

The Geometry of Knots

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MathFest 2016

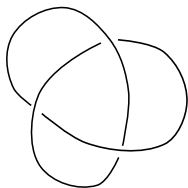
Knots and Links

Definition

A *Knot* is an embedding of the circle in the 3-sphere, S^3 without self intersections.

Definition

A *Link* is an embedding of a finite number of circles in S^3

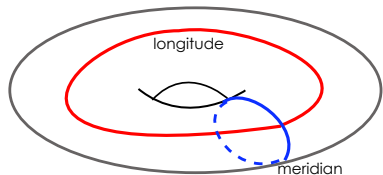


Trefoil Knot



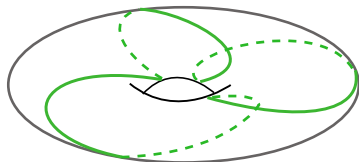
5 Chain

Dehn Filling



Definition

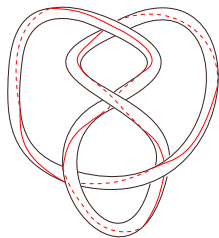
The (p,q) curve on a torus is the curve corresponding to the curve that wraps p times around the meridian and q times around the longitude.



A $(3,1)$ curve on a torus

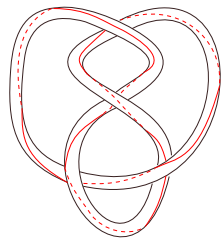
Definition

(p,q) Dehn Filling on a knot in the 3-sphere is 'drilling' out a small torus-shaped neighborhood of the knot, and gluing a solid torus back in such that its meridian is glued to the (p,q) curve of the missing torus



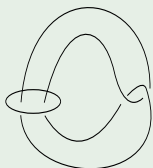
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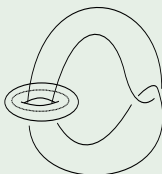


Example

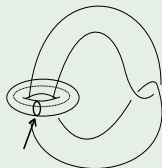
$(1,0)$ Dehn filling



Whitehead Link



Small neighborhood



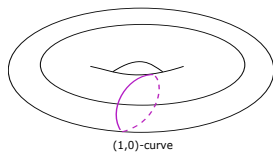
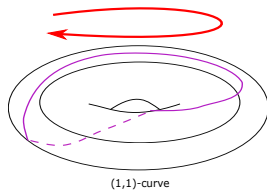
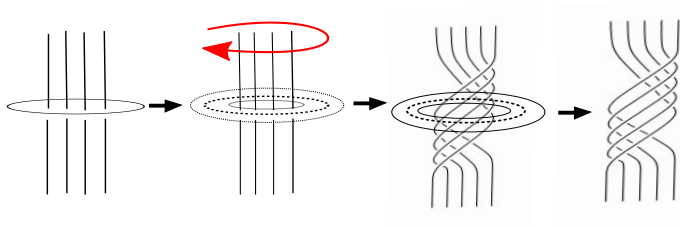
Glue the meridian along the $(1,0)$ curve



The resulting knot

Dehn Filling on Links

(1,1) Dehn filling on a trivial component



Fact

$(1, q)$ Dehn filling on an **unknotted** component of a link complement gives a link complement.

In fact, it will be the complement of the original link, without the trivial component, and the strands through it twisted q times.

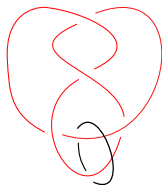
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Fact

Dehn Filling on Knotted components give 3-manifolds, however, they won't necessarily be complements of links or knots



We would call them 'cusped' manifolds because they still have boundary homeomorphic to tori, corresponding to the cusps that don't get filled in the link complement.

Fact

The Lickorish Wallace theorem states that every compact, orientable 3-manifold can be obtained by a Dehn filling on a knot or link complement.

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We will look at ways to use Dehn filling to study some fascinating hyperbolic knot invariants.

Hyperbolic Knots

Definition

A *hyperbolic knot or link* is a knot or link whose complement in the 3-sphere is a 3-manifold that admits a hyperbolic metric.

This gives us a very useful invariant for hyperbolic knots: Volume (V) of the hyperbolic knot complement

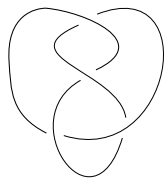


Figure 8 Knot
Volume=2.0298...

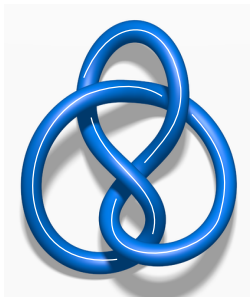


5 Chain
Volume=10.149.....

Cusps of Hyperbolic Knots

Definition

A *Cusp* of a knot or link in S^3 is defined as a tubular neighborhood of the knot or link in the complement.



Definition

The *Cusp Volume* (V_c) of a **hyperbolic** knot or link is the hyperbolic volume of the maximal cusp in the complement.

Definition

Cusp Density (D_C) of a knot or link is the ratio: $\frac{V_C}{V}$ where V_C is the total cusp volume and V is the hyperbolic volume of the complement.

Example

The highest cusp density a hyperbolic manifold can have is $0.853\dots$, the cusp density of the figure 8 knot and the minimally twisted 5-chain.

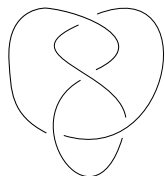


Figure 8 Knot
Volume= $2.0298\dots$
Cusp Volume= $\sqrt{3}$



5 Chain
Volume= $10.149\dots$
Cusp Volume = $5\sqrt{3}$

Restricted Cusp Density

Definition

Restricted Cusp Density of a subset of the components of a link is the ratio of the total cusp volume of just those components to the volume of the complement.

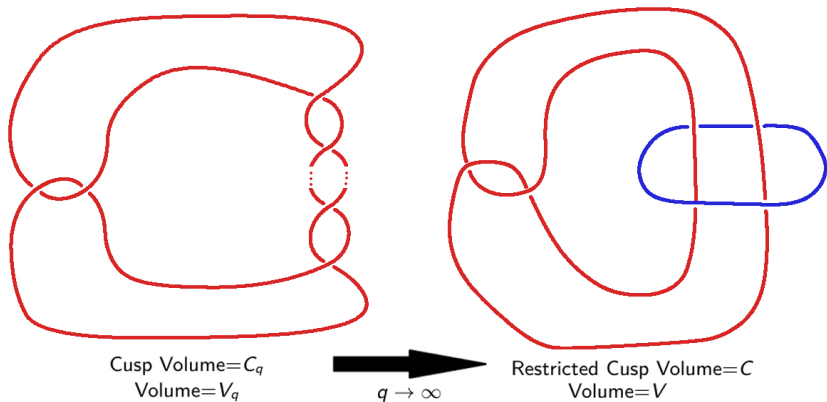
Example

The volume of a single maximized cusp in the 5-chain is $4\sqrt{3}$, so the restricted cusp density of that cusp is $4\sqrt{3}/10.149\dots = 0.6826\dots$



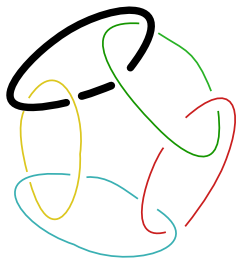
Dehn Filling on Hyperbolic Links

- As q approaches infinity, if a component of a hyperbolic link L is $(1, q)$ Dehn filled, the volume of the resulting manifold and the cusp volumes of the remaining components approach their original values in the complement of L .



Dehn Filling on Hyperbolic Links

- Given a link complement where a subset of the components have restricted cusp density C , if all other components are $(1, q)$ Dehn filled, as q approaches infinity the resulting manifold will have cusp density approaching C .



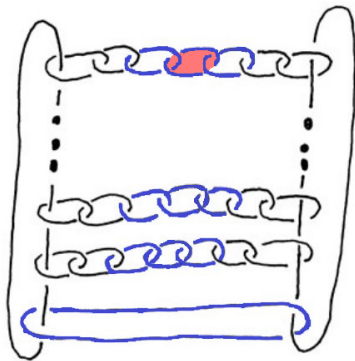
Theorem (SMALL 2016)

For any $x \in [0, 0.853\dots]$, there exist hyperbolic link complements with cusp density arbitrarily close to x .

- In 2002, Adams proved this result for hyperbolic manifolds in general, but we show that the construction in the proof actually uses only link complements.

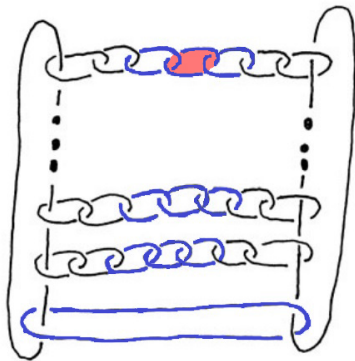
Cusp Density of Hyperbolic Links

- Choose $x \in [0, 0.853\dots]$.
- Adams constructs links of the form below, with additional components attached by belted sum along the red disk.
- The restricted cusp density of the blue components, including those not pictured, is arbitrarily close to x .



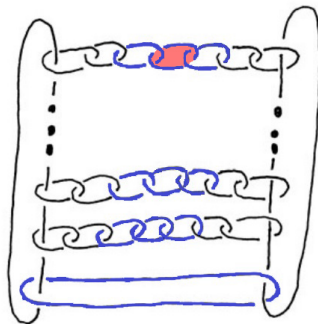
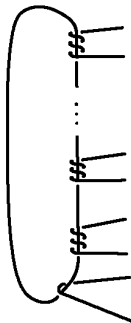
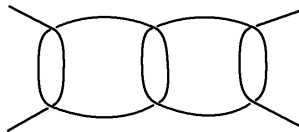
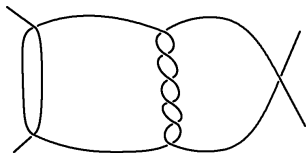
Cusp Density of Hyperbolic Links

- For large q , $(1, q)$ Dehn filling on all remaining components gives manifolds with *cusp density* arbitrarily close to x .
- But are they link complements?



Cusp Density of Hyperbolic Links

- Yes they are!
- The components can be filled in the order indicated below.



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Thank You!

- 1 Colin Adams (2002). "Cusp Densities of Hyperbolic 3-Manifolds" *Proceedings of the Edinburgh Mathematical Society* **45**, 277-284
- 2 W. Thurston (1978). "The geometry and topology of 3-manifolds", *Princeton University lecture notes* (<http://www.msri.org/gt3m>).
- 3 R. Meyerhoff (1978). "Geometric Invariants for 3-Manifolds" *The Mathematical Intelligencer* **14** 37-52.