Cusp Density: Dense or Knot?

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Olivetti Club 10/3/17
Axiom of Euclid: If a straight line $c$ intersects two other straight lines $a$ and $b$ and makes with them two interior angles on the same side whose sum is less than two right angles, then $a$ and $b$ meet on that side of $c$ on which the angles lie.
One model for hyperbolic geometry is the Poincare disk with metric 
\[ g = \frac{2g_{Euc}}{(1-|\vec{x}|^2)^2} \], where straight lines are modeled as lines 
and circle arcs perpendicular to the boundary.

Higher dimensional disks (like the unit ball in \( \mathbb{R}^3 \)) similarly 
model higher dimensional hyperbolic space.
Hyperbolic Geometry

- The hyperbolic plane has a greater variety of isometries than the Euclidean plane.
- Parabolic isometries fix only a point on the boundary.
- A horocycle is the orbit of a point under the set of isometries fixing only a particular boundary point.
Another model of hyperbolic space is the upper half-space, with metric \( g = \frac{g_{Euc}}{z^2} \).

Geodesics vertical lines or circles perpendicular to boundary.

Similar to Poincare model, but with boundary flattened (except point at infinity).
Hyperbolic Geometry

- Horospheres ‘centered’ on the xy-plane look like those in the Poincare model.
- Horospheres ‘centered’ at infinity are flat horizontal planes.
- These planes inherit a Euclidean metric, thus so do all horospheres(!)
Hyperbolic Geometry

- Polyhedra can have ‘ideal’ vertices at infinity.
- The faces of polyhedra are totally geodesic planes.
A hyperbolic 3-manifold is a 3-manifold with a Riemannian metric having constant curvature -1.

The volume of a hyperbolic 3-manifold is the integral of the volume form given by the metric over the entire manifold.

By the Mostow Rigidity Theorem, a hyperbolic metric on a manifold is unique up to isometry, so hyperbolic volume is a manifold invariant.
Any hyperbolic 3-manifold is the quotient of hyperbolic 3-space by the orbits of a discrete group of fixed point free isometries.

Such a manifold has finite volume if a fundamental domain of that discrete group of isometries is made up of finitely many hyperbolic tetrahedra.
Hyperbolic Links

- A knot is a smooth embedding of $S^1$ in $S^3$.
- A link is a smooth embedding of the disjoint union of any number of copies of $S^1$ in $S^3$.
- A link is hyperbolic if the complement of its image in $S^3$ is a hyperbolic manifold.
Hyperbolic Links

- The figure 8 knot has volume $2v_t$, where $v_t$ is the volume of an ideal regular tetrahedron.
- The minimally twisted 5-chain has volume $10v_t$.
- Computing these hyperbolic structures is hard, but it can be done by a computer (SnapPea/SnapPy)
Hyperbolic Links

- A cusp in a hyperbolic 3-manifold is a $T^2 \times [0, \infty)$ neighborhood of a boundary component.
- For hyperbolic links, a cusp is a solid torus neighborhood of a component, intersected with the complement.
- A cusp is maximal when it is tangent to itself and thus cannot be further expanded.
- The cusp volume of a manifold is the maximal total volume among all configurations of nonintersecting cusps around its boundary components.
A cusp in a hyperbolic manifold lifts to a disjoint union of horoballs in hyperbolic space, all of which are identified in the manifold by the covering transformations.

Cusps arise from quotients of hyperbolic space by a group generated by two parabolic isometries about the same boundary point.

Letting that boundary point be the point at infinity and considering the horoball centered at infinity helps make clear why this gives a solid torus (sans core curve).
Covers and Gluings

- An n-fold cyclic cover of a link, unwinding around some component, is the link formed by cutting the complement open along a (punctured) surface bounded by the link and gluing together n copies of the manifold in a cycle.
- This can also be done for knots and general 3-manifolds, but the cover may not be a knot complement.
- Taking an n-fold cover multiplies both volume and cusp volume by n.

Twisted Borromean Rings

3-fold cover
The belted sum of two links is formed by cutting each open along a twice-punctured disk and gluing them together. The resulting link complement is a hyperbolic manifold as any twice-punctured disk in a hyperbolic manifold is isotopic to a totally geodesic surface with unique hyperbolic structure (Adams). The volume of the sum is the sum of the volumes, but the cusp volumes may not add.

\[ L_1 \quad L_2 \quad \text{Belted Sum: } L_{1+2} \]
Covers and Gluings

- There are special cases where cusp volumes do add in belted sums.
- Any belted sum of ‘tetrahedral’ manifolds built only out of ideal regular tetrahedra has cusp volume the sum of those of its summands.
- The figure 8 knot and the minimally twisted 5-chain are tetrahedral.
(p,q) Dehn filling on a torus shaped boundary component of a manifold is the operation of attaching a solid torus to the manifold by gluing the meridian of the boundary of the solid torus to a (p,q) curve on the manifold boundary component.
Dehn Filling

- (1,q) Dehn filling on an unknotted component of a hyperbolic link complement gives the complement of the link with the filled component removed and q full twists applied to the strands passing through it.
Dehn Filling

- As $q$ approaches infinity, if a component of a hyperbolic link $L$ is $(1, q)$ Dehn filled, the volume of the resulting manifold and the cusp volumes of the remaining components approach their original values in the complement of $L$. 

Cusp Volume = $C_q$
Volume = $V_q$

$q \to \infty$

Restricted Cusp Volume = $C$
Volume = $V$
Cusp Density

- The *Cusp Density* of a hyperbolic 3-manifold is the ratio of the cusp volume to the volume of the manifold.
- The *Restricted Cusp Density* of a subset of the cusps of a manifold is the ratio of the cusp volume from just those cusps to the volume of the manifold.
- Results on horosphere packing in hyperbolic space show that cusp density is bounded above by \(0.853... = \frac{\sqrt{3}}{2v_t}\).
All tetrahedral manifolds have cusp density .853...
The minimally twisted 5-chain then has cusp volume .853...
The restricted cusp density of just one cusp is .68...
Cusp Density

- $D_n$ is the alternating daisy chain with $n$ components
- It is always possible to have volume of at least $\sqrt{3}/4$ in all cusps of a manifold at once, thus the total volume of $D_n$ goes to infinity.
- The maximal cusp volume of a single cusp approaches that of a component of the Borromean rings, which is 4, so as $n$ goes to $\infty$ the restricted cusp density of one cusp approaches zero.
Theorem (Adams 2001): The set of values of cusp density for finite-volume hyperbolic 3-manifolds is dense in the interval $[0, .853...]$.

To prove this, choose any $x \in [0, .853...]$ and construct a sequence of 3-manifolds with cusp density approaching $x$. 
Density Construction for Manifolds

- The minimally twisted 5-chain has cusp density $0.853...$ with up to $4\sqrt{3}$ volume per cusp.
- A 2-fold cyclic cover $L$ of the 5-chain has the same cusp density.
- Let $L_k$ be a k-fold cyclic cover of $L$ about the component $C'$

![Diagram showing density construction for manifolds with labels for dimensions such as $8\sqrt{3}$ and $16\sqrt{3}$]
Density Construction for Manifolds

- $n$ can be chosen so that the restricted cusp density of the labelled components is arbitrarily close to 0.
- The same then holds for the lifts of those components in the $m$-fold cyclic cover $D_{n,m}$
Define $F_{k,n,m}$ as the belted sum of $L_k$ and $D_{n,m}$ along the highlighted disk and $E$.

Choose even $n$ large enough so that the maximal volume in each cusp is within .1 of 4 and the restricted cusp density of the cusps $C_1, \ldots, C_4$ is less than $x$.

The cusp volumes add, up to a constant $p < 16\sqrt{3} + 12.3$. 

Shapiro Cusp Density
The cusp density of $F_{k,n,m}$ restricted to the cusps in $L_k$ and those in black in $D_{n,m}$ is then, in terms of the volumes $V_L, V_D$ of $L$ and $D_n$ and cusp volumes $CV_L, CV_D$, is

$$\frac{kCV_L + mCV_D - p}{kV_L + mV_D} = \frac{k}{m} \frac{CV_L + CV_D}{V_L + V_D} - \frac{p}{kV_L + mV_D}$$

$k$ and $m$ can be made arbitrarily large, making the second term negligible without affecting the first.
Replace $\frac{k}{m}$ with a real variable $t$:

$$f(t) = \frac{tCV_L + CV_D}{tV_L + V_D} - \epsilon$$

As $t$ goes to 0, $f(t)$ goes to the restricted cusp density of $D_n$ (less than $x$), and as $t$ goes to infinity $f(t)$ goes to $0.853...$

Thus $t$ can be chosen such that $f(t) = x$, and as $f$ is continuous this value can be approached by the images of a sequence of rationals approaching $t$. 
Density Construction for Manifolds

- We now have a means of constructing manifolds with a restricted cusp density arbitrarily close to $x$.
- Now just do $(1, q)$ Dehn filling on each of the remaining (blue) components, which for high enough $q$ has volume and cusp volumes arbitrarily close to those in the original manifold.
- The filled manifolds now have cusp density arbitrarily close to $x$, completing the proof.
This construction shows that the cusp densities of finite volume hyperbolic 3-manifolds are dense in $[0,.853...]$.

However, it is not hard to show that the manifolds constructed are link complements.

This would then show that the cusp densities of link complements are dense in the same interval.
Density Construction for Links

- Use result on Dehn filling an unknotted component:
Density Construction for Links

- This shows the manifold obtained by filling on the blue components below is a link complement.
- The same argument applies whenever the unknotted components have no cycles in their adjacencies.
Density Construction for 1-Cusped Manifolds
Find examples to extend result to entire interval for 1-cusped manifolds.

Do the same for knots (harder).

Find sequences of knots with cusp density approaching .853...
Acknowledgements

- Professor Colin Adams
- Josh, Michael, Rosie, & Shruthi
- SMALL
- National Science Foundation REU Grant DMS - 1347804
- Williams College Science Center
- SnapPy
