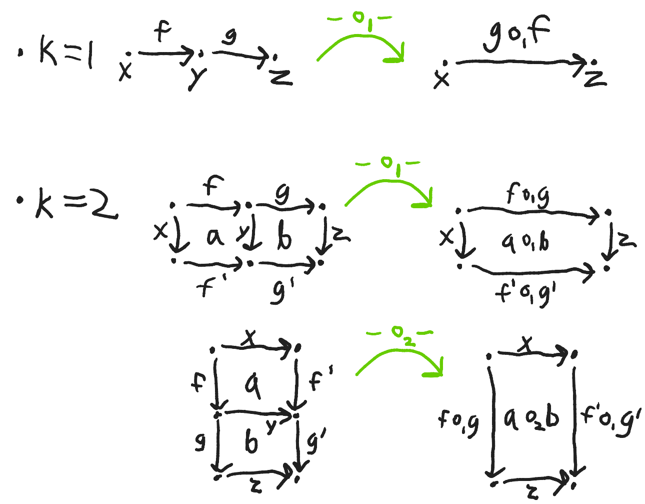
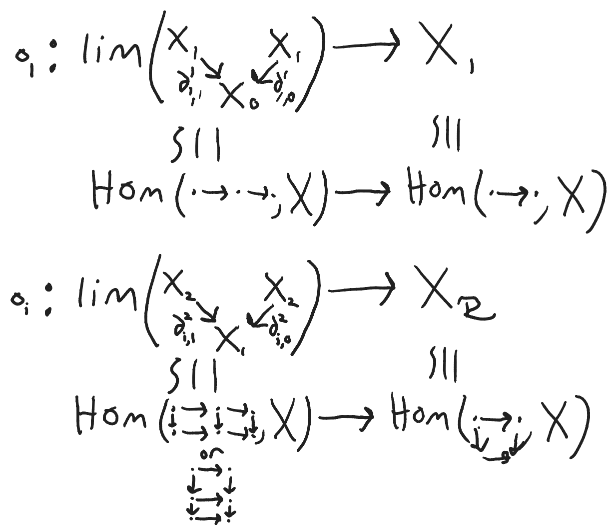
Cubical n-Categories and Cubical Theta

Thursday, May 14, 2020 7:07 PM

Cubical n-categories (n=1,2,3,…,ω)

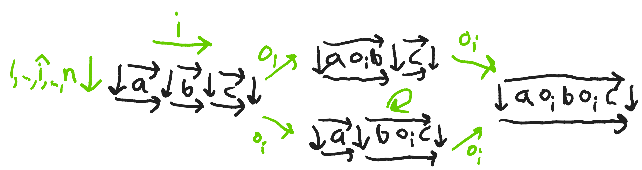
— A (strict) cubical n-category is an n-truncated cubical set X (of any sort with degeneracies) equipped with k different composition operations between k-cubes (k≤n):



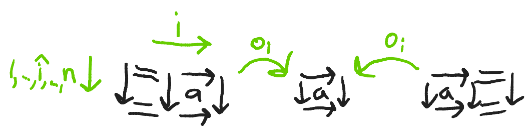


Such that

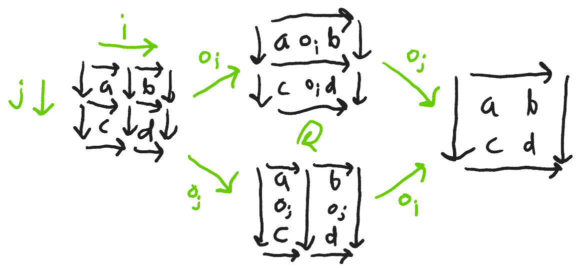
• Each $\circ_i$ is associative



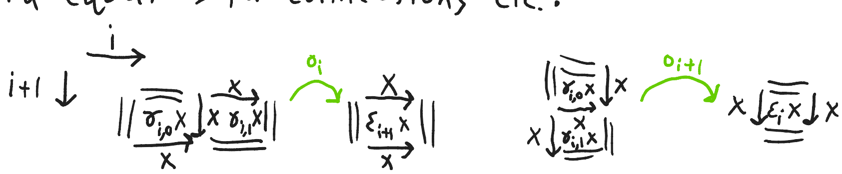
• $\varepsilon_i : X_{n-1} \to X_n$ provides $\circ_i$ with units



• Interchange law



• Extra equations for connections etc.:



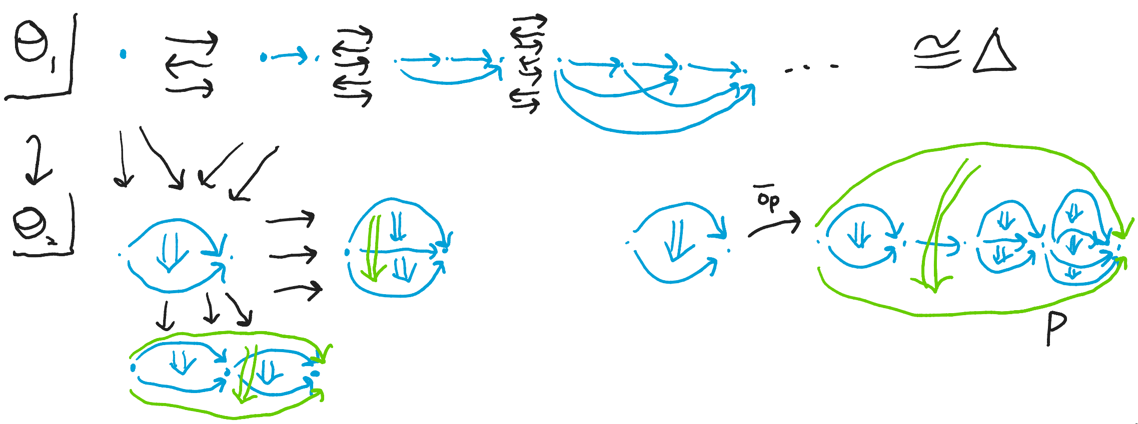
Theorem (Al-Agl, Brown, Steiner, "Multiple categories: the equivalence of a globular and a cubical approach")

The category of cubical n-categories with connections is equivalent to that of globular n-categories for $n = 1, 2, \ldots, \omega$

Proof (Movie version)

Globular n-categories & Θ_n

— Θ_n is the full subcategory of strict n-categories containing the free n-categories on globular pasting diagrams

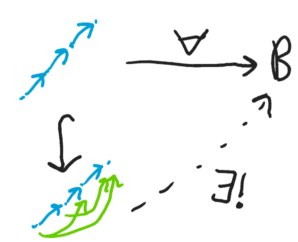


— An n-category A has composition maps $\mathrm{Hom}_{n\text{-cat}}(P, A) \xrightarrow{op} \mathrm{Hom}_{n\text{-cat}}(\text{(diagram)}^{(n)}, A)$ for each n-pasting diagram P, induced by $P \xleftarrow{op} \text{(diagram)}^{(n)}$

— $P \mapsto \mathrm{Hom}_{n\text{-cat}}(P, A)$ defines a functor $NA : \Theta_n^{op} \to \mathrm{Set}$, "the nerve of A"

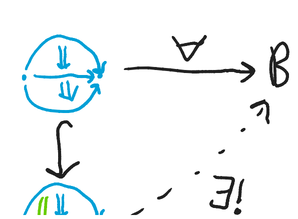
— $N : n\text{-Cat} \to \hat{\Theta}_n$ is fully faithful, and a Θ_n-set B is the nerve of an n-category if for all pasting diagrams P (such as $\to\to\to$ or (diagram)) we have

$B_{\to\to\to} \cong \lim\left(B_{\to} \to B_1 \leftarrow B_{\to} \to B_1 \leftarrow B_{\to}\right) \Leftrightarrow \mathrm{Hom}(\text{(diagram)}, B) \cong \mathrm{Hom}(\to\to\to, B) \Leftrightarrow$



$B_3 \cong B_1 \times_{B_0} B_1 \times_{B_0} B_1$ in $\hat{\Delta}$, the "Segal condition"

$B_{\text{(diagram)}} \cong \lim\left(B_{\text{(diagram)}} \to B_{\to} \leftarrow B_{\text{(diagram)}}\right) \Leftrightarrow \mathrm{Hom}(\text{(diagram)}, B) \cong \mathrm{Hom}(\text{(diagram)}, B) \Leftrightarrow$



Call such a B "Segal"

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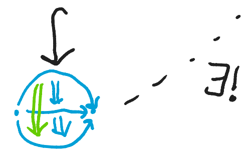


— ⇓ n-categories can be equivalently defined as Segal functors $\Theta_n^{op} \to Set$

— A category equipped with a class of distinguished limits (here Θ_n^{op} & limits above) is called a "limit sketch", and a "model" of the sketch is a functor to Set preserving those limits.

Models of this sketch on Θ_n^{op} are n-categories.

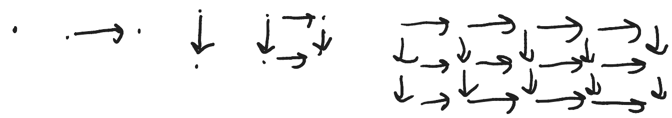
The category $\boxminus_n$

— Want an analogous description of cubical n-categories

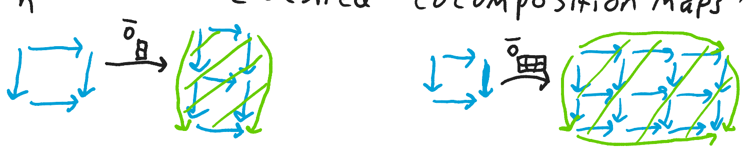
— $\boxminus_n$ should be a full subcategory of n-$\square$Cat containing free cubical n-cats on cubical pasting diagrams

— Natural choice of cubical n-pasting diagrams: n-dimensional grids

$n=2$



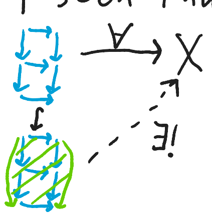
— $\boxminus_n$ has the desired "cocomposition maps"



— For X a cubical n-cat, define its nerve $N_\square X : (\boxminus_n)^{op} \to Set$

$P \mapsto Hom_{n\text{-}\square Cat}(P, X)$

— $N_\square$ is fully faithful, so n-$\square$Cat can be identified with the full subcat of $\hat{\boxminus}_n$ containing Y such that for all P (such as ⊞)



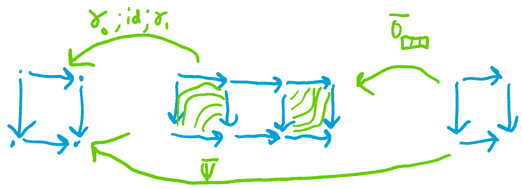
call such X Segal

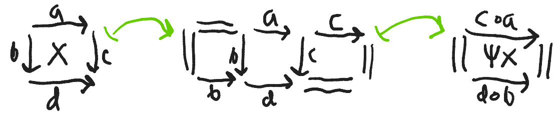
$\boxminus_n^r$ and $\boxtimes_n^r$

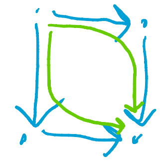
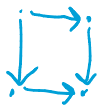
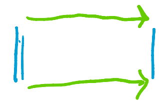
— n-Pasting diagrams are all of the diagrams that can

— n-Pasting diagrams are all of the diagrams that can compose into an n-cube

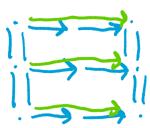
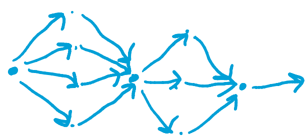
— With both connections, we get some extra pasting diagrams:



In X: 

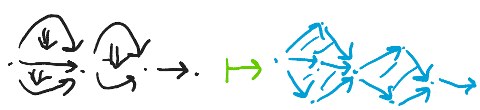
The free cubical 2-cat on a single square looks like 
with an outer square  and inner square 

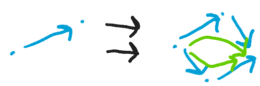
we can then use vertical composition to compose these "diagonally"

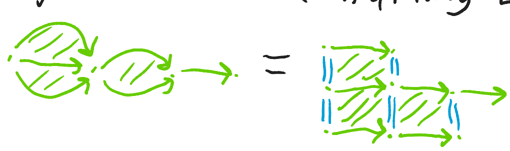
 

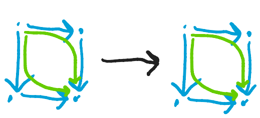
— Call $\boxminus_n^\gamma$ the extension of $\boxminus_n$ to include these additional objects in n-□cat

— Cubical n-categories with connections similarly form the full subcategory of $\widehat{\boxminus}_n^\gamma$ satisfying (now a few extra) Segal conditions

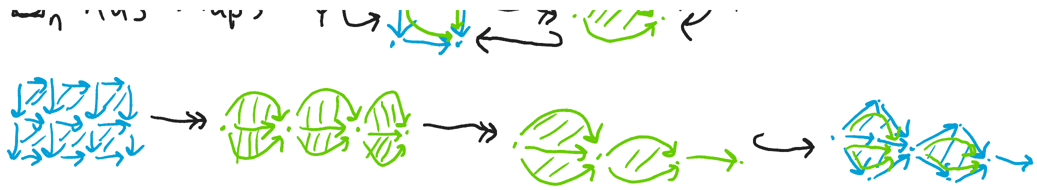
— There is a functor $\Theta_n \to \boxminus_n^\gamma$: 

Not full : 

— Define $\boxslash_n^\gamma$ as the full subcategory of n-□cat containing $\boxminus_n^\gamma$ and the globular pasting diagrams 

— Ψ:  is idempotent with image 

$\boxslash_n^\gamma$ has maps Ψ 



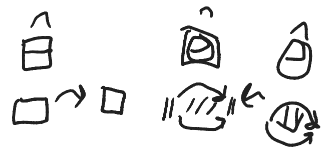
Comparing Cubical and Globular n-categories

— There are fully faithful functors $\square_n^{(\gamma)} \xrightarrow{\beta} \square\Theta_n^{\gamma} \xleftarrow{\alpha} \Theta_n$

and forgetful functors with right adjoints $\widehat{\square}_n^{(\gamma)} \underset{\beta_*}{\overset{\beta^*}{\rightleftarrows}} \widehat{\square\Theta}_n^{\gamma} \underset{\alpha_*}{\overset{\alpha^*}{\rightleftarrows}} \widehat{\Theta}_n$

— For p in $\widehat{\square\Theta}_n^{\gamma}$ $\quad (\alpha_* A)_p = \mathrm{Hom}_{\widehat{\Theta}_n}(\alpha^* y(P), A)$

$(\beta_* X)_p \cong \mathrm{Hom}_{\widehat{\square}_n^{(\gamma)}}(\beta^* y(P), X)$



— As α, β fully faithful, $\alpha^* y(\alpha P_\circ) \cong y(P_\circ)$ for $P_\circ$ in Θ_n

$\beta^* y(\beta P_\square) \cong y(P_\square)$ for $P_\square$ in $\square_n^{(\gamma)}$

so $(\alpha^*\alpha_* A)_{P_\circ} \cong \mathrm{Hom}_{\widehat{\Theta}_n}(\alpha^* y(\alpha P_\circ), A) \cong \mathrm{Hom}_{\widehat{\Theta}_n}(y(P_\circ), A) \cong A_{P_\circ}$

$(\beta^*\beta_* X)_{P_\square} \cong \mathrm{Hom}_{\widehat{\square}_n^{(\gamma)}}(\beta^* y(\beta P_\square), X) \cong \mathrm{Hom}_{\widehat{\square}_n^{(\gamma)}}(y(P_\square), X) \cong X_{P_\square}$

counits iso $\Rightarrow$ $\widehat{\square}_n^{(\gamma)}$ and $\widehat{\Theta}_n$ reflective subcategories of $\widehat{\square\Theta}_n^{\gamma}$

— Remains to show their Segal subcategories agree in $\widehat{\square\Theta}_n^{\gamma}$

— Z in $\widehat{\square\Theta}_n^{\gamma}$ is Segal if both $\alpha^* Z$, $\beta^* Z$ are Segal

— We will show that if Z is Segal the units of the adjunctions $\widehat{\square}_n^{(\gamma)} \underset{\beta_*}{\overset{\beta^*}{\rightleftarrows}} \widehat{\square\Theta}_n^{\gamma} \underset{\alpha_*}{\overset{\alpha^*}{\rightleftarrows}} \widehat{\Theta}_n$ are iso at Z

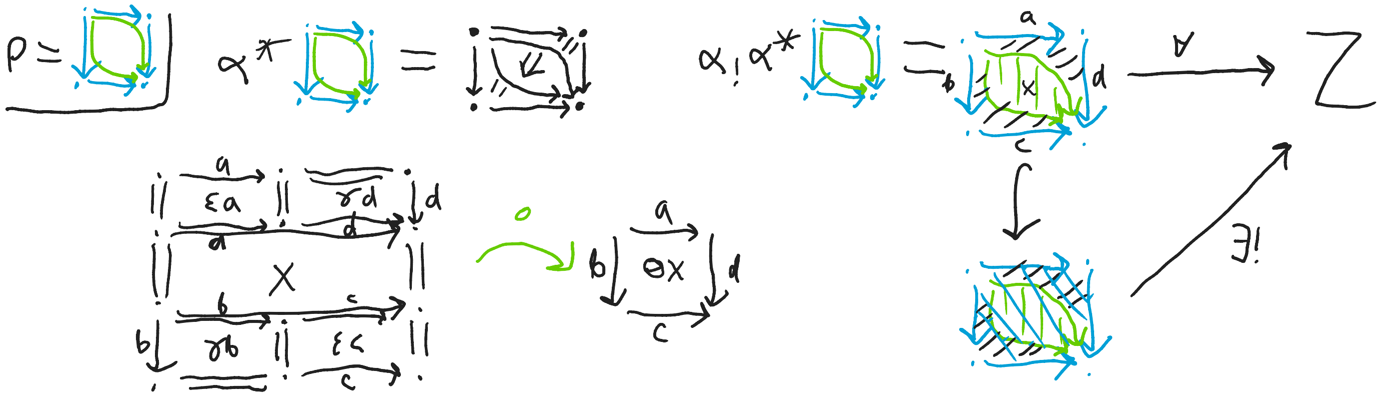
—α— $Z_p \cong \mathrm{Hom}_{\widehat{\square\Theta}_n^{\gamma}}(y(P), Z) \to \mathrm{Hom}_{\widehat{\Theta}_n}(\alpha^* y(P), \alpha^* Z) = (\alpha_*\alpha^* Z)_p$

$\cong \mathrm{Hom}_{\widehat{\square\Theta}_n^{\gamma}}(\alpha_!\alpha^* y(P), Z)$ $\qquad \widehat{\square\Theta}_n^{\gamma} \underset{\alpha_*}{\overset{\alpha^*}{\rightleftarrows}} \widehat{\Theta}_n$, $\alpha_! \dashv \alpha^* \dashv \alpha_*$

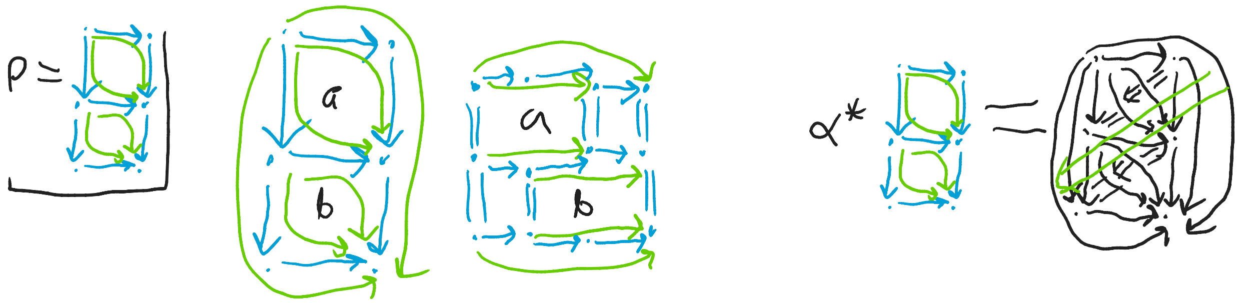
So we need $\alpha_!\alpha^* y(P) \xrightarrow{\forall} Z$

So we need $\alpha_! \alpha^* y(P) \xrightarrow{\forall} Z$, with $\alpha_! \alpha^* y(P) \to y(P)$ and $y(P) \dashrightarrow Z$, $\exists!$

$P = \alpha P_0$ | $\alpha_! \alpha^* y(\alpha P_0) \simeq \alpha_! y(P_0) \simeq y(\alpha P_0)$



This is actually sufficient for all remaining P in Θ_2^γ



—— $\Rightarrow$ $\mathrm{Segal}(\hat{\Theta}_n^\gamma) \simeq n\text{-cat}$

β —— $Z_P \cong \mathrm{Hom}_{\hat{\Theta}_n^\gamma}(y(P), Z) \to \mathrm{Hom}_{\hat{\square}_n^{\gamma}}(\beta^* y(P), \beta^* Z) = (\beta_* \beta^* Z)_P$

$P = \beta(P_\square)$ | Automatic, as above

$P =$ [pic] | Recall Θ_n^γ has maps ΨG

$\Rightarrow$ Any Z has $Z_{\circ} \cong \mathrm{im} Z_\Psi \subset Z_\square$

So as $\beta_* \beta^* Z_\square \cong Z_\square$, $\beta_* \beta^* Z_{\circ} \cong Z_{\circ}$

$P =$ [pic] | ... vertical composition $\Rightarrow$ globular

$j_{*}^{!!} \mathcal{L}_{\square} = \mathcal{L}_{\square}$, $j_{*}^{!!} \mathcal{L}_{\circledcirc} = \mathcal{L}_{\circledcirc}$

P = (globular composite) Follows from ↑ as cubical composition ⇒ globular

— ↳ $\omega\square\mathrm{cat} \simeq \mathrm{Segal}(\hat{\square}_n^{:})$

— conclusion: $\omega\square\mathrm{cat} \simeq \mathrm{Segal}(\hat{\square}_n^{:}) \simeq n\text{-cat}$

□

— All of the equations in Al-Agl-Brown-Steiner can be interpreted as equations in $\hat{\square}_n^{\sigma}$