(x)_e \cong E\mu(t, f)_e.$$

□

We now want to prove that $\Theta_{T \wr T'} \simeq \Theta_T \wr \Theta_{T'}$, but Lemma 11.23 is not quite strong enough; in general, it does not show that $\Theta_{T \wr T', a} \simeq \Theta_{T, a} \wr \Theta_{T', a}$, as an object in $\Theta_{T, a} \wr \Theta_{T', a}$ could be of the form $(t, (t_x)_{x \in Et})$ where the operations t_x output different cell shapes from \mathcal{D} , which is not allowed in $\Theta_{T \wr T', a}$. However, when T' has *enough units* (Definition 6.46), as is the case in many common examples, this is not an issue as any such collection of operations outputting different cell shapes from \mathcal{D} can be “equivalently” regarded as operations outputting a single cell shape from \mathcal{D} .

When T' has P -units, let P' be the poset on $\text{Ob}(\mathcal{C} \wr \mathcal{D})$ with $c \leq c'$ for all c in $\mathcal{C} \setminus e$ and $d^k \leq (d')^k$ whenever $d \leq d'$ in P . $T \wr T'$ is easily seen to have P' -units with $\eta_{d^k}((d')^k) = (\eta(e_k), (\eta_d(d')))$ and $\eta_{d^k, (d')^k} = \text{id}_{\eta(e_k)} \wr (\eta_d, d')$. $\overline{\Theta}_{T \wr T', a}$ then denotes the subcategory of $\Theta_{T \wr T', a}$ generated by $\Theta_{T \wr T', a}$ and the inert isomorphisms

$$E(t, (t_x)_{x \in Et_e}) \cong Et \wr (E't_x) \xrightarrow{Et \wr (\eta_d, t_x)} Et \wr (E'\eta_d(t_x)) \cong E(t, (\eta_d(t_x))),$$

along with their inverses, for $t_x \in S'd'$ and $d' \leq d$ in P (as in Proposition 6.41).

Definition 11.24. T is (e, κ) -ary for a regular cardinal κ if it is e -graded and for all $t \in Se$, Et_e is κ -small.

Proposition 11.25. For T, T' as above, if T is (e, κ) -ary and T' has κ -enough units, we have

$$\overline{\Theta}_{T \wr_e T', a} \simeq \Theta_{T, a} \wr \overline{\Theta}_{T', a}.$$

The essence of this result is that $\Theta_{T \wr_e T', a}$ agrees with $\Theta_{T, a} \wr \Theta_{T', a}$ when both sides account for the fact that any collection of operations of T' can be treated as if they all output a single cell shape, which is typically top-dimensional or of “arbitrarily high dimension” as in Example 6.48.

Proof. By definition of P and the coproduct decomposition of active subcategories, $\overline{\Theta}_{T \wr_e T', a}$ is equivalent to the coproduct of the categories $\Theta_{T \wr_e T', a}^c$ for c in $\mathcal{C} \setminus e$ and for each k the category $\operatorname{colim}_{d \in P} \Theta_{T \wr_e T', a}^{d^k}$, where the inclusion functors for $d' \leq d$ arise analogously to those in Lemma 6.36.

$\Theta_{T, a} \wr \overline{\Theta}_{T', a}$ is (by Proposition 9.18 and the coproduct decomposition of $\Theta_{T, a}$) the coproduct of the categories $\Theta_{T, a}^c \wr \overline{\Theta}_{T', a} \cong \Theta_{T, a}^c$ for c in $\mathcal{C} \setminus e$ (as τ_e sends $\Theta_{T, a}^c$ to the emptyset) and for each k the category $\Theta_{T, a}^{e_k} \wr \overline{\Theta}_{T', a} = \Theta_{T, a}^{e_k} \wr (\operatorname{colim}_{d \in P} \Theta_{T', a}^d)$. By Lemma 11.23 and Proposition 9.19, as P is by assumption a κ -filtered category, this component is equivalent to

$$\operatorname{colim}_{d \in P} (\Theta_{T, a}^{e_k} \wr \Theta_{T', a}^d) \cong \operatorname{colim}_{d \in P} \Theta_{T \wr_e T', a}^{d^k}.$$

Therefore as $\Theta_{T, a}^c \cong \Theta_{T \wr_e T', a}^c$, all of the components agree and so we have the desired equivalence $\overline{\Theta}_{T \wr_e T', a} \simeq \Theta_{T, a} \wr \overline{\Theta}_{T', a}$. \square

Example 11.26. When T is the free n -category monad and T' is the free m -category monad, this shows that the active subcategory of Θ_{n+m} is the wreath product of the active subcategories of Θ_n and Θ_m . This shows that for any n the active subcategory of Θ_n is $\Delta_+^{op} \wr \cdots \wr \Delta_+^{op}$. This lets the active subcategory of $\Theta_n = \Delta \wr \cdots \wr \Delta$ be constructed out of the active subcategory of Δ analogously to how the active subcategory of $\Delta \times \cdots \times \Delta$ is constructed as $\Delta_+^{op} \times \cdots \times \Delta_+^{op}$.

Finally, we show that, in an appropriate sense, factorization systems (as in Definition 6.44) preserve equivalences.

Proposition 11.27. *If \mathcal{A}, \mathcal{B} are categories with factorization systems $(\mathcal{A}_1, \mathcal{A}_2)$ and $(\mathcal{B}_1, \mathcal{B}_2)$, and $F : \mathcal{A} \rightarrow \mathcal{B}$ is a factorization-preserving functor which restricts to equivalences $F_1 : \mathcal{A}_1 \rightarrow \mathcal{B}_1$ and $F_2 : \mathcal{A}_2 \rightarrow \mathcal{B}_2$, then F is an equivalence.*

Proof. It suffices to show that F is fully faithful and essentially surjective. Fullness and faithfulness of F follow from the same for F_1 and F_2 (which imply that the restriction of F to a functor $\mathcal{A}_1 \cap \mathcal{A}_2 \rightarrow \mathcal{B}_1 \cap \mathcal{B}_2$ is fully faithful as well), and F is essentially surjective as F_1 and F_2 are as functors between wide subcategories of \mathcal{A}, \mathcal{B} . □

Theorem 11.28. *For T an (e, κ) -ary familial monad on $\widehat{\mathcal{C}}$ and T' a familial monad on $\widehat{\mathcal{D}}$ with κ -enough units and connected arities, we have*

$$\Theta_{T \wr T'} \simeq \Theta_T \wr_{\tau_e} \Theta_{T'}.$$

Proof. As τ_e satisfies the conditions of Proposition 9.22, $\Theta_T \wr_{\tau_e} \Theta_{T'}$ has a weak factor-

ization system induced by the strict active-inert factorization system on Θ_T (Theorem 6.31) and the weak active-inert factorization system on $\Theta_{T'}$ accounting for units (Theorem 6.43). The equivalences

$$\Theta_{T \wr_e T', 0} \cong \Theta_{T, 0} \wr \Theta_{T', 0} \qquad \overline{\Theta}_{T \wr_e T', a} \cong \Theta_{T, a} \wr \overline{\Theta}_{T', a}$$

from Proposition 11.21 and Proposition 11.25 assemble into a factorization-preserving functor $\Theta_{T \wr_e T'} \cong \Theta_T \wr_{\tau_e} \Theta_{T'}$, where the factorization system on $\Theta_{T \wr_e T'}$ is given by $\overline{\Theta}_{T \wr_e T', a}$ and $\Theta_{T \wr_e T', 0}$ (this is entirely analogous to Theorem 6.43 even though $T \wr_e T'$ need not have enough units). By Proposition 11.27, this is an equivalence of categories. \square

Example 11.29. This recovers the previously known fact that $\Theta_n \wr \Theta_m \simeq \Theta_{n+m}$, which we now know corresponds to the fact that n -categories enriched in $m - \mathbf{Cat}$ are $(n + m)$ -categories.

Example 11.30. When T' is the free ω -category monad on globular sets, as T' has only ω -enough units, for this result to apply T must be e -graded with arities having only finitely many e -cells. However this is the case in all of the examples we have considered, so Theorem 11.28 describes what kind of cell shapes appear in the nerve of an $\omega - \mathbf{Cat}$ enriched multicategory $(\Omega \wr \Theta)$ or double category $(\Delta^{\times 2} \wr \Theta)$. Of course, an $\omega - \mathbf{Cat}$ enriched n -category is simply an ω -category, which agrees with the theories as $\Theta_n \wr \Theta \simeq \Theta$.

Example 11.31. When T is the free monoid monad and T' has connected arities and ω -enough units, monoidal T' -algebras (the algebras of $T \wr_E T'$) have a fully faithful nerve functor to $\widehat{\mathbf{L}_{mon} \wr \Theta_{T'}}$, diagrams over the category with objects finite

lists of objects in $\Theta_{T'}$ and morphisms certain arrangements of morphisms in $\Theta_{T'}$ (see Example 9.13).

The Segal condition for such a diagram X says that the restriction to the subcategory (isomorphic to $\Theta_{T'}$) of single objects (t) and single morphisms ($g : t \rightarrow t'$) between them satisfies the Segal condition for nerves of T' -algebras, and the restriction to the subcategory isomorphic to \mathbf{L}_{mon} , of objects $(\eta(d), \dots, \eta(d))$ for each fixed d in \mathcal{D} and morphisms made up only of identities, satisfies the Segal condition for monoids. This breakdown of the Segal condition for $\Theta_{T \circ T'}$ into those for Θ_T and $\Theta_{T'}$ is an example of a more general phenomenon that we will explore in future work.

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